Component Analysis Methods for Signal Processing

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Component Analysis for PR

- Computer Vision & Image Processing
  - Structure from motion.
  - Spectral graph methods for segmentation.
  - Appearance and shape models.
  - Fundamental matrix estimation and calibration.
  - Compression.
  - Classification.
  - Dimensionality reduction and visualization.
- Signal Processing
  - Spectral estimation, system identification (e.g. Kalman filter), sensor array processing (e.g. cocktail problem, eco cancellation), blind source separation, …
- Computer Graphics
  - Compression (BRDF), synthesis,…
- Speech, bioinformatics, combinatorial problems.

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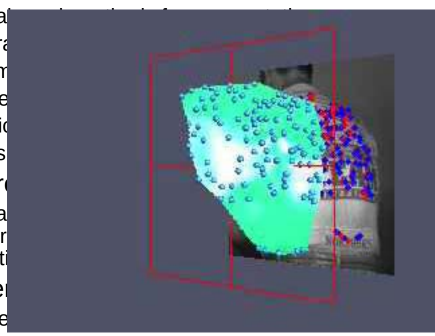

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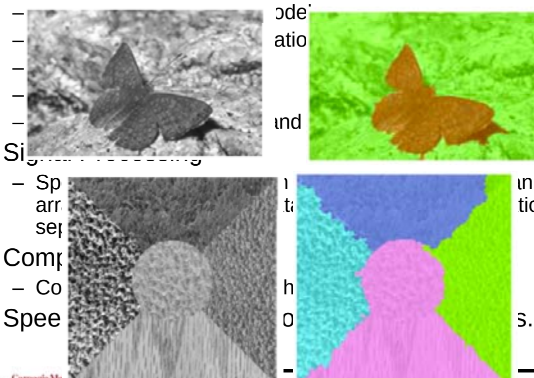

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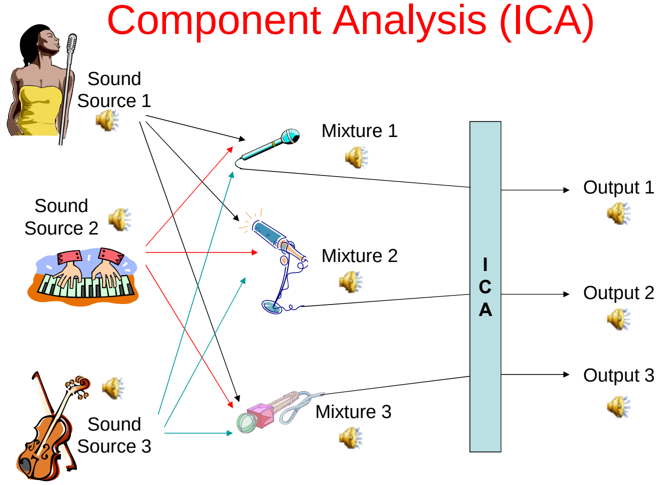
Component Analysis (ICA)



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Why Component Analysis for PR?

- Learn from high dimensional data and few samples.
  - Useful for dimensionality reduction.
- Easy to incorporate
  - Robustness to noise, missing data, outliers (de la Torre & Black, 2003a)
  - Invariance to geometric transformations (de la Torre & Black, 2003b; de la Torre & Nguyen,2007)
  - Non-linearities (Kernel methods) (Scholkopf & Smola,2002; Shawe-Taylor & Cristianini,2004)
  - Probabilistic (latent variable models) (Everitt,1984)
  - Multi-factorial (tensors) (Paatero & Tapper, 1994 ;O'Leary & Peleg,1983; Vasilescu & Terzopoulos,2002; Vasilescu & Terzopoulos,2003)
  - Exponential family PCA (Gordon,2002; Collins et al. 01)
- Efficient methods $O(\underbrace{d}_{\text{features}}\quad \underbrace{n < < n^2}_{\text{samples}})$

Are CA Methods Popular/Useful/Used?

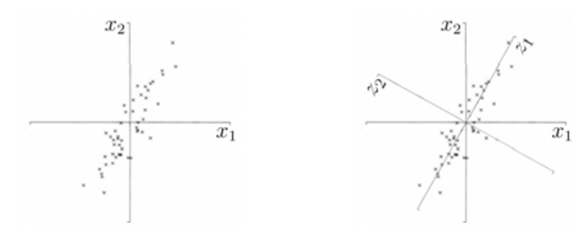
- About 20% of CVPR-06 papers use CA.
- Google:
  - Results **1** - **10** of about **1,870,000** for **"principal component analysis"**.
  - Results **1** - **10** of about **506,000** for **"independent component analysis"**.
  - Results **1** - **10** of about **273,000** for **"linear discriminant analysis"**.
  - Results **1** - **10** of about **46,100** for **"negative matrix factorization"**.
  - Results **1** - **10** of about **491,000** for **"kernel methods"**.
- Still work to do
  - Results **1** - **10** of about **65,300,000** for **"Britney Spears"**.

