Fundamentals of Linear Algebra

Class 2-3. 6 Sep 2011

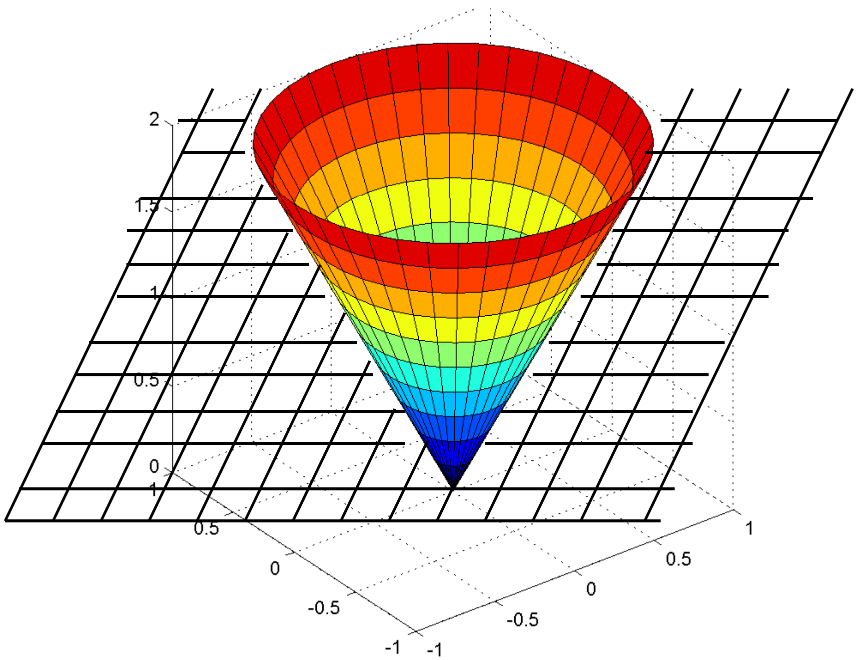
Instructor: Bhiksha Raj

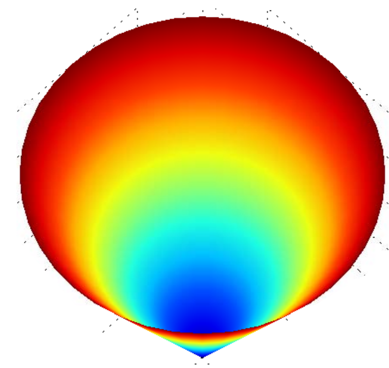
Administrivia

- TA Times:
  - Anoop Ramakrishna: Thursday 12.30-1.30pm
  - Manuel Tragut: Friday 11am – 12pm.

- HW1: On the webpage

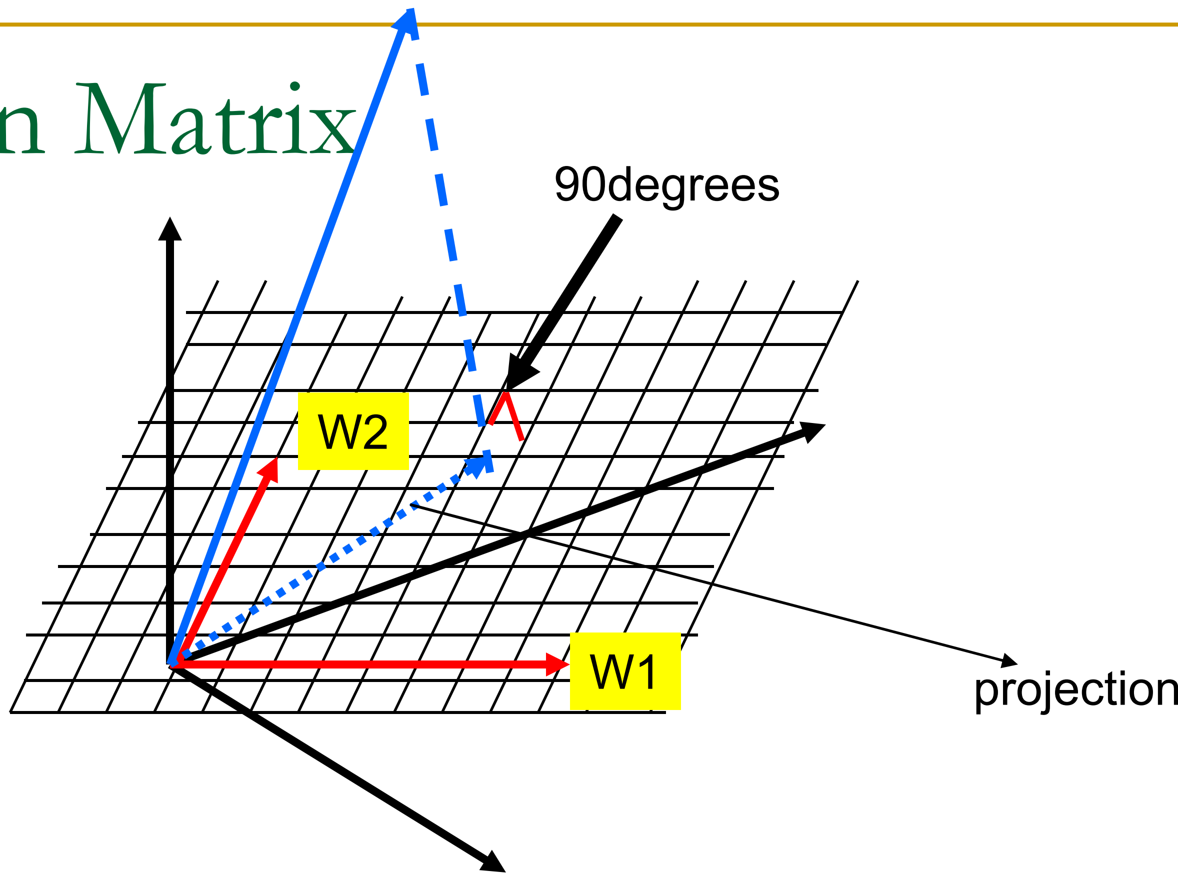
Projections





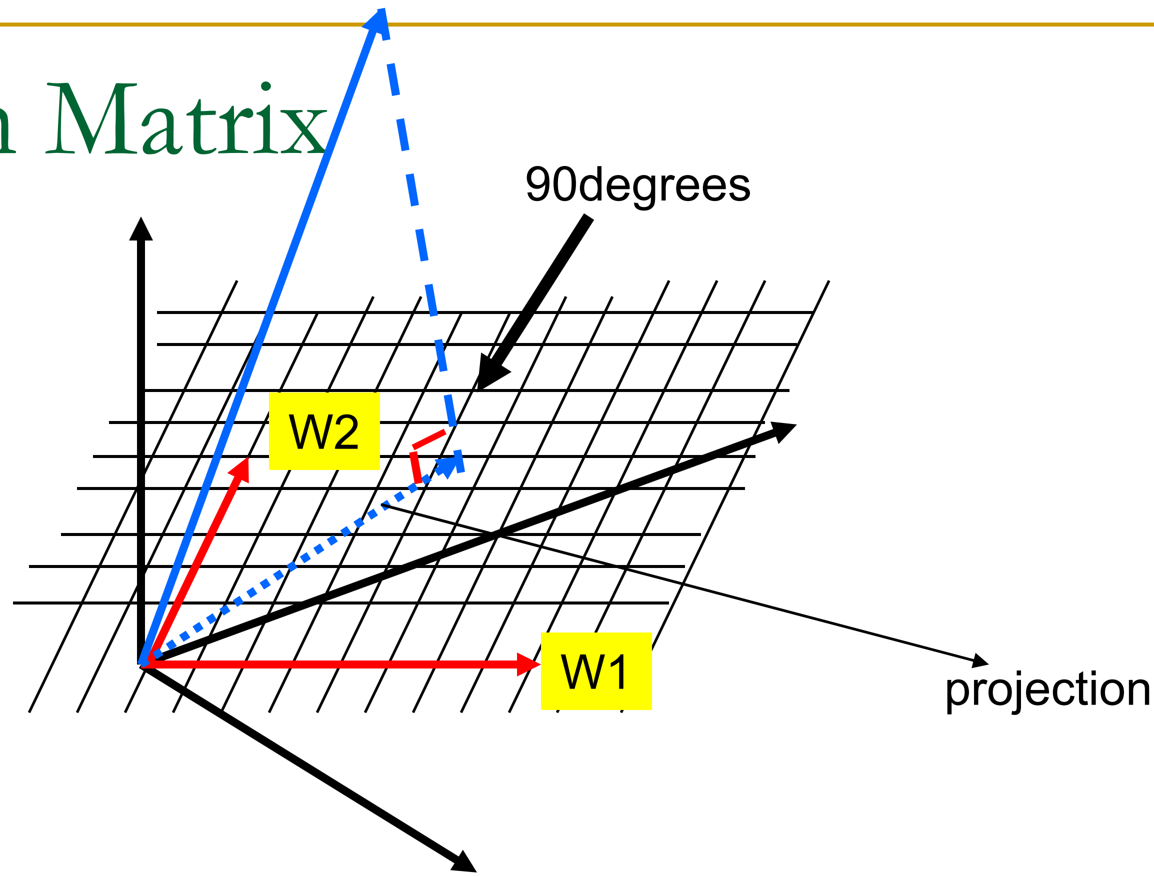
- What would we see if the cone to the left were transparent if we looked at it along the normal to the plane
  - The plane goes through the origin
  - Answer: the figure to the right
- How do we get this? Projection

Projection Matrix



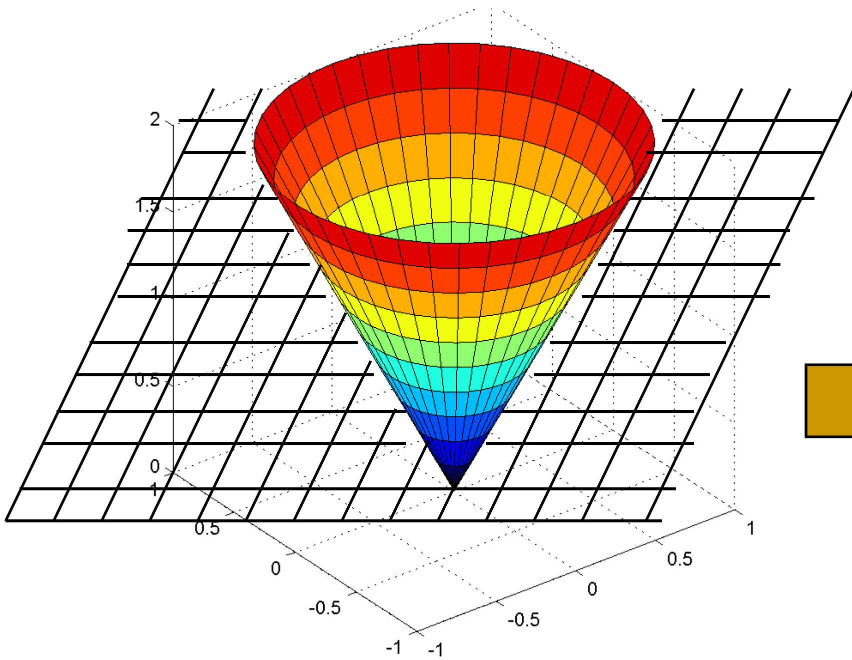
- Consider any plane specified by a set of vectors W_1, W_2..
  - Or matrix [W_1 W_2 ..]
  - Any vector can be projected onto this plane
  - The matrix A that rotates and scales the vector so that it becomes its projection is a projection matrix

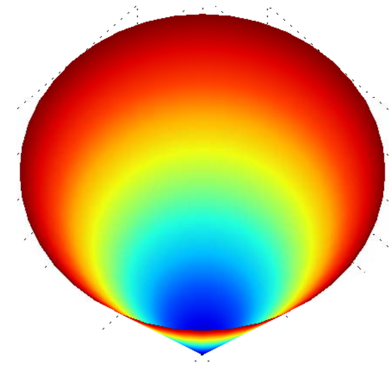
Projection Matrix



- Given a set of vectors W1, W2, which form a matrix W = [W1 W2..]
- The projection matrix that transforms any vector X to its projection on the plane is
  - $P = W (W^T W)^{-1} W^T$
    - We will visit matrix inversion shortly
- Magic – any set of vectors from the same plane that are expressed as a matrix will give you the same projection matrix
  - $P = V (V^T V)^{-1} V^T$

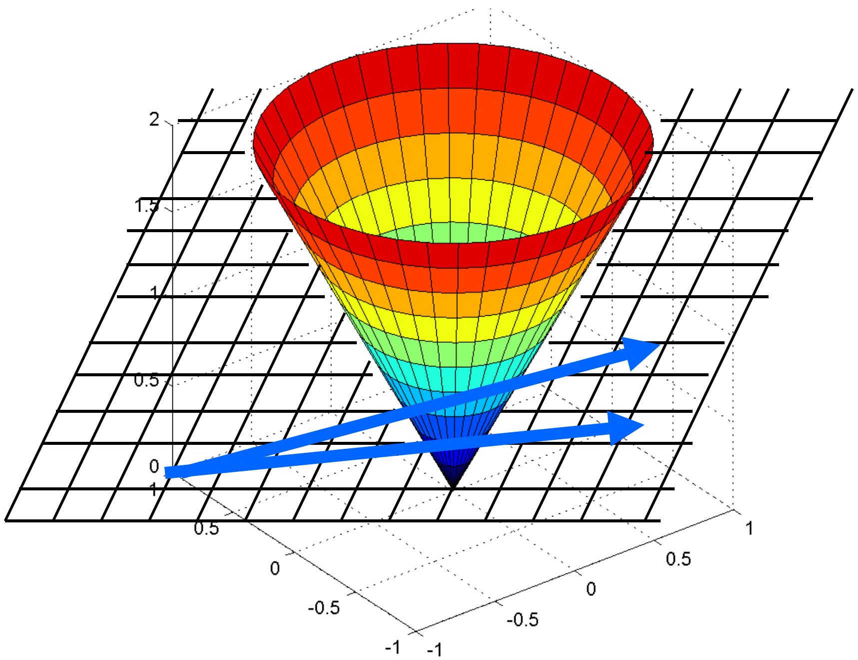
Projections

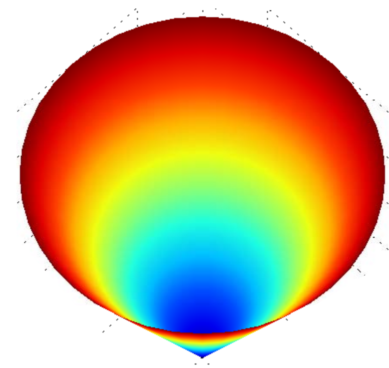




- HOW?