Outline

- Introduction
- Generative models
  - Principal Component Analysis (PCA)
  - Non-negative Matrix Factorization (NMF)
  - Independent Component Analysis (ICA)
  - Multidimensional Scaling (MDS)
- Discriminative models
  - Linear Discriminant Analysis (LDA).
  - Oriented Component Analysis (OCA).
  - Canonical Correlation Analysis (CCA).
- Standard extensions of linear models
  - Kernel methods.
  - Latent variable models.
  - Tensor factorization

Principal Component Analysis (PCA)

(Pearson, 1901; Hotelling, 1933;Mardia et al., 1979; Jolliffe, 1986; Diamantaras, 1996)

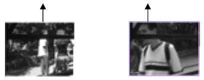


- PCA finds the directions of maximum variation of the data based on linear correlation.
- PCA decorrelates the original variables.

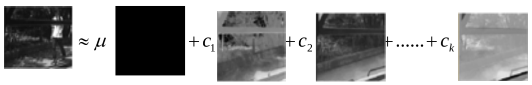
PCA

$d =$ pixels, $n =$ images

$$\mathbf{D} = [\mathbf{d}_1\ \mathbf{d}_2\ \ldots\ \mathbf{d}_n] \approx \mathbf{BC} + \boldsymbol{\mu}\mathbf{1}_n^T$$



$$\mathbf{D} \in \Re^{d\times n} \quad \mathbf{B} \in \Re^{d\times k}$$
$$\mathbf{C} \in \Re^{k\times n} \quad \boldsymbol{\mu} \in \Re^{d\times 1}$$



$\approx \mu + c_1 + c_2 + \ldots\ldots + c_k$

•Assuming 0 mean data, the basis **B** that preserve the maximum variation of the signal is given by the eigenvectors of **DD**^{T}.

$$\underbrace{\mathbf{DD}^T}_{d\times d}\mathbf{B} = \mathbf{B}\Lambda$$

Snap-shot Method & SVD

- If d>>n (e.g. images 100*100 vs. 300 samples) no **DD**^{T}.
- **DD**^{T} and **D**^{T}**D** have the same eigenvalues (energy) and related eigenvectors (by **D**).
- **B** is a linear combination of the data! (Sirovich, 1987)

$$\mathbf{DD}^T\mathbf{B} = \mathbf{B}\Lambda \quad \mathbf{B} = \mathbf{D}\boldsymbol{\alpha} \quad \cancel{\mathbf{D}^T}\mathbf{D}\mathbf{D}^T\mathbf{D}\boldsymbol{\alpha} = \cancel{\mathbf{D}^T}\mathbf{D}\boldsymbol{\alpha}\Lambda$$

- [**α**,**L**]=eig(**D**^{T}**D**) **B**=**D** **α**(diag(diag(**L**))) $^{-0.5}$
- SVD factorizes the data matrix **D** as: $\mathbf{DD}^T = \mathbf{U}\Lambda\mathbf{U}^T$

(Beltrami, 1873; Schmidt, 1907; Golub & Loan, 1989)

PCA	SVD
$\mathbf{D} = \mathbf{BC}$	$\mathbf{D} = \mathbf{U}\Sigma\mathbf{V}^T$ $\quad \mathbf{D}^T\mathbf{D} = \mathbf{V}\Lambda\mathbf{V}^T$
$\mathbf{B} \in \Re^{d\times k}$ $\quad \mathbf{C} \in \Re^{k\times n}$	$\mathbf{U} \in \Re^{d\times k}$ $\quad \Sigma \in \Re^{k\times n}$ $\quad \mathbf{V} \in \Re^{n\times n}$
$\mathbf{B}^T\mathbf{B} = \mathbf{I}$ $\quad \mathbf{CC}^T = \Lambda$	$\mathbf{U}^T\mathbf{U} = \mathbf{I}$ $\quad \mathbf{V}^T\mathbf{V} = \mathbf{I}$ $\quad \Sigma$ *diagonal*

Error Function for PCA

- PCA minimizes the following **CONVEX** function.

(Eckardt & Young, 1936; Gabriel & Zamir, 1979; Baldi & Hornik, 1989; Shum et al., 1995; de la Torre & Black, 2003a)

$$E_1(\mathbf{B},\mathbf{C}) = \sum_{i=1}^{n} \|\mathbf{d}_i - \mathbf{Bc}_i\|_2^2 = \|\mathbf{D} - \mathbf{BC}\|_F$$

- Not unique solution: $\mathbf{BRR}^{-1}\mathbf{C} = \mathbf{BC} \qquad \mathbf{R} \in \Re^{k\times k}$
- To obtain same PCA solution **R** has to satisfy:

$$\hat{\mathbf{B}} = \mathbf{BR} \qquad \hat{\mathbf{C}} = \mathbf{R}^{-1}\mathbf{C}$$
$$\hat{\mathbf{B}}^T\hat{\mathbf{B}} = \mathbf{I} \qquad \hat{\mathbf{C}}\hat{\mathbf{C}}^T = \Lambda$$

- **R** is computed as a generalized k×k eigenvalue problem.

(de la Torre, 2006)

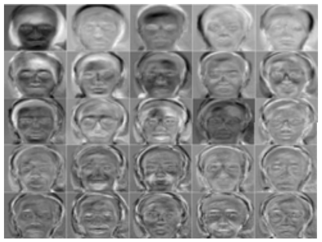
$$(\mathbf{CC}^T)^{-1}\mathbf{R} = \mathbf{B}^T\mathbf{BR}\Lambda^{-1}$$

PCA/SVD in Computer Vision

- PCA/SVD has been applied to:
  - Recognition (eigenfaces:Turk & Pentland, 1991; Sirovich & Kirby, 1987; Leonardis & Bischof, 2000; Gong et al., 2000; McKenna et al., 1997a)
  - Parameterized motion models (Yacoob & Black, 1999; Black et al., 2000; Black, 1999; Black & Jepson, 1998)
  - Appearance/shape models (Cootes & Taylor, 2001; Cootes et al., 1998; Pentland et al., 1994; Jones & Poggio, 1998; Casia & Sclaroff, 1999; Black & Jepson, 1998; Blanz & Vetter, 1999; Cootes et al., 1995; McKenna et al., 1997; de la Torre et al., 1998b; de la Torre et al., 1998b)
  - Dynamic appearance models (Soatto et al., 2001; Rao, 1997; Orriols & Binefa, 2001; Gong et al., 2000)
  - Structure from Motion (Tomasi & Kanade, 1992; Bregler et al., 2000; Sturm & Triggs, 1996; Brand, 2001)
  - Illumination based reconstruction (Hayakawa, 1994)
  - Visual servoing (Murase & Nayar, 1995; Murase & Nayar, 1994)
  - Visual correspondence (Zhang et al., 1995; Jones & Malik, 1992)
  - Camera motion estimation (Hartley, 1992; Hartley & Zisserman, 2000)
  - Image watermarking (Liu & Tan, 2000)
  - Signal processing (Moonen & de Moor, 1995)
  - Neural approaches (Oja, 1982; Sanger, 1989; Xu, 1993)
  - Bilinear models (Tenenbaum & Freeman, 2000; Marimont & Wandell, 1992)
  - Direct extensions (Welling et al., 2003; Penev & Atick, 1996)

“Intercorrelations among variables are the bane of the multivariate researcher’s struggle for meaning”

Cooley and Lohnes, 1971



Part-based Representation

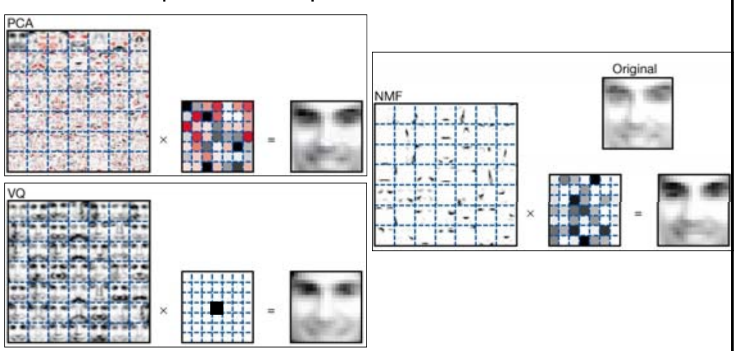
- The firing rates of neurons are never negative.
- Independent representations.

NMF & ICA

Non-negative Matrix Factorization

- Positive factorization.

  $$E(\mathbf{B},\mathbf{C}) = \| \mathbf{D} - \mathbf{BC} \|_F \quad \mathbf{B},\mathbf{C} \geq 0$$
- Leads to part-based representation.



Nonnegative Factorization

(Lee & Seung, 1999;Lee & Seung, 2000)

$$\min_{\mathbf{B}\geq 0,\mathbf{C}\geq 0} F = \sum_{ij} \left| d_{ij} - (\mathbf{BC})_{ij} \right|^2$$

Derivatives:

$$\frac{\partial F}{\partial \mathbf{C}_{ij}} = (\mathbf{B}^T\mathbf{BC})_{ij} - (\mathbf{B}^T\mathbf{C})_{ij}$$

$$\frac{\partial F}{\partial \mathbf{B}_{ij}} = (\mathbf{BCC}^T)_{ij} - (\mathbf{DC}^T)_{ij}$$

Inference:

$$\boxed{\mathbf{C}_{ij} \leftarrow \mathbf{C}_{ij} \frac{(\mathbf{B}^T\mathbf{D})_{ij}}{(\mathbf{B}^T\mathbf{BV})_{ij}}}$$

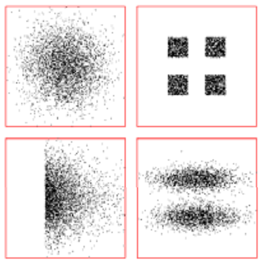
Learning:

$$\boxed{\mathbf{B}_{ij} \leftarrow \mathbf{B}_{ij} \frac{(\mathbf{DC}^T)_{ij}}{(\mathbf{BCC}^T)_{ij}}}$$

- Multiplicative algorithm can be interpreted as diagonally rescaled gradient descent.