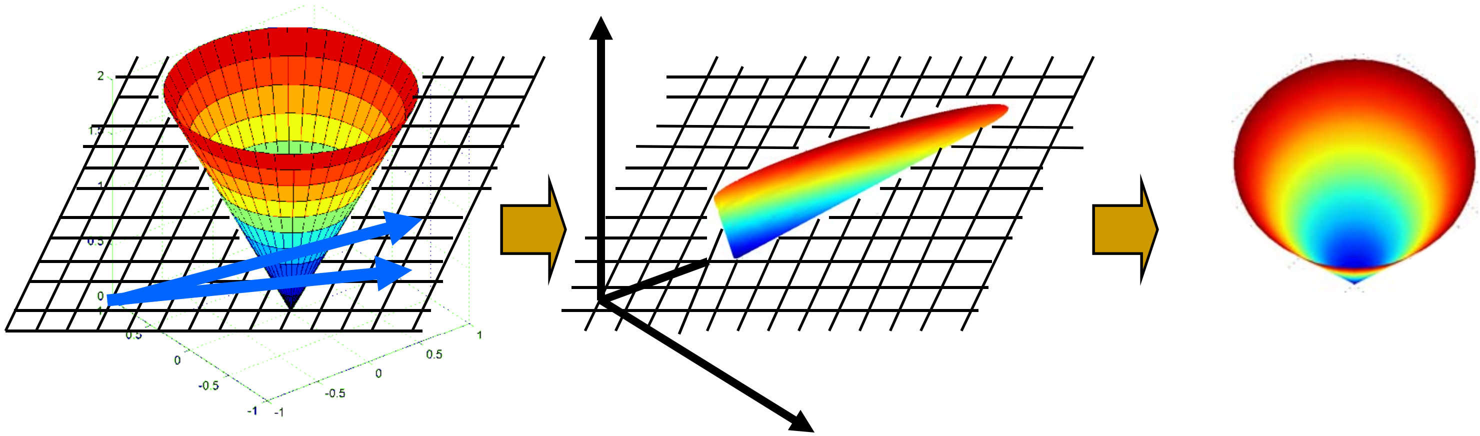
Projections





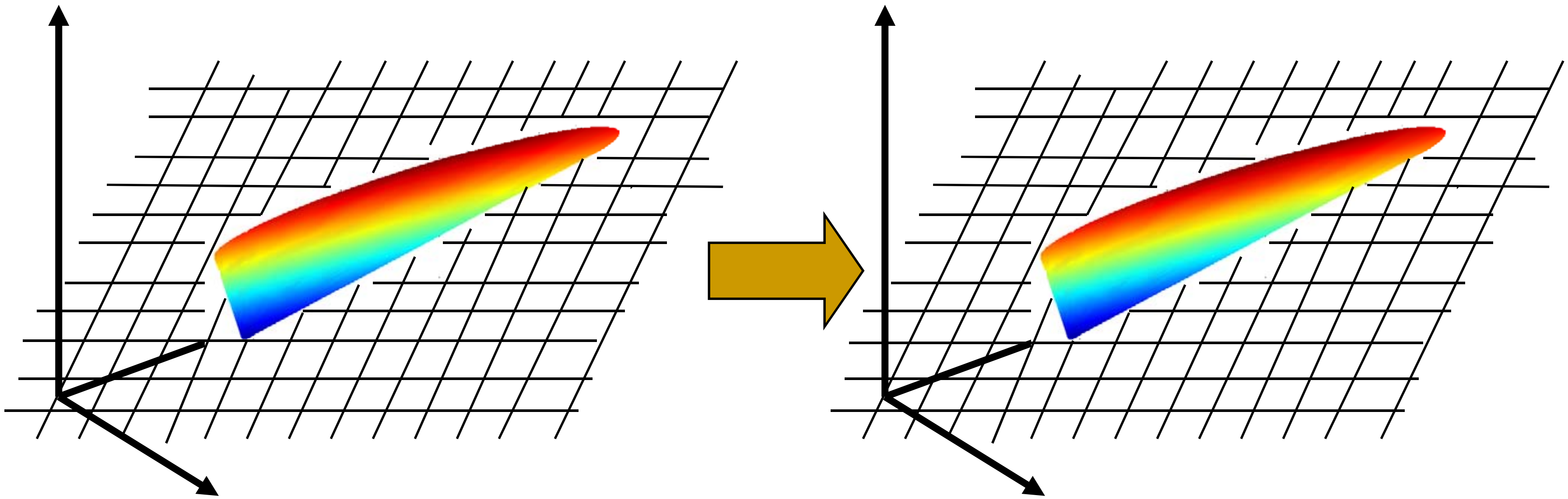
- Draw any two vectors W1 and W2 that lie on the plane
  - ***ANY two*** **so long as they have different angles**
- Compose a matrix W = [W1 W2]
- Compose the projection matrix $P = W (W^T W)^{-1} W^T$
- Multiply every point on the cone by P to get its projection
- View it ☺
  - I'm missing a step here – what is it?

Projections



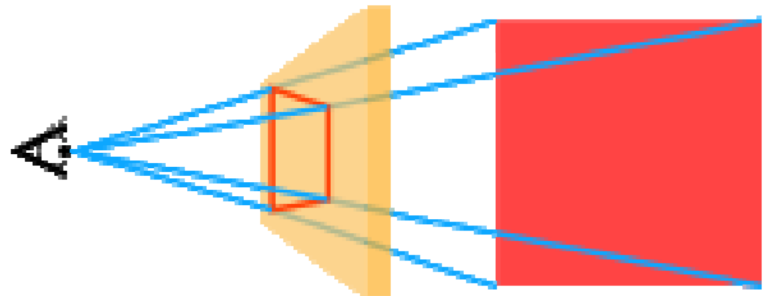
- The projection actually projects it onto the plane, but you're still seeing the plane in 3D
  - The result of the projection is a 3-D vector
  - $P = W (W^T W)^{-1} W^T = 3x3$, $P*Vector = 3x1$
  - The image must be rotated till the plane is in the plane of the paper
    - The Z axis in this case will always be zero and can be ignored
    - How will you rotate it? (remember you know W1 and W2)

Projection matrix properties



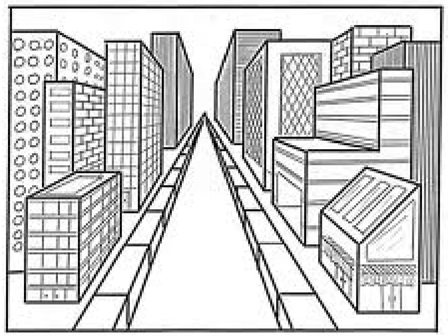
- The projection of any vector that is already on the plane is the vector itself
  - Px = x if x is on the plane
  - If the object is already on the plane, there is no further projection to be performed
- The projection of a projection is the projection
  - P (Px) = Px
  - That is because Px is already on the plane
- Projection matrices are *idempotent*
  - $P^2 = P$
    - Follows from the above

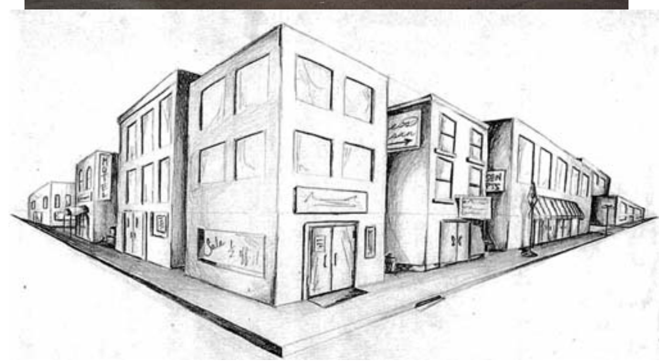
Perspective



- The picture is the equivalent of “painting” the viewed scenery on a glass window
- Feature: The lines connecting any point in the scenery and its projection on the window merge at a common point
  - The eye

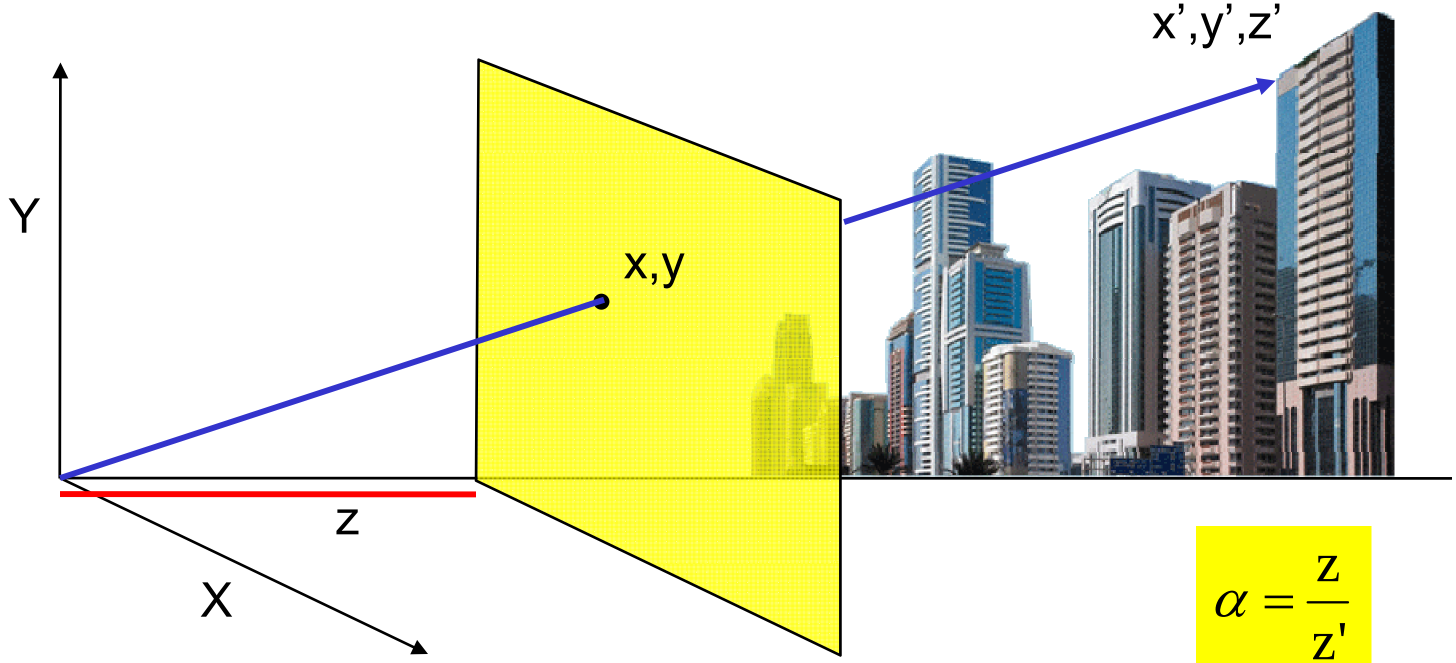
An aside on Perspective..





- Perspective is the result of convergence of the image to a point
- Convergence can be to multiple points
  - Top Left: One-point perspective
  - Top Right: Two-point perspective
  - Right: Three-point perspective

Central Projection



$$\frac{x}{x'} = \frac{y}{y'} = \frac{z}{z'}$$

Property of a line through origin

$$\alpha = \frac{z}{z'}$$
$$x = \alpha x'$$
$$y = \alpha y'$$

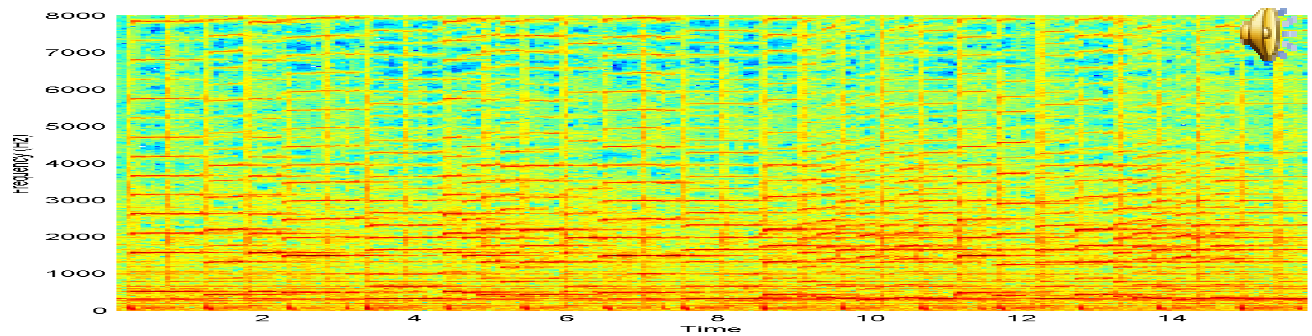
- The positions on the "window" are scaled along the line
- To compute (x,y) position on the window, we need z (distance of window from eye), and (x',y',z') (location being projected)

Projections: A more physical meaning

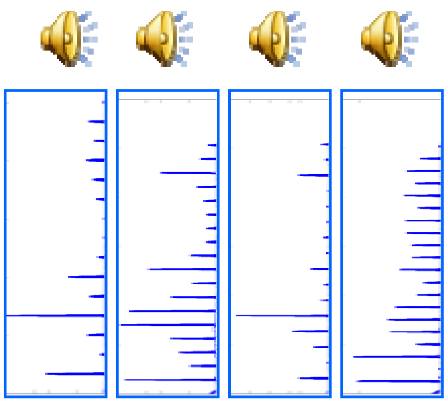
- Let W_1, W_2 .. W_k be "bases"
- We want to explain our data in terms of these "bases"
  - We often cannot do so
  - But we can explain a significant portion of it

- The portion of the data that can be expressed in terms of our vectors W_1, W_2, .. W_k, is the projection of the data on the W_1 .. W_k (hyper) plane
  - In our previous example, the "data" were all the points on a cone
  - The interpretation for volumetric data is obvious

Projection : an example with sounds

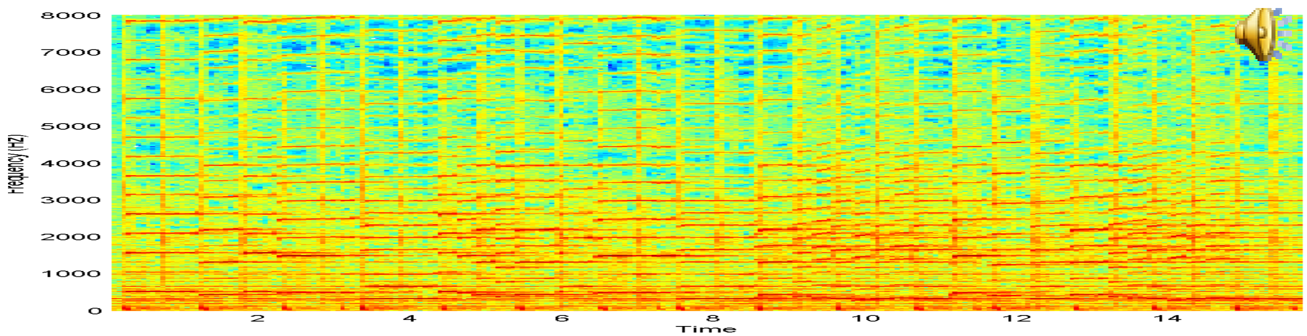


- The spectrogram (matrix) of a piece of music

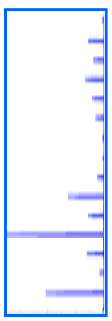


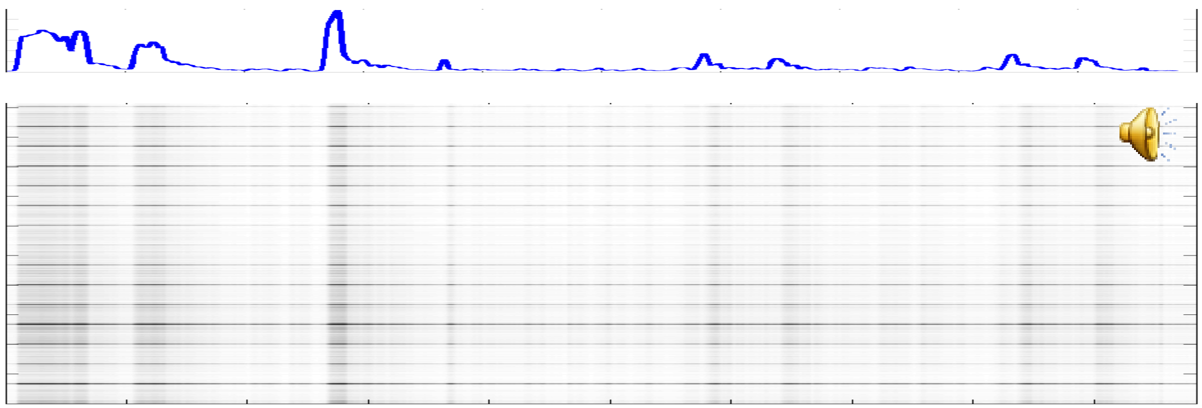
- How much of the above music was composed of the above notes
  - I.e. how much can it be explained by the notes

Projection: one note

M = 

- The spectrogram (matrix) of a piece of music

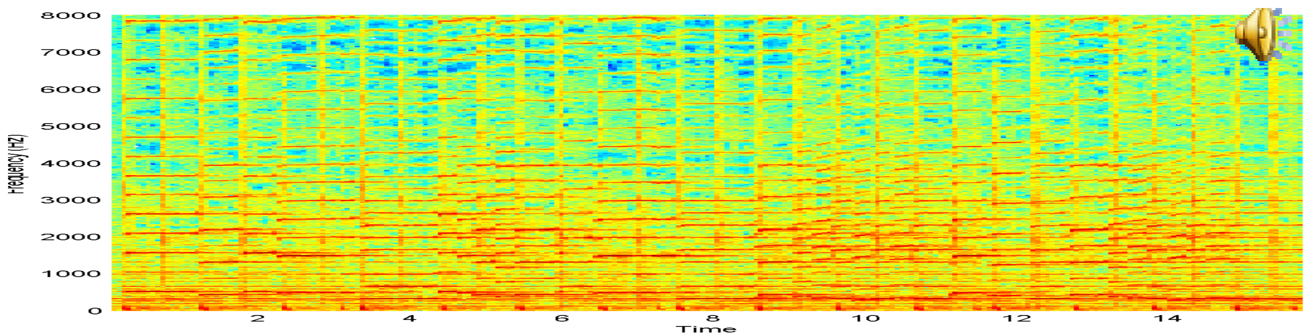
W = 



- M = spectrogram;   W = note
- $P = W (W^T W)^{-1} W^T$
- Projected Spectrogram = P * M

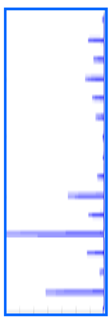
Projection: one note – cleaned up

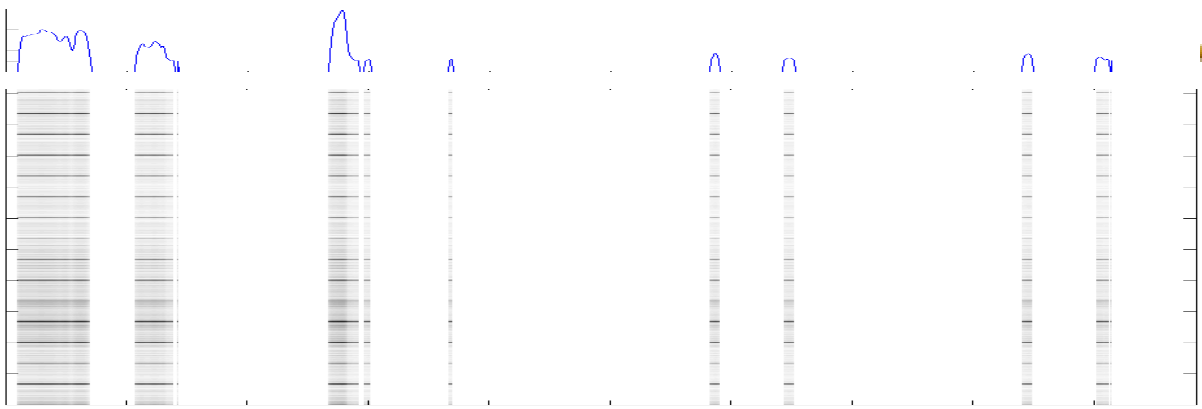
M =



- The spectrogram (matrix) of a piece of music

W =

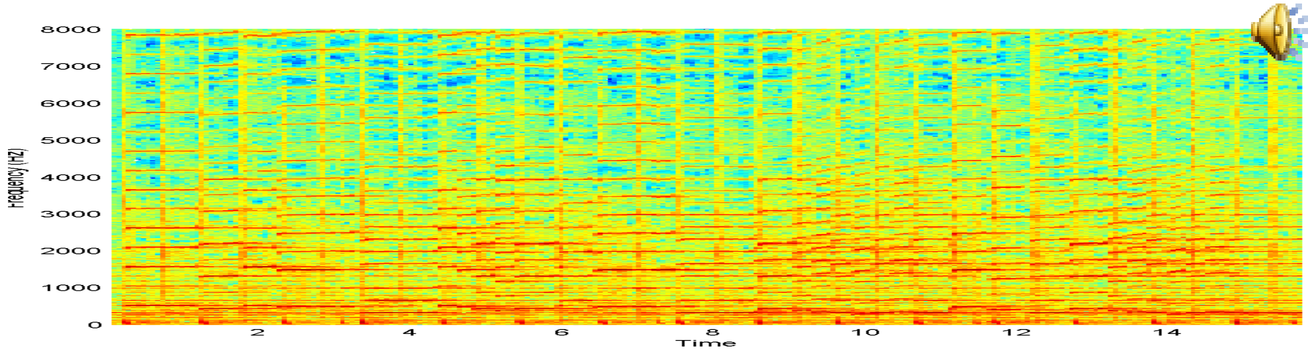




- Floored all matrix values below a threshold to zero

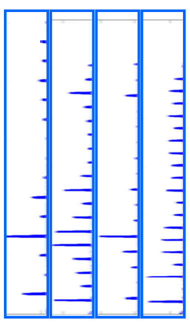
Projection: multiple notes

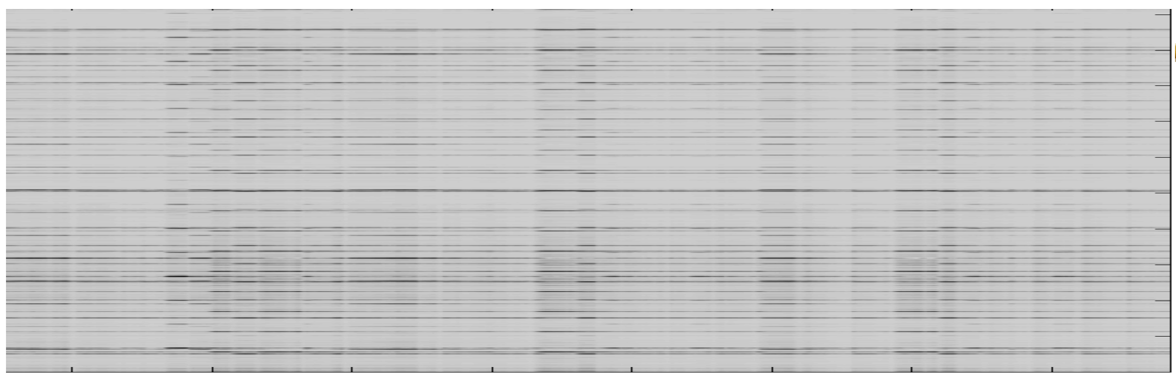
M =



- The spectrogram (matrix) of a piece of music

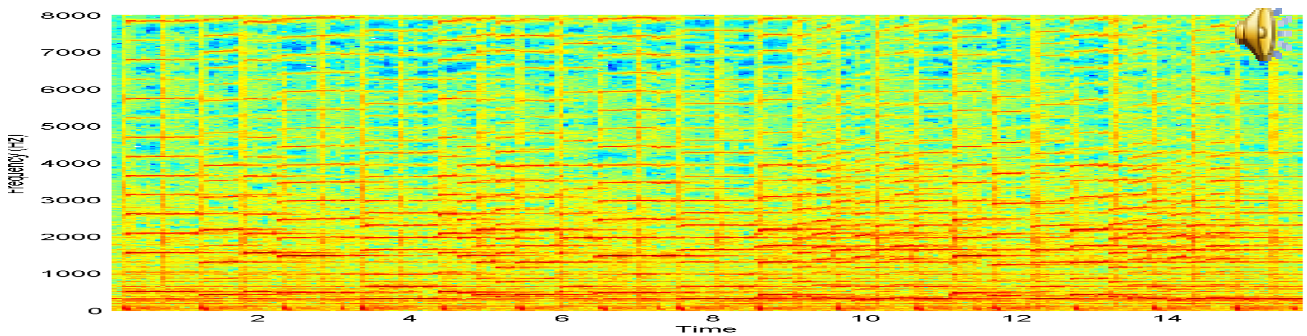
W =



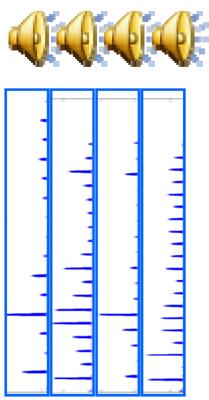


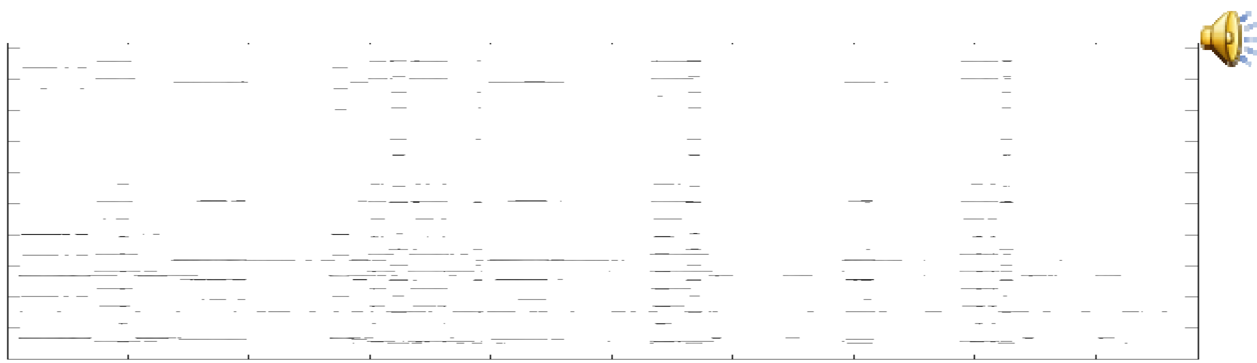
- $P = W (W^T W)^{-1} W^T$
- Projected Spectrogram = P * M