Independent Component Analysis

- We need more than second order statistics to represent the signal.



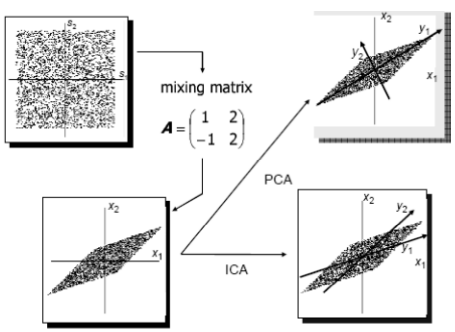
ICA

(Hyvrinen et al., 2001)

$$\mathbf{D} = \mathbf{BC} \quad \mathbf{C} \approx \mathbf{S} = \mathbf{WD} \quad \mathbf{W} \approx \mathbf{B}^{-1}$$

- Look for s_i that are independent.
- PCA finds uncorrelated variables, the independent components have non Gaussian distributions.
- Uncorrelated $E(s_i s_j) = E(s_i)E(s_j)$
- Independent $E(g(s_i)f(s_j)) = E(g(s_i))E(f(s_j))$ for any non-linear f,g

ICA vs PCA



Many optimization criteria

- Minimize high order moments: e.g. kurtosis

  kurt(W) = $E\{s^4\} - 3(E\{s^2\})^2$
- Many other information criteria.
- Also an error function: (Olhausen & Field, 1996)

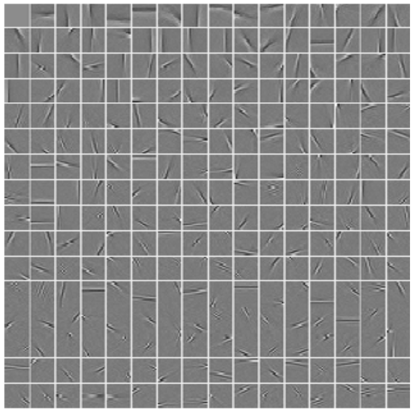
$$\sum_{i=1}^{n} \|\mathbf{d}_i - \mathbf{B}\mathbf{c}_i\| + \sum_{i=1}^{n} S(\mathbf{c}_i)$$

Sparseness (e.g. S=| |)

- Other sparse PCA.

  (Chennubhotla & Jepson, 2001b; Zou et al., 2005; dAspremont et al., 2004;)

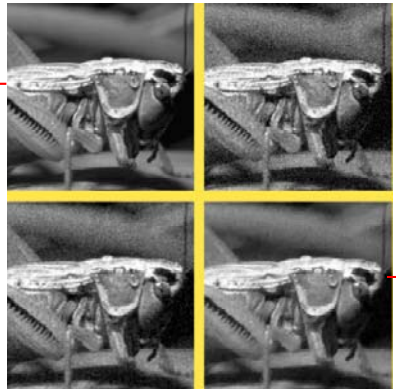
Basis of natural images



Denoising

Original image

Noisy Image (30% noise)



Denoise (Wiener filter)

ICA

Multidimensional Scaling (MDS)

- MDS takes a matrix of pair-wise distances and finds an embedding that preserves the interpoint distances.

An example: map of the US

	Chicago	Raleigh	Boston	Seattle	S.F.	Austin	Orlando
Chicago	0						
Raleigh	641	0					
Boston	851	608	0				
Seattle	1733	2363	2488	0			
S.F.	1855	2406	2696	684	0		
Austin	972	1167	1691	1764	1495	0	
Orlando	994	520	1105	2565	2458	1015	0

MDS(II)

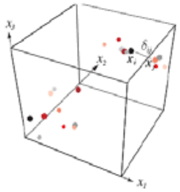
Optimize w.r.t y_i

$$\sum_i \sum_j (\delta_{ij} - d_{ij})^2$$

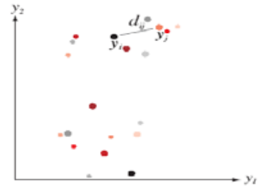
Observed distance between points i and j in p-space

Distance between the points in two-dimensional space

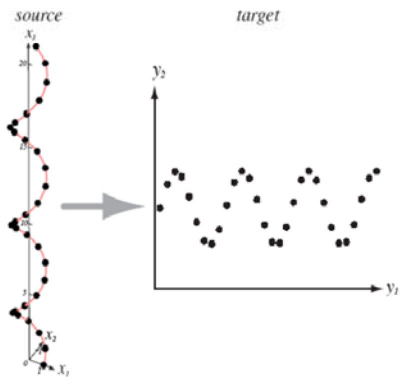
Criterion is invariant wrt rotations and translations.
However it is not invariant to scaling
Better criterion is $\frac{\sum_i \sum_j (\delta_{ij} - d_{ij})^2}{\sum_{i,j} d_{ij}^2}$ or $\sqrt{\frac{\sum_i \sum_j (\delta_{ij} - d_{ij})^2}{\sum_{i,j} d_{ij}^2}}$ Called stress



mapping



MDS (III)