Projection: multiple notes, cleaned up

M = 

- The spectrogram (matrix) of a piece of music

W = 



- $P = W (W^T W)^{-1} W^T$
- Projected Spectrogram = P * M

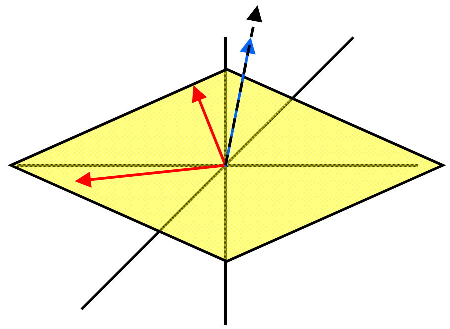
Projection and Least Squares

- Projection actually computes a *least squared error* estimate
- For each vector V in the music spectrogram matrix
  - Approximation: V_{approx} = a*note1 + b*note2 + c*note3..

$$V_{approx} = \begin{bmatrix} \text{note1} & \text{note2} & \text{note3} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

  - Error vector E = V – V_{approx}
  - Squared error energy for V e(V) = norm(E)^2
  - Total error = sum_over_all_V { e(V) } = Σ_V e(V)
- Projection computes V_{approx} for all vectors such that Total error is minimized
  - It does not give you "a", "b", "c".. Though
    - That needs a different operation – the inverse / pseudo inverse

Orthogonal and Orthonormal matrices



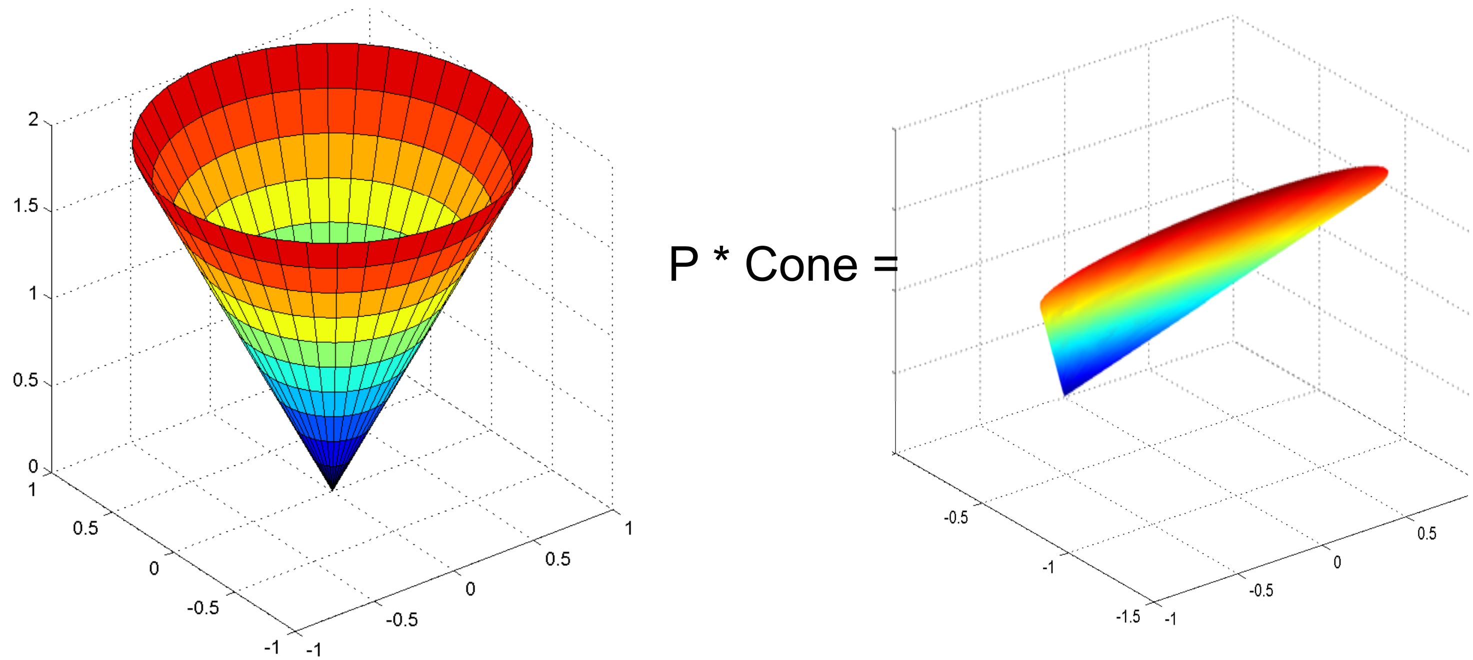
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0.707 & -0.354 & 0.612 \\ 0.707 & 0.354 & -0.612 \\ 0 & 0.866 & 0.5 \end{bmatrix}$$

- Orthogonal Matrix : AA^T = diagonal
  - Each row vector lies exactly along the normal to the plane specified by the rest of the vectors in the matrix
- Orthonormal Matrix: $AA^T = A^TA = I$
  - In additional to be orthogonal, each vector has length exactly = 1.0
  - Interesting observation: In a square matrix if the length of the row vectors is 1.0, the length of the column vectors is also 1.0

Orthogonal and Orthonormal Matrices

- Orthonormal matrices will retain the relative angles between transformed vectors
  - Essentially, they are combinations of rotations, reflections and permutations
  - Rotation matrices and permutation matrices are all orthonormal matrices
  - The vectors in an orthonormal matrix are at 90degrees to one another.
- Orthogonal matrices are like Orthonormal matrices with stretching
  - The product of a diagonal matrix and an orthonormal matrix

Matrix Rank and Rank-Deficient Matrices

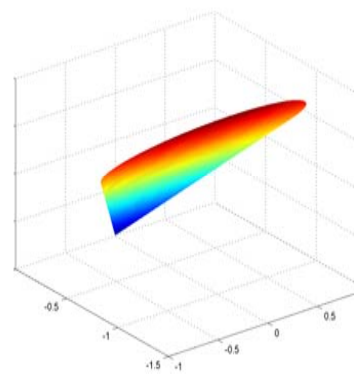


- Some matrices will eliminate one or more dimensions during transformation
  - These are *rank deficient* matrices
  - The rank of the matrix is the dimensionality of the trasnsformed version of a full-dimensional object

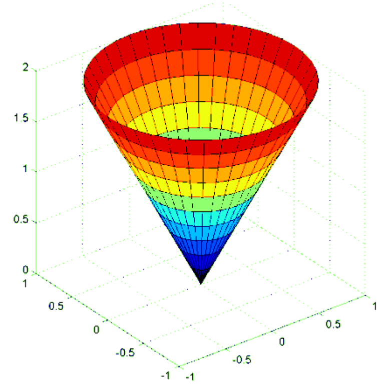
Matrix Rank and Rank-Deficient Matrices

P =

1.0000	0	0
0	0.2500	-0.4330
0	-0.4330	0.7500

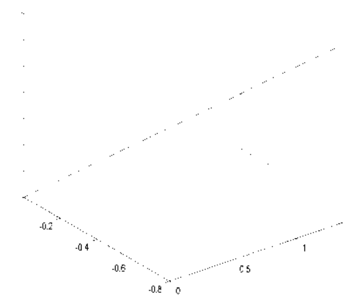


Rank = 2



P2 =

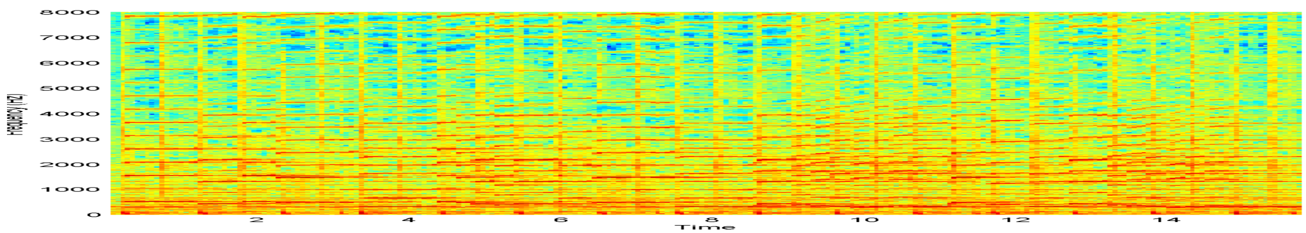
0.5000	-0.2500	0.4330
-0.2500	0.1250	-0.2165
0.4330	-0.2165	0.3750

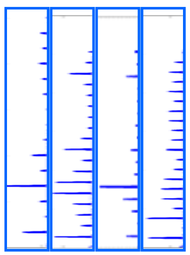


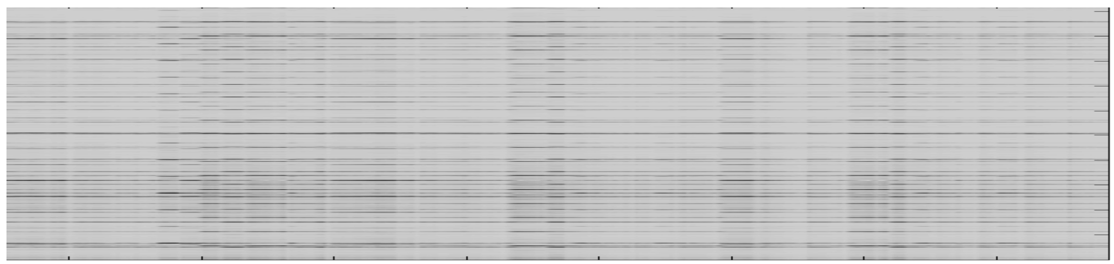
Rank = 1

- Some matrices will eliminate one or more dimensions during transformation
  - These are *rank deficient* matrices
  - The rank of the matrix is the dimensionality of the transformed version of a full-dimensional object

Projections are often examples of rank-deficient transforms

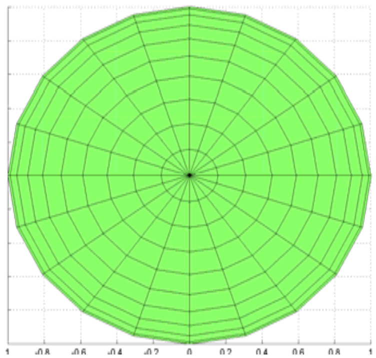
M = 

W = 



- $P = W (W^T W)^{-1} W^T$; Projected Spectrogram = P * M
- The original spectrogram can never be recovered
  - P is rank deficient
- P explains all vectors in the new spectrogram as a mixture of only the 4 vectors in W
  - There are only 4 ***independent*** bases
  - Rank of P is 4

Non-square Matrices

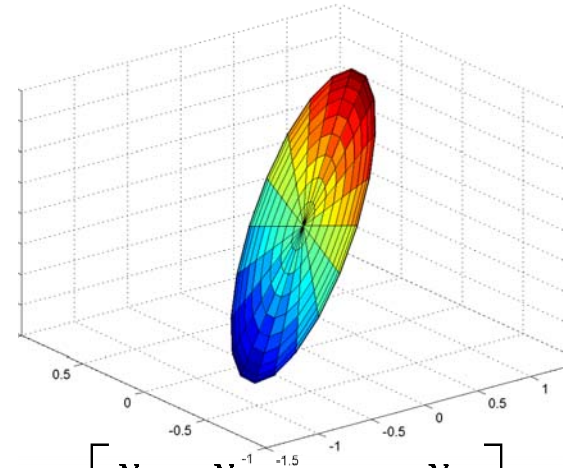


$$\begin{bmatrix} x_1 & x_2 & . & . & x_N \\ y_1 & y_2 & . & . & y_N \end{bmatrix}$$

X = 2D data

$$\begin{bmatrix} .8 & .9 \\ .1 & .9 \\ .6 & 0 \end{bmatrix}$$

P = transform

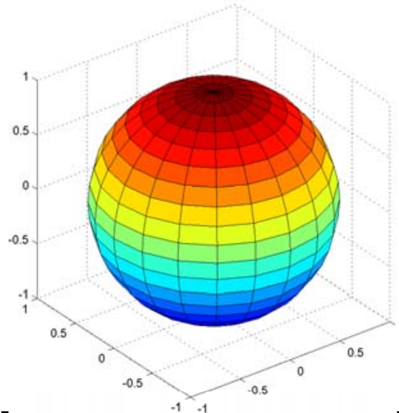


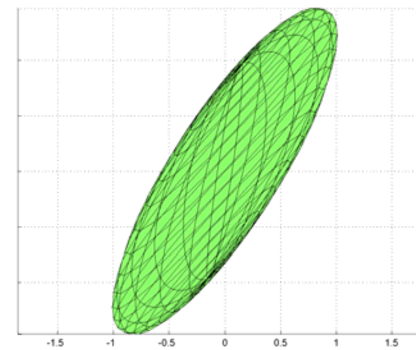
$$\begin{bmatrix} x_1 & x_2 & . & . & x_N \\ y_1 & y_2 & . & . & y_N \\ z_1 & z_2 & . & . & z_N \end{bmatrix}$$