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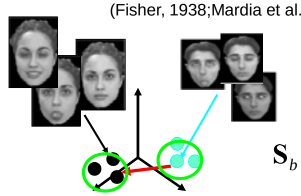
Linear Discriminant Analysis (LDA)

(Fisher, 1938;Mardia et al., 1979; Bishop, 1995)



$$\mathbf{S}_b = \sum_{i=1}^{C}\sum_{j=1}^{C}(\boldsymbol{\mu}_i - \boldsymbol{\mu}_j)(\boldsymbol{\mu}_i - \boldsymbol{\mu}_j)^T$$

$$\mathbf{S}_t = \mathbf{D}\mathbf{D}^T = \sum_{i=1}^{n}\mathbf{d}_i\mathbf{d}_i^T$$

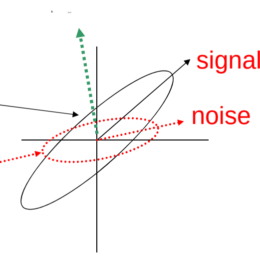
$$J(\mathbf{B}) = \frac{|\mathbf{B}^T\mathbf{S}_b\mathbf{B}|}{|\mathbf{B}^T\mathbf{S}_t\mathbf{B}|} \qquad \mathbf{S_b B} = \mathbf{S}_t\mathbf{B}\boldsymbol{\Lambda}$$

$$\mathbf{S}_w = \sum_{j=1}^{c}\sum_{i=1}^{C_j}(\mathbf{d}_i - \boldsymbol{\mu}_j)(\mathbf{d}_i - \boldsymbol{\mu}_j)^T$$

- Optimal linear dimensionality reduction if classes are Gaussian with equal covariance matrix.

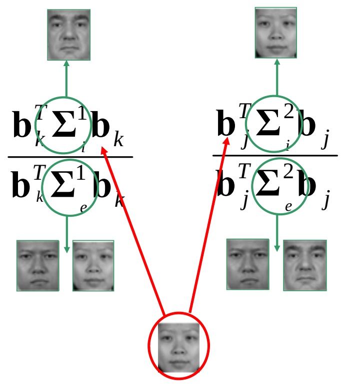
Oriented Component Analysis (OCA)

$$\frac{\mathbf{b}_{OCA}^T\boldsymbol{\Sigma}_{signal}\mathbf{b}_{OCA}}{\mathbf{b}_{OCA}^T\boldsymbol{\Sigma}_{noise}\mathbf{b}_{OCA}}$$



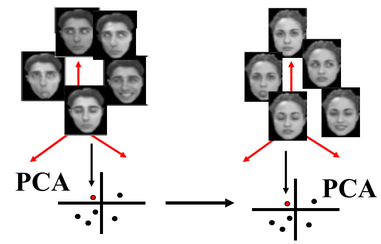
- Generalized eigenvalue problem: $\boldsymbol{\Sigma}_i\mathbf{b}_k = \boldsymbol{\Sigma}_e\mathbf{b}_k\lambda$
- $\mathbf{b}_{oca}$ is steered by the distribution of noise.

OCA for face recognition



Canonical Correlation Analysis (CCA)

- PCA independently and general mapping



- Signals dependent signals with small energy can be lost.

Canonical Correlation Analysis (CCA)

(Mardia et al., 1979; Borga)

- Learn relations between multiple data sets? (e.g. find features in one set related to another data set)
- Given two sets $\mathbf{X} \in \Re^{d_1 \times n}$ *and* $\mathbf{Y} \in \Re^{d_2 \times n}$, CCA finds the pair of directions $\mathbf{w_x}$ and $\mathbf{w_y}$ that maximize the correlation between the projections (assume zero mean data)

$$\rho = \frac{\mathbf{w}_x^T \mathbf{X}^T \mathbf{Y} \mathbf{w}_y}{\sqrt{\mathbf{w}_x^T \mathbf{X}^T \mathbf{X} \mathbf{w}_x^T \mathbf{w}_y^T \mathbf{Y}^T \mathbf{Y} \mathbf{w}_y^T}}$$

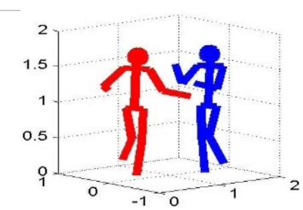
- Several ways of optimizing it:

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{X}^T\mathbf{Y} \\ \mathbf{X}^T\mathbf{Y} & \mathbf{0} \end{bmatrix} \in \Re^{(d_1+d_2)\times(d_1+d_2)}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{X}^T\mathbf{X} & \mathbf{0} \\ \mathbf{0} & \mathbf{Y}^T\mathbf{Y} \end{bmatrix} \in \Re^{(d_1+d_2)\times(d_1+d_2)} \quad \mathbf{w} = \begin{bmatrix} \mathbf{w}_x \\ \mathbf{w}_y \end{bmatrix}$$

- An stationary point of r is the solution to CCA.