PX = 3D, rank 2

- Non-square matrices add or subtract axes
  - More rows than columns → add axes
    - But does not increase the dimensionality of the data

Non-square Matrices





$$\begin{bmatrix} x_1 & x_2 & . & . & x_N \\ y_1 & y_2 & . & . & y_N \\ z_1 & z_2 & . & . & z_N \end{bmatrix}$$

X = 3D data, rank 3

$$\begin{bmatrix} .3 & 1 & .2 \\ .5 & 1 & 1 \end{bmatrix}$$

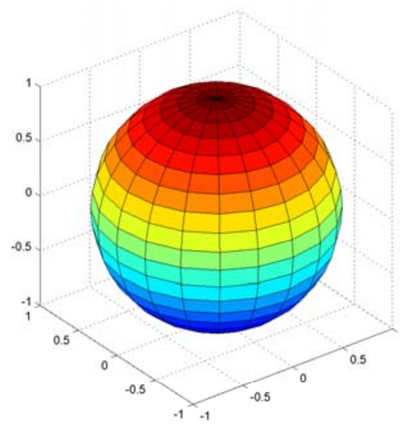
P = transform

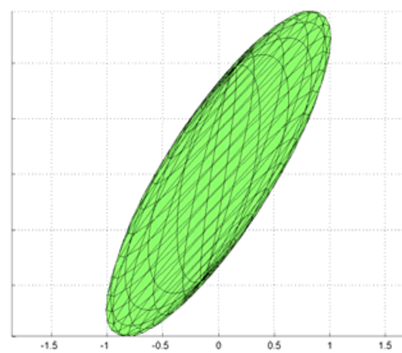
$$\begin{bmatrix} x_1 & x_2 & . & . & x_N \\ y_1 & y_2 & . & . & y_N \end{bmatrix}$$

PX = 2D, rank 2

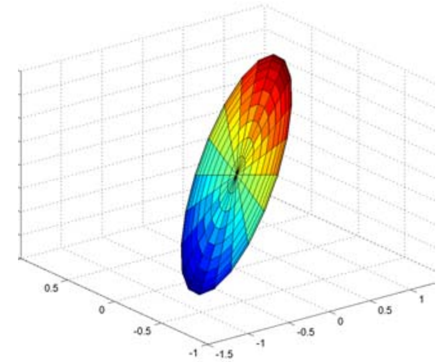
- Non-square matrices add or subtract axes
  - Fewer rows than columns → reduce axes
    - May reduce dimensionality of the data

The Rank of a Matrix





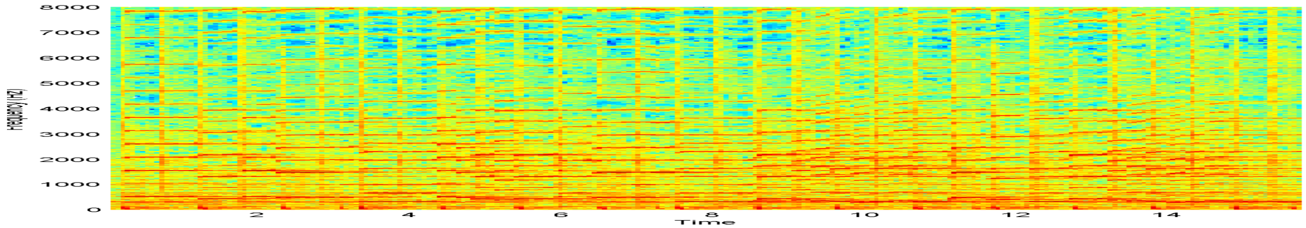
$$\begin{bmatrix} .3 & 1 & .2 \\ .5 & 1 & 1 \end{bmatrix}$$

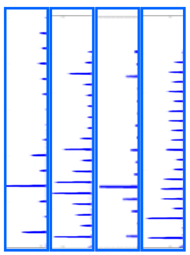


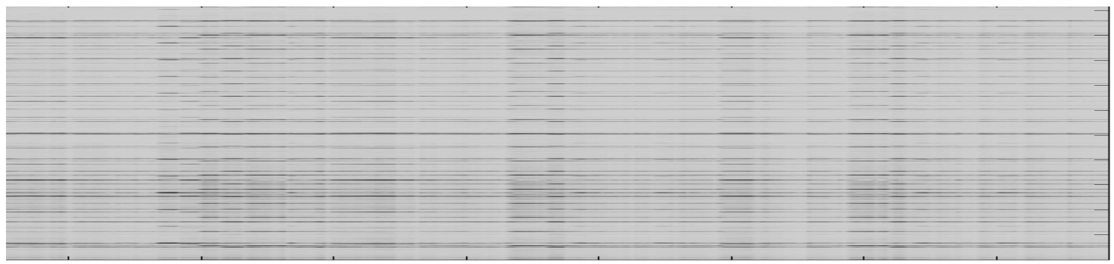
$$\begin{bmatrix} .8 & .9 \\ .1 & .9 \\ .6 & 0 \end{bmatrix}$$

- The matrix rank is the dimensionality of the transformation of a full-dimensioned object in the original space
- The matrix can never *increase* dimensions
  - Cannot convert a circle to a sphere or a line to a circle
- The rank of a matrix can never be greater than the lower of its two dimensions

The Rank of Matrix

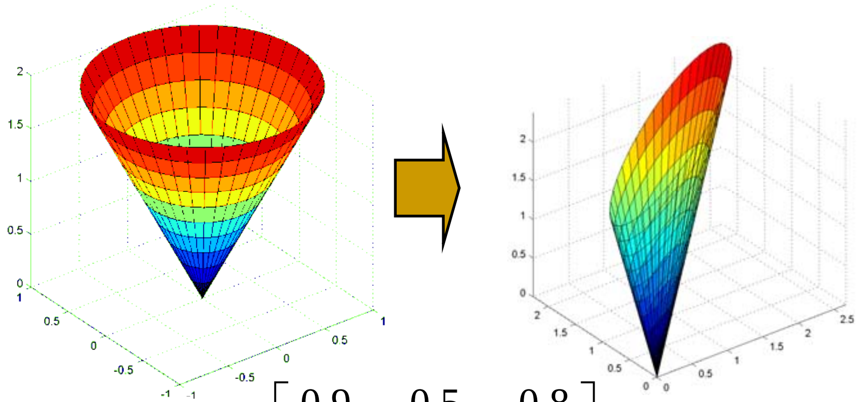
M = 



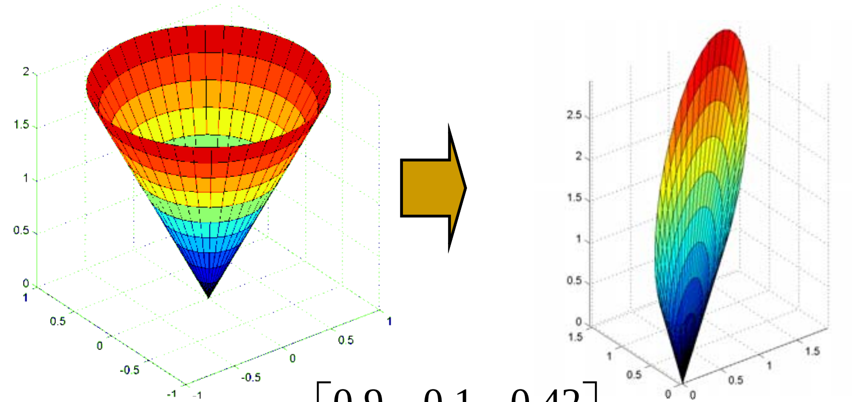


- Projected Spectrogram = P * M
  - Every vector in it is a combination of only 4 bases
- The rank of the matrix is the *smallest* no. of bases required to describe the output
  - E.g. if note no. 4 in P could be expressed as a combination of notes 1,2 and 3, it provides no additional information
  - Eliminating note no. 4 would give us the same projection
  - The rank of P would be 3!

Matrix rank is unchanged by transposition



$$\begin{bmatrix} 0.9 & 0.5 & 0.8 \\ 0.1 & 0.4 & 0.9 \\ 0.42 & 0.44 & 0.86 \end{bmatrix}$$

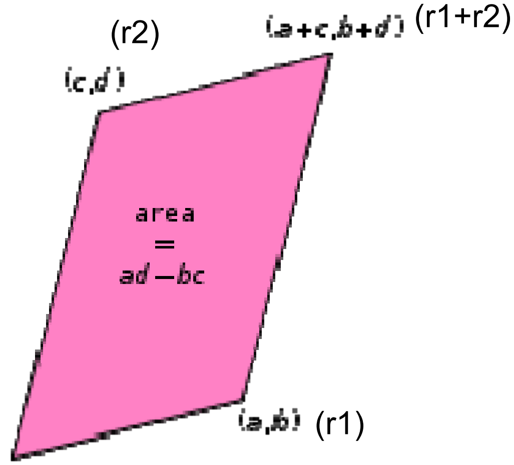


$$\begin{bmatrix} 0.9 & 0.1 & 0.42 \\ 0.5 & 0.4 & 0.44 \\ 0.8 & 0.9 & 0.86 \end{bmatrix}$$

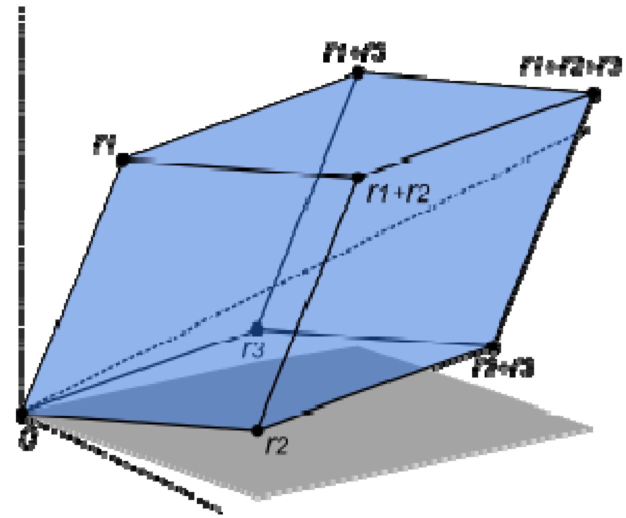
- If an N-D object is compressed to a K-D object by a matrix, it will also be compressed to a K-D object by the transpose of the matrix

Matrix Determinant

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{matrix} \text{(r1)} \\ \text{(r2)} \end{matrix}$$



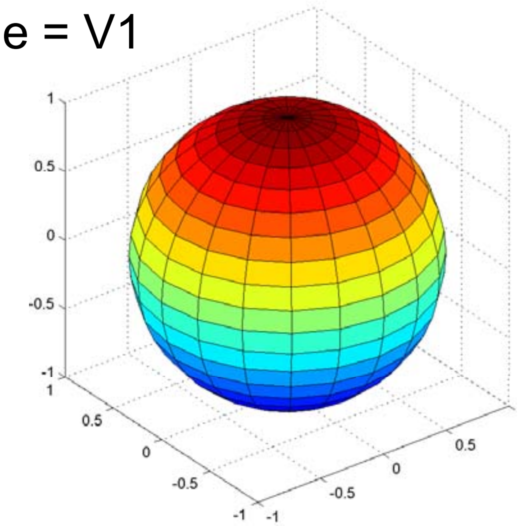
$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$



- The determinant is the "volume" of a matrix
- Actually the volume of a parallelepiped formed from its row vectors
  - Also the volume of the parallelepiped formed from its column vectors
- Standard formula for determinant: in text book

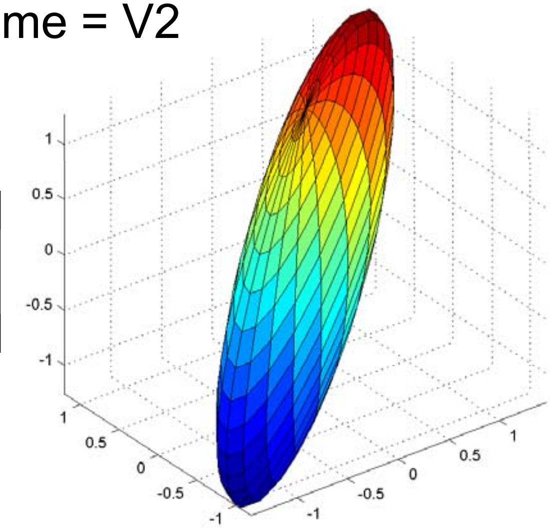
Matrix Determinant: Another Perspective

Volume = V1



$$\begin{bmatrix} 0.8 & 0 & 0.7 \\ 1.0 & 0.8 & 0.8 \\ 0.7 & 0.9 & 0.7 \end{bmatrix}$$

Volume = V2



- The determinant is the ratio of N-volumes
  - If V_1 is the volume of an N-dimensional object "O" in N-dimensional space
    - O is the complete set of points or vertices that specify the object
  - If V_2 is the volume of the N-dimensional object specified by A*O, where A is a matrix that transforms the space
  - $|A| = V_2 / V_1$

Matrix Determinants

- Matrix determinants are *only defined for square matrices*
  - They characterize volumes in linearly transformed space of the same dimensionality as the vectors
- Rank deficient matrices have determinant 0
  - Since they compress full-volumed N-D objects into zero-volume N-D objects
    - E.g. a 3-D sphere into a 2-D ellipse: The ellipse has 0 volume (although it does have area)
- Conversely, all matrices of determinant 0 are rank deficient
  - Since they compress full-volumed N-D objects into zero-volume objects

Multiplication properties

- Properties of vector/matrix products
  - Associative

$$\mathbf{A}\cdot(\mathbf{B}\cdot\mathbf{C}) = (\mathbf{A}\cdot\mathbf{B})\cdot\mathbf{C}$$

  - Distributive

$$\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C}$$

  - NOT commutative!!!

$$\mathbf{A}\cdot\mathbf{B} \neq \mathbf{B}\cdot\mathbf{A}$$

    - *left multiplications ≠ right multiplications*
  - Transposition

$$(\mathbf{A}\cdot\mathbf{B})^T = \mathbf{B}^T\cdot\mathbf{A}^T$$

Determinant properties

- Associative for square matrices $$|\mathbf{A}\cdot\mathbf{B}\cdot\mathbf{C}| = |\mathbf{A}|\cdot|\mathbf{B}|\cdot|\mathbf{C}|$$
  - Scaling volume sequentially by several matrices is equal to scaling once by the product of the matrices
- Volume of sum != sum of Volumes $$|(\mathbf{B}+\mathbf{C})| \neq |\mathbf{B}| + |\mathbf{C}|$$
  - The volume of the parallelepiped formed by row vectors of the sum of two matrices is not the sum of the volumes of the parallelepipeds formed by the original matrices
- Commutative for square matrices!!!

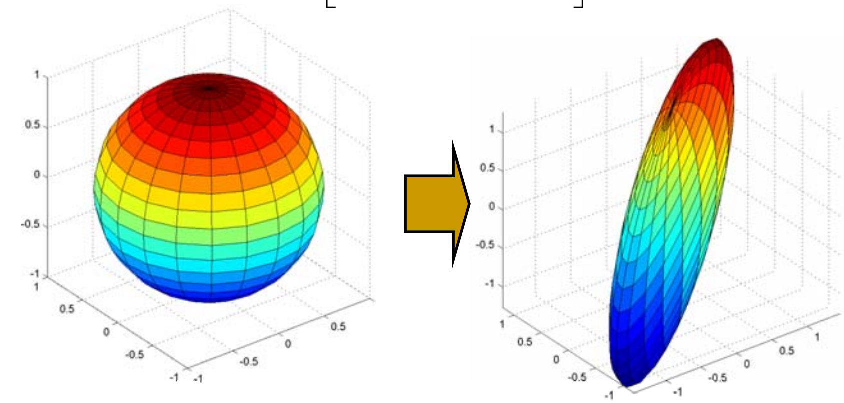
$$|\mathbf{A}\cdot\mathbf{B}| = |\mathbf{B}\cdot\mathbf{A}| = |\mathbf{A}|\cdot|\mathbf{B}|$$

  - The order in which you scale the volume of an object is irrelevant

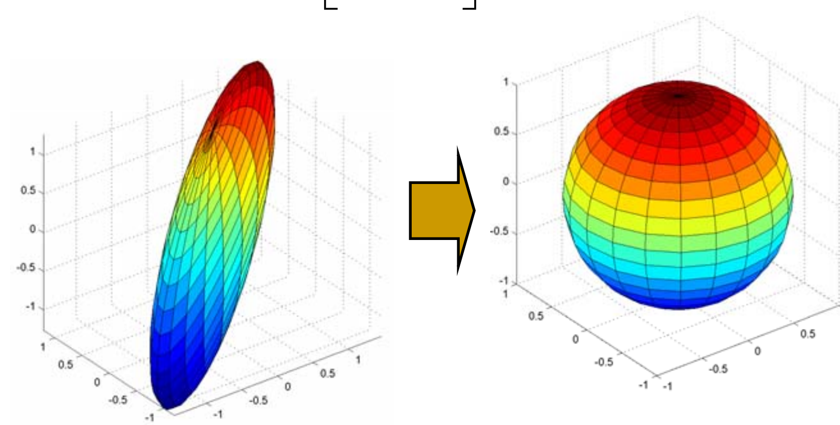
Matrix Inversion

- A matrix transforms an N-D object to a different N-D object
- What transforms the new object back to the original?
  - The *inverse transformation*
- The inverse transformation is called the matrix inverse

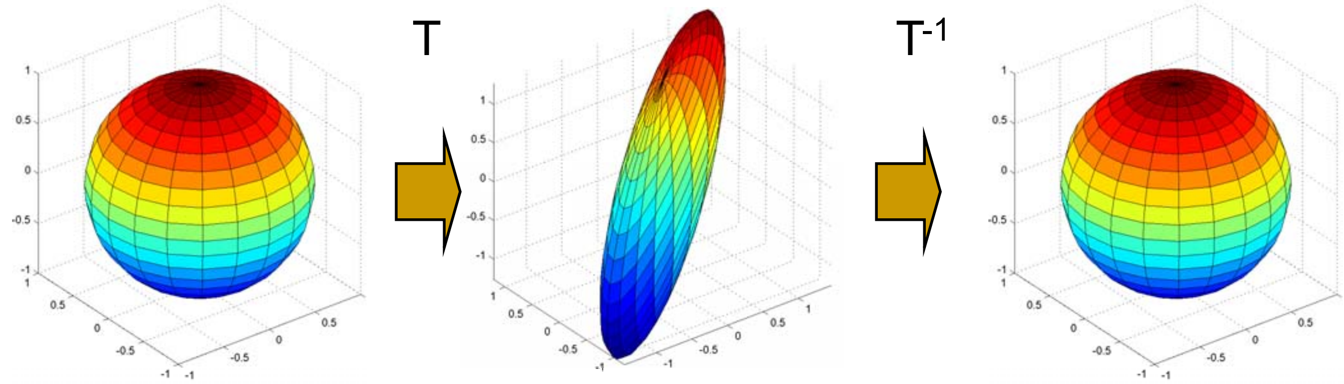
$$T = \begin{bmatrix} 0.8 & 0 & 0.7 \\ 1.0 & 0.8 & 0.8 \\ 0.7 & 0.9 & 0.7 \end{bmatrix}$$



$$Q = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} = T^{-1}$$



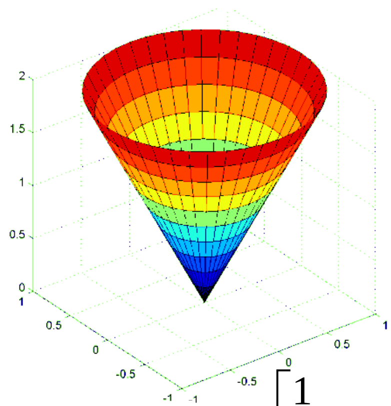
Matrix Inversion

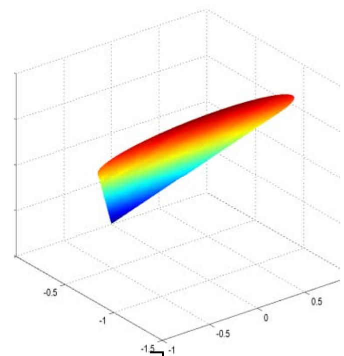


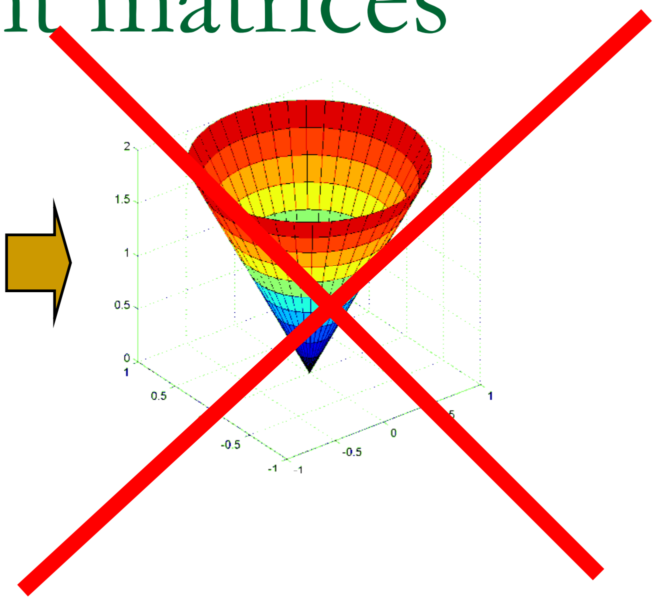
$T^{-1}T = I$

- The product of a matrix and its inverse is the identity matrix
  - Transforming an object, and then inverse transforming it gives us back the original object

Inverting rank-deficient matrices







$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & .25 & -0.433 \\ 0 & -0.433 & 0.75 \end{bmatrix}$$

- Rank deficient matrices "flatten" objects
  - In the process, multiple points in the original object get mapped to the same point in the transformed object
- It is not possible to go "back" from the flattened object to the original object
  - Because of the many-to-one forward mapping
- Rank deficient matrices have no inverse

Revisiting Projections and Least Squares

- Projection computes a *least squared error* estimate
- For each vector V in the music spectrogram matrix
  - Approximation: V_{approx} = a*note1 + b*note2 + c*note3..

$$T = \begin{bmatrix} \text{note1} & \text{note2} & \text{note3} \end{bmatrix} \qquad V_{approx} = T \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

  - Error vector E = $V - V_{approx}$
  - Squared error energy for V $e(V) = \text{norm}(E)^2$
  - Total error = Total error + e(V)
- Projection computes V_{approx} for all vectors such that Total error is minimized
- **But WHAT ARE "a" "b" and "c"?**

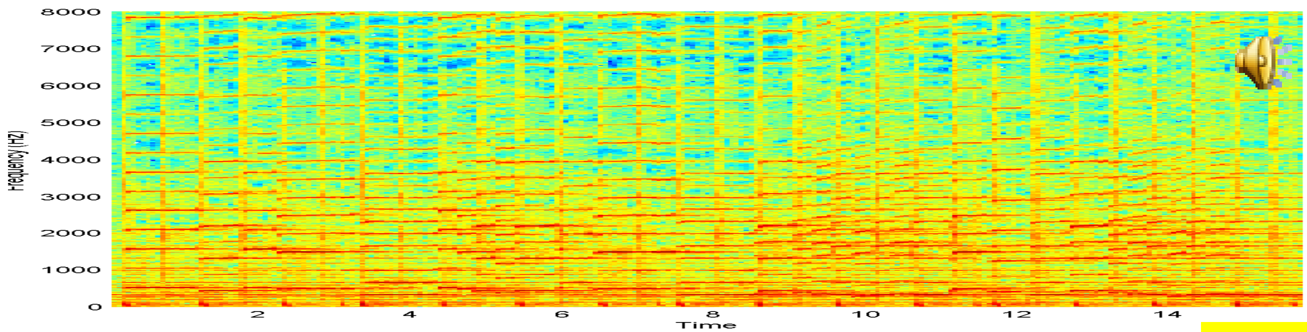
The Pseudo Inverse (PINV)

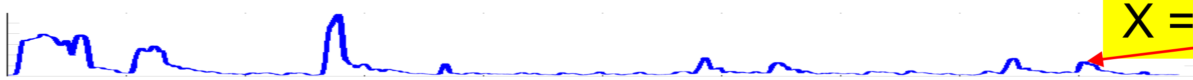
$$V_{approx} = T\begin{bmatrix} a \\ b \\ c \end{bmatrix} \Rightarrow V \approx T\begin{bmatrix} a \\ b \\ c \end{bmatrix} \Rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} = PINV(T) * V$$

- We are approximating spectral vectors V as the transformation of the vector $[a\ b\ c]^T$
  - Note – we're viewing the collection of bases in T as a transformation
- The solution is obtained using the *pseudo inverse*
  - This give us a *LEAST SQUARES* solution
    - If T were square and invertible Pinv(T) = T^{-1}, and V=V_{approx}