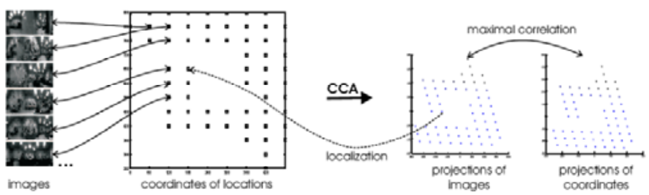
$$\mathbf{Aw} = \lambda \mathbf{Bw}$$

Dynamic Coupled Component Analysis



Robot localization with Canonical Correlation Analysis

(Skocaj & Leonardis, 2000)



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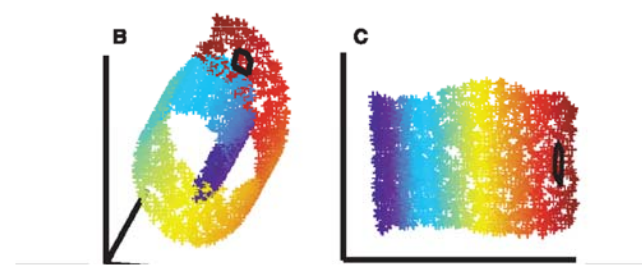
Kernel Methods

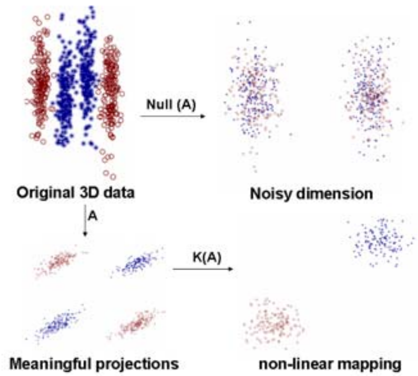
Linear methods fail

- When data points sit on a non-linear manifold
  - We won't find a good **linear** mapping from the data points to a plane, because there isn't any
  - In the end, linear methods do nothing more than rotate/translate/scale data

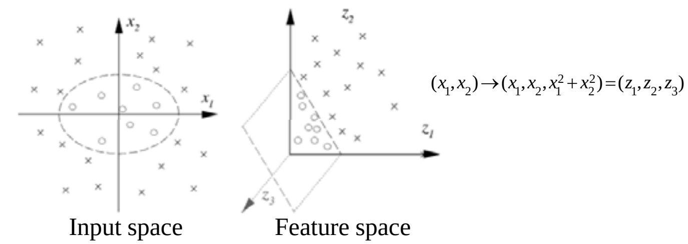


Linear methods fail

- Learning a non-linear representation for classification



Kernel Methods for Classification



- The kernel defines an implicit mapping (usually high dimensional and non-linear) from input to feature space, so the data becomes linearly separable.
- Computation in the feature space can be costly because it is (usually) high dimensional
  - The feature space is typically infinite-dimensional!

Kernel Methods

- Suppose $\phi(.)$ is given as follows

$$\phi(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = (1, \sqrt{2}x_1, \sqrt{2}x_2, x_1^2, x_2^2, \sqrt{2}x_1 x_2)$$

- An inner product in the feature space is

$$\langle \phi(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}), \phi(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}) \rangle = (1 + x_1 y_1 + x_2 y_2)^2$$

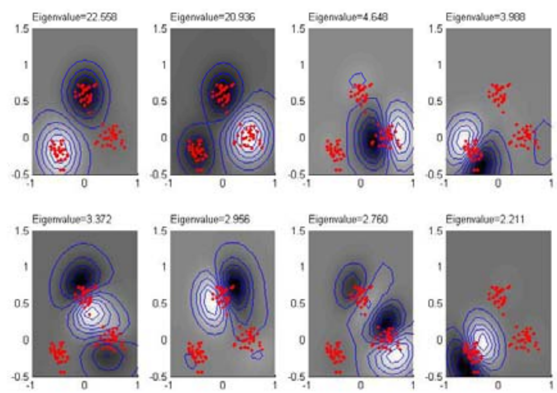
- So, if we define the kernel function as follows, there is no need to carry out $\phi(.)$ explicitly

$$K(\mathbf{x}, \mathbf{y}) = (1 + x_1 y_1 + x_2 y_2)^2$$

- This use of kernel function to avoid carrying out $\phi(.)$ explicitly is known as the kernel trick. In any linear algorithm that can be expressed by inner products can be made nonlinear by going to the feature space

Kernel PCA

(Scholkopf et al., 1998)



Kernel PCA

(Scholkopf et al., 1998)

- Eigenvectors of the cov. Matrix in feature space.

$$\overline{\mathbf{C}} = \frac{1}{n}\sum_{i=1}^{n}\Phi(\mathbf{d}_i)\Phi(\mathbf{d}_i)^{\mathrm{T}} \qquad \overline{\mathbf{C}}\,\mathbf{b}_1 = \mathbf{b}_1\lambda$$

- Eigenvectors lie in the span of data in feature space.

$$\mathbf{b}_1 = \sum_{i=1}^{n}\alpha_i\Phi(\mathbf{d}_i)$$

$$\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\Phi(\mathbf{d}_i)K(\mathbf{d}_i,\mathbf{d}_j) = [\sum_{i=1}^{n}\alpha_i\Phi(\mathbf{d}_i)]\lambda$$

$$\mathbf{K}\boldsymbol{\alpha} = \boldsymbol{\alpha}\lambda$$

Latent Variable Models

Factor Analysis

- A Gaussian distribution on the coefficients and noise is added to PCA➔ Factor Analysis. (Mardia et al., 1979)

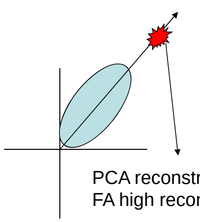
$$\mathbf{d} = \boldsymbol{\mu} + \mathbf{B}\mathbf{c} + \boldsymbol{\eta}$$

$$p(\mathbf{c}) = N(\mathbf{c}\,|\,\mathbf{0},\mathbf{I}_k) \qquad p(\mathbf{d}\,|\,\mathbf{c},\mathbf{B}) = N(\mathbf{d}\,|\,\boldsymbol{\mu}+\mathbf{B}\mathbf{c},\Psi)$$

$$p(\boldsymbol{\eta}) = N(\mathbf{c}\,|\,\mathbf{0},\Psi) \qquad \Psi = diag(\eta_1,\eta_2,\ldots,\eta_d)$$

$$\mathrm{cov}(\mathbf{d}) = E((\mathbf{d}-\boldsymbol{\mu})(\mathbf{d}-\boldsymbol{\mu})^T) = \mathbf{B}\mathbf{B}^T + \Psi$$

- Inference (Roweis & Ghahramani, 1999;Tipping & Bishop, 1999a)



$p(\mathbf{c},\mathbf{d})$ Jointly Gaussian

$$p(\mathbf{c}\,|\,\mathbf{d}) = N(\mathbf{c}\,|\,\mathbf{m},\mathbf{V})$$

$$\mathbf{m} = \mathbf{B}^T(\mathbf{B}\mathbf{B}^T+\Psi)^{-1}(\mathbf{d}-\boldsymbol{\mu})$$

$$\mathbf{V} = (\mathbf{I}+\mathbf{B}^T\Psi^{-1}\mathbf{B})^{-1}$$

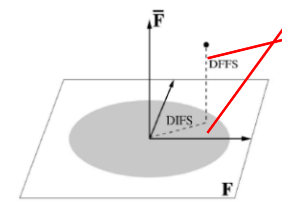
PCA reconstruction low error.
FA high reconstruction error (low likelihood).

Ppca

- If $\Psi = E(\boldsymbol{\eta}\boldsymbol{\eta}^T) = \varepsilon\mathbf{I}_d$ PPCA.
- If $\varepsilon \rightarrow 0$ is equivalent to PCA. $\varepsilon \rightarrow 0 \quad \mathbf{B}^T(\mathbf{B}\mathbf{B}^T+\Psi)^{-1} = (\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T$
- Probabilistic visual learning (Moghaddam & Pentland, 1997;)

$$p(\mathbf{d}) = \int p(\mathbf{d}\,|\,\mathbf{c})p(\mathbf{c})d\mathbf{c} = \frac{e^{-\frac{1}{2}(\mathbf{d}-\boldsymbol{\mu})^T\Sigma^{-1}(\mathbf{d}-\boldsymbol{\mu})}}{(2\pi)^{\frac{d}{2}}|\Sigma|^{\frac{1}{2}}} = \frac{e^{-\frac{1}{2}(\mathbf{d}-\boldsymbol{\mu})^T(\mathbf{B}\mathbf{B}^{\mathrm{T}}+\varepsilon\mathbf{I})^{-1}(\mathbf{d}-\boldsymbol{\mu})}}{(2\pi)^{\frac{d}{2}}|\Sigma|^{\frac{1}{2}}} = \left[\frac{e^{-\frac{1}{2}\sum_{i=1}^{k}\frac{c_i^2}{\lambda_i}}}{(2\pi)^{\frac{d}{2}}\prod_{i=1}^{k}\lambda_i^{1/2}}\right]\left[\frac{e^{-\frac{\varepsilon^2(\mathbf{d})}{2\rho}}}{(2\pi\rho)^{\frac{(d-k)}{2}}}\right]$$