Explaining music with one note

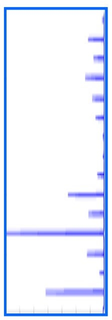
M =

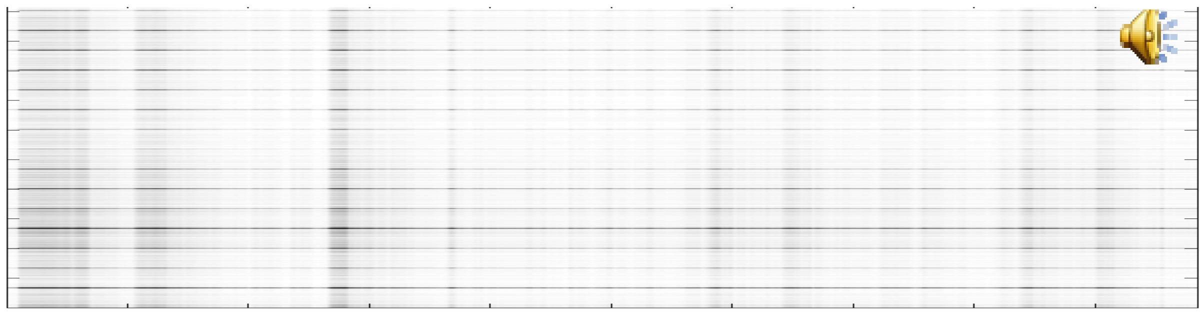




X =PINV(W)*M

W =



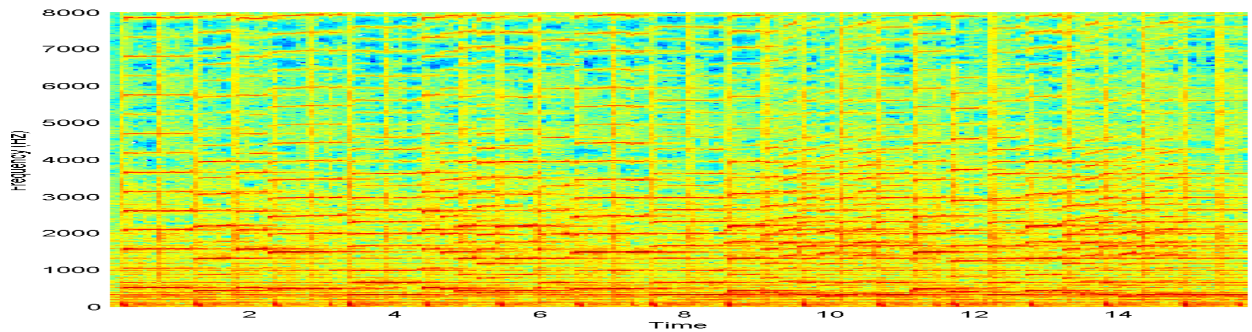


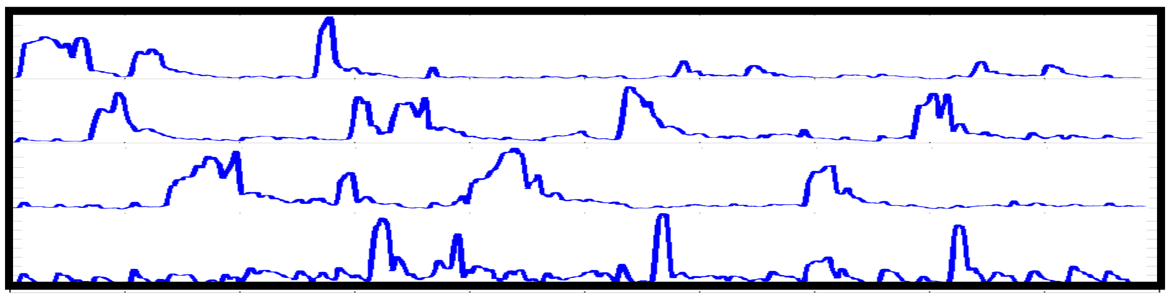
- Recap: P = W $(W^TW)^{-1}$ W^T, Projected Spectrogram = P*M
- **Approximation: M = W*X**
- The amount of W in each vector = X = PINV(W)*M
- W*Pinv(W)*M = Projected Spectrogram
  - W*Pinv(W) = Projection matrix!!

PINV(W) = $(W^TW)^{-1}W^T$

Explanation with multiple notes

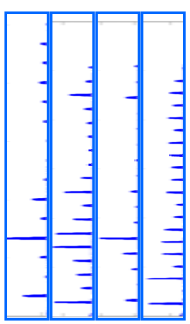
M =

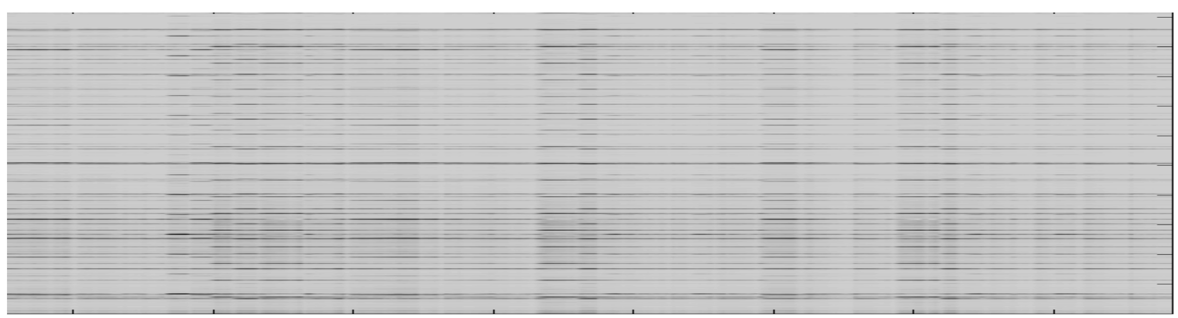




X=PINV(W)M

W =

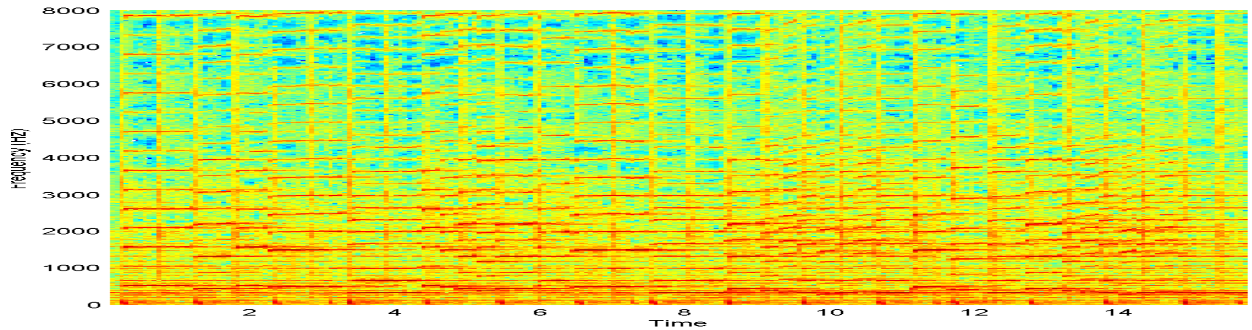




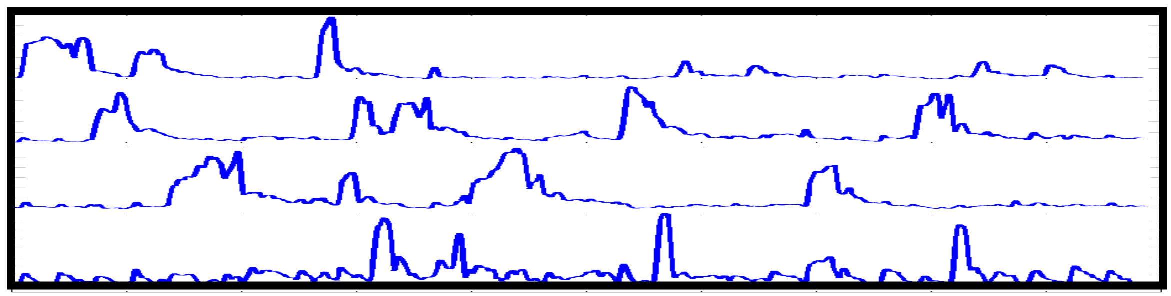
- X = Pinv(W) * M; Projected matrix = W*X = W*Pinv(W)*M

How about the other way?

M =



V =



W = ?

U = ?

- WV \approx M   W = M * Pinv(V)   U = WV

Pseudo-inverse (PINV)

- Pinv() applies to non-square matrices
- Pinv (Pinv (A))) = A
- A*Pinv(A)= projection matrix!
  - Projection onto the columns of A

- If A = K x N matrix and K > N, A projects N-D vectors into a higher-dimensional K-D space
- Pinv(A)*A = I in this case

Matrix inversion (division)

- The inverse of matrix multiplication
  - Not element-wise division!!
- Provides a way to "undo" a linear transformation
  - Inverse of the unit matrix is itself
  - Inverse of a diagonal is diagonal
  - Inverse of a rotation is a (counter)rotation (its transpose!)
  - Inverse of a rank deficient matrix does not exist!
    - But pseudoinverse exists
- Pay attention to multiplication side!

$$\mathbf{A}\cdot\mathbf{B}=\mathbf{C},\ \ \mathbf{A}=\mathbf{C}\cdot\mathbf{B}^{-1},\ \ \mathbf{B}=\mathbf{A}^{-1}\cdot\mathbf{C}$$

- Matrix inverses defined for square matrices only
  - If matrix not square use a matrix pseudoinverse:

$$\mathbf{A}\cdot\mathbf{B}=\mathbf{C},\ \ \mathbf{A}=\mathbf{C}\cdot\mathbf{B}^{+},\ \ \mathbf{B}=\mathbf{A}^{+}\cdot\mathbf{C}$$

- MATLAB syntax: `inv(a), pinv(a)`

What is the Matrix ?

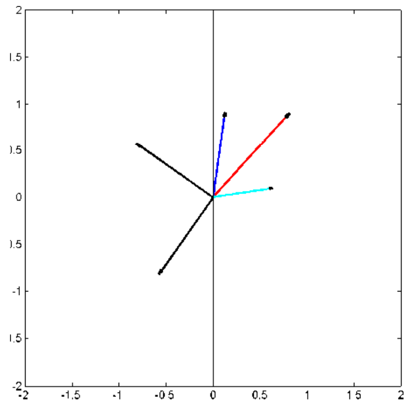
- Duality in terms of the matrix identity
  - Can be a container of data
    - An image, a set of vectors, a table, etc …
  - Can be a linear transformation
    - A process by which to transform data in another matrix
- We'll usually start with the first definition and then apply the second one on it
  - Very frequent operation
  - Room reverberations, mirror reflections, etc …
- Most of signal processing and machine learning are a matrix multiplication!

Eigenanalysis

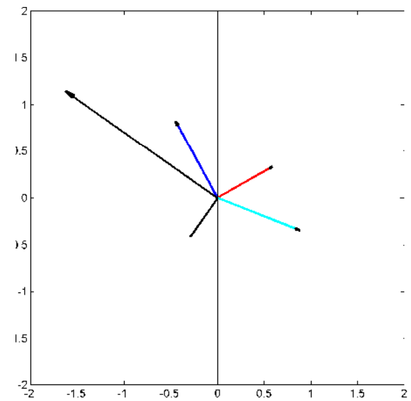
- If something can go through a process mostly unscathed in character it is an *eigen*-something
  - Sound example:
- A vector that can undergo a matrix multiplication and keep pointing the same way is an *eigenvector*
  - Its length can change though
- How much its length changes is expressed by its corresponding *eigenvalue*
  - Each eigenvector of a matrix has its eigenvalue
- Finding these “eigenthings” is called eigenanalysis

EigenVectors and EigenValues

Black vectors are eigen vectors



$$A = \begin{bmatrix} 1.5 & -0.7 \\ -0.7 & 1.0 \end{bmatrix}$$

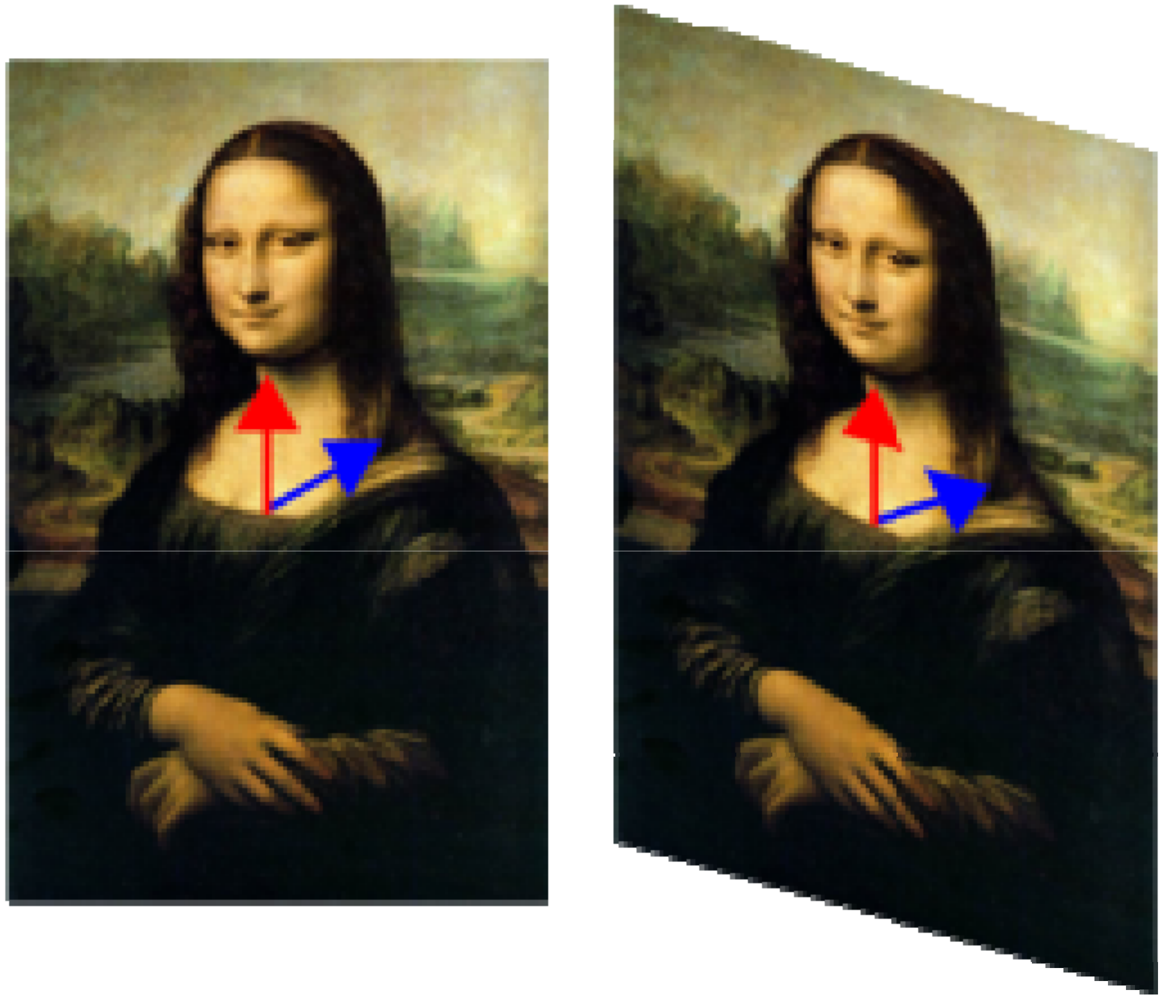


- Vectors that do not change angle upon transformation
  - They may change length

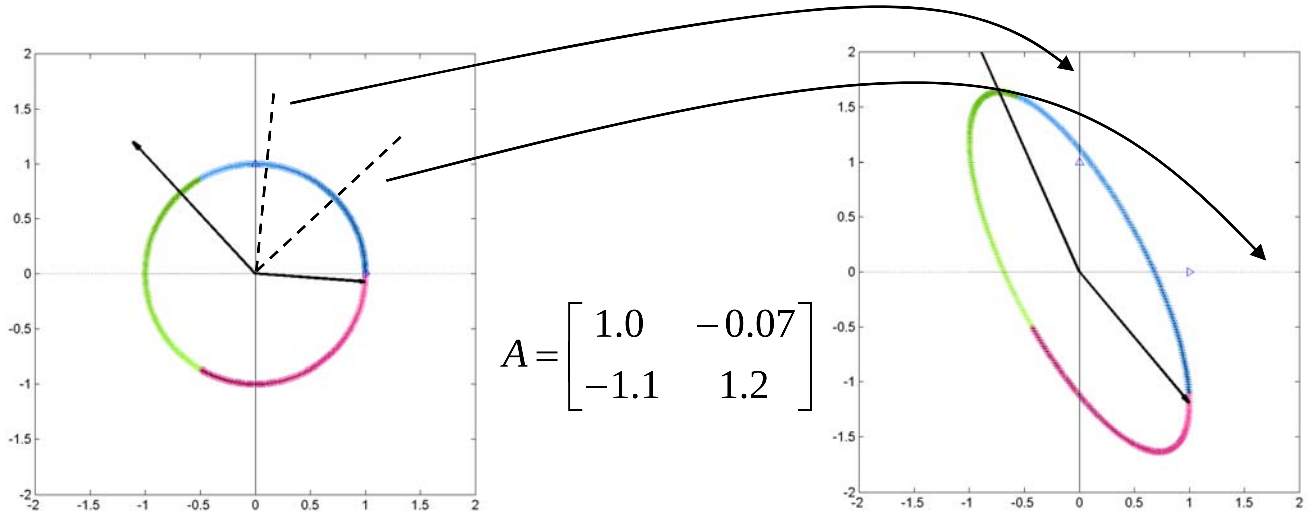
$$MV = \lambda V$$

  - V = eigen vector
  - λ = eigen value
  - Matlab: `[V, L] = eig(M)`
    - L is a diagonal matrix whose entries are the eigen values
    - V is a maxtrix whose columns are the eigen vectors

Eigen vector example

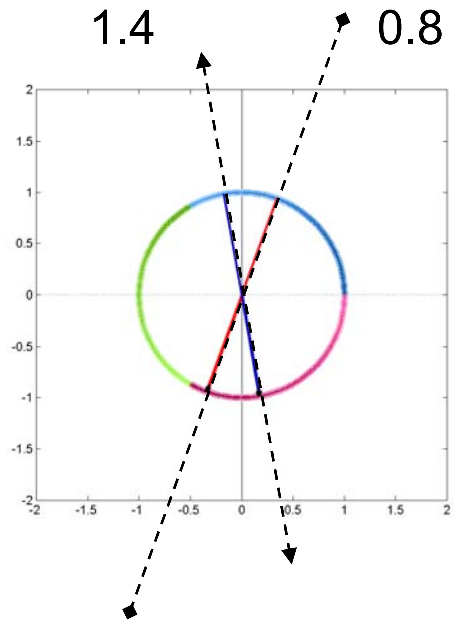


Matrix multiplication revisited



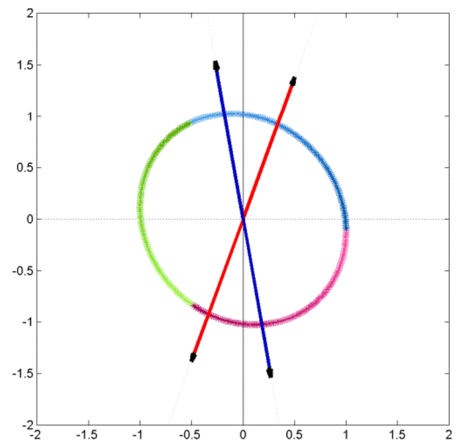
- Matrix transformation "transforms" the space
  - Warps the paper so that the normals to the two vectors now lie along the axes

A stretching operation



- Draw two lines
- Stretch / shrink the paper along these lines by factors λ_1 and λ_2
  - The factors could be negative – implies flipping the paper
- The result is a transformation of the space

A stretching operation



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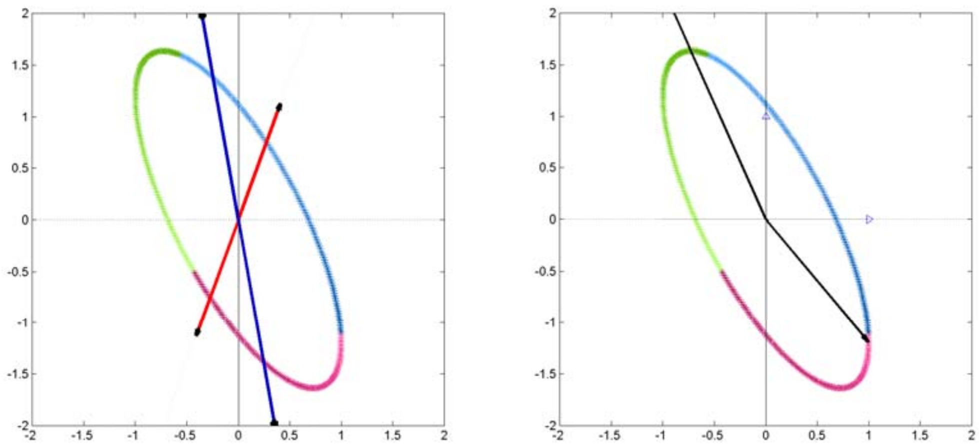
Physical interpretation of eigen vector



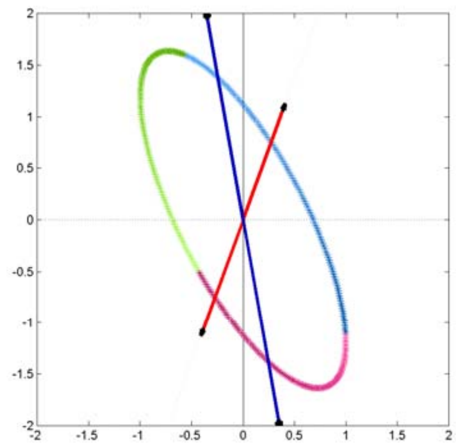
- The result of the stretching is exactly the same as transformation by a matrix
- The axes of stretching/shrinking are the eigenvectors
  - The degree of stretching/shrinking are the corresponding eigenvalues
- The EigenVectors and EigenValues convey all the information about the matrix

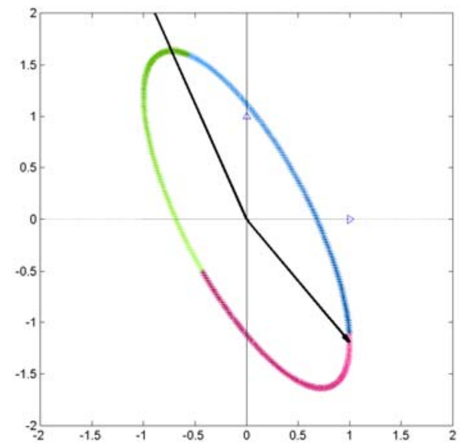
Physical interpretation of eigen vector

$$V = \begin{bmatrix} V_1 & V_2 \end{bmatrix}$$

$$L = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$M = VLV^{-1}$$





- The result of the stretching is exactly the same as transformation by a matrix
- The axes of stretching/shrinking are the eigenvectors
  - The degree of stretching/shrinking are the corresponding eigenvalues
- The EigenVectors and EigenValues convey all the information about the matrix

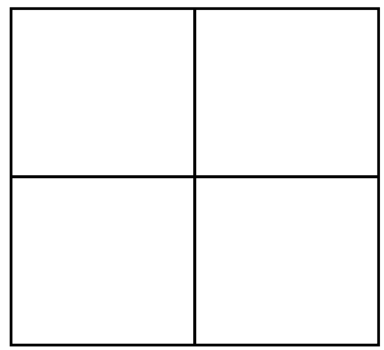
Eigen Analysis

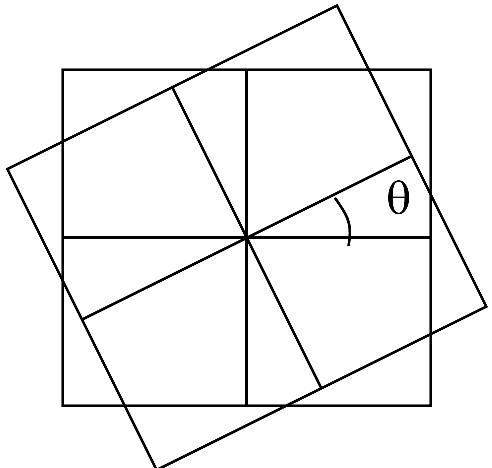
- Not all square matrices have nice eigen values and vectors
  - E.g. consider a rotation matrix

$$\mathbf{R}_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$X = \begin{bmatrix} x \\ y \end{bmatrix}$$

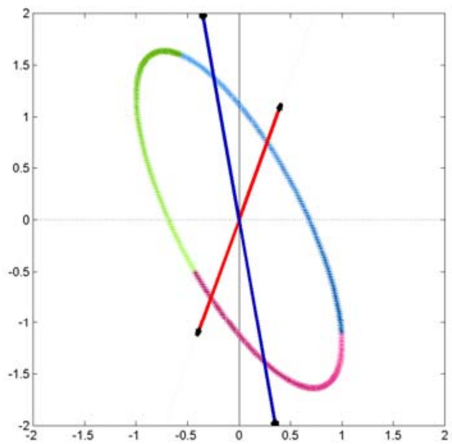
$$X_{new} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$



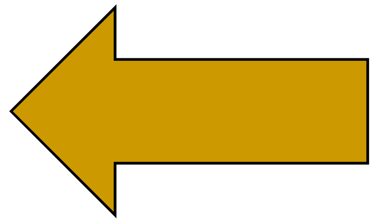


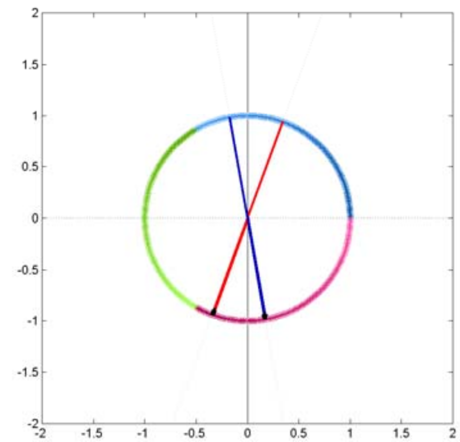
  - This rotates every vector in the plane
    - No vector that remains unchanged
- In these cases the Eigen vectors and values are complex
- Some matrices are special however..

Singular Value Decomposition



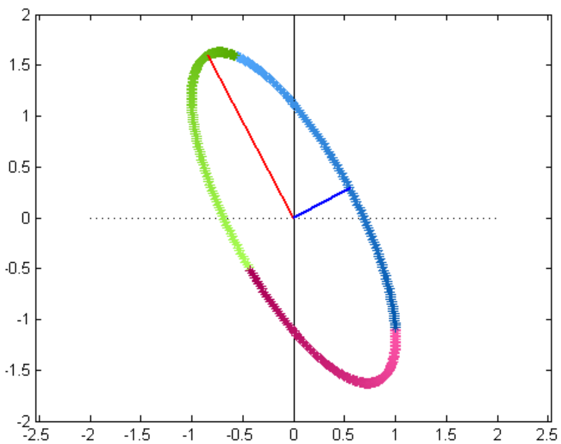
$$A = \begin{bmatrix} 1.0 & -0.07 \\ -1.1 & 1.2 \end{bmatrix}$$





- Matrix transformations convert circles to ellipses
- Eigen vectors are vectors that do not change direction in the process
- There is another key feature of the ellipse to the right that carries information about the transform
  - Can you identify it?

Singular Value Decomposition



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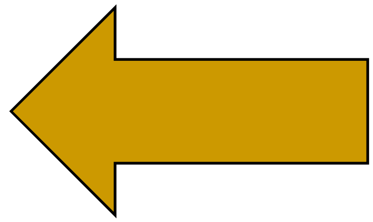
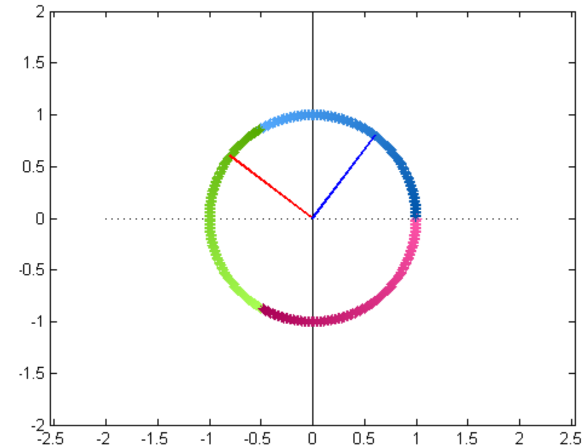