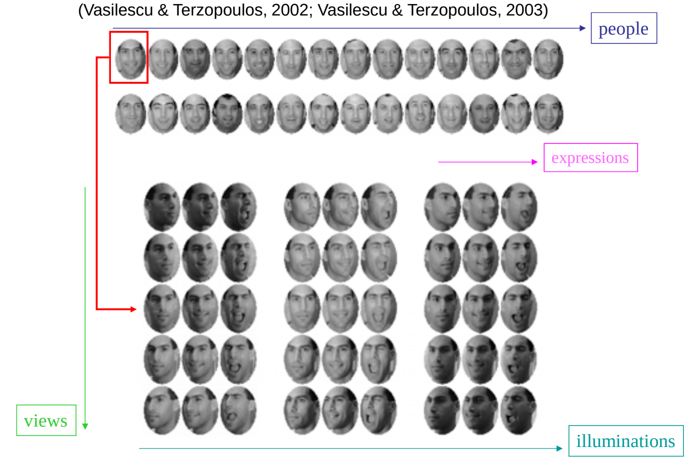
$$\mathbf{c}_i = \mathbf{B}^T\mathbf{d}_i$$



Tensor Factorization

Tensor faces

(Vasilescu & Terzopoulos, 2002; Vasilescu & Terzopoulos, 2003)



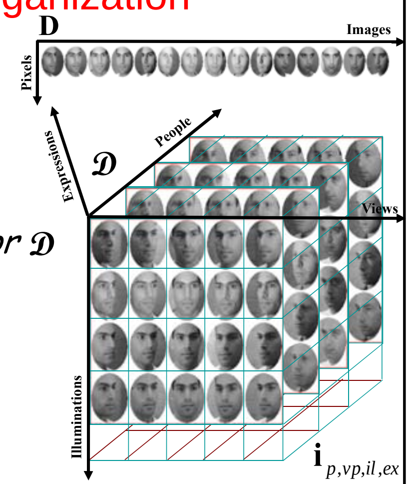
Eigenfaces

- Facial images (identity change)

- Eigenfaces bases vectors capture the variability in facial appearance (do not decouple pose, illumination, …)

Data Organization

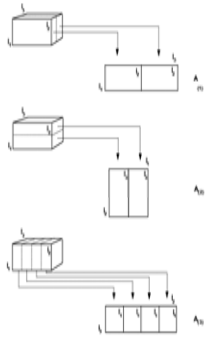
- Linear/PCA: Data Matrix
  - $\mathbb{R}^{\text{pixels x images}}$
  - a matrix of image vectors
- Multilinear: *Data Tensor* $\mathcal{D}$
  - $\mathbb{R}^{\text{people x views x illums x express x pixels}}$
  - N-dimensional matrix
  - 28 people, 45 images/person
  - 5 views, 3 illuminations, 3 expressions per person

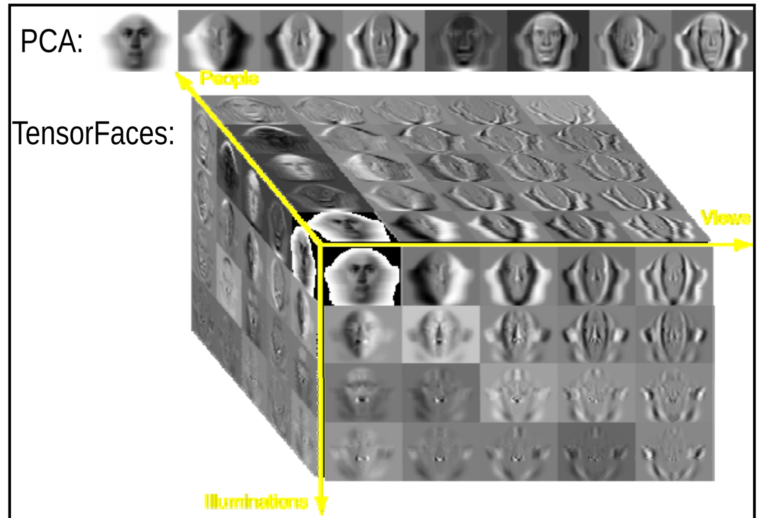


N-Mode SVD Algorithm

$$\mathcal{D} = \mathcal{Z} \times_1 \mathbf{U}_{\text{people}} \times_2 \mathbf{U}_{\text{views}} \times_3 \mathbf{U}_{\text{illums.}} \times_4 \mathbf{U}_{\text{express.}} \times_5 \mathbf{U}_{\text{pixels}}$$

$N = 3$





Strategic Data Compression = Perceptual Quality

- TensorFaces data reduction in illumination space primarily degrades illumination effects (cast shadows, highlights)
- PCA has *lower mean square error* but *higher perceptual error*

Original	TensorFaces	TensorFaces	PCA
		Mean Sq. Err. = 409.15	Mean Sq. Err. = 85.75
	6 illum + 11 people param.	3 illum + 11 people param.	33 parameters
176 basis vectors	66 basis vectors	33 basis vectors	33 basis vectors

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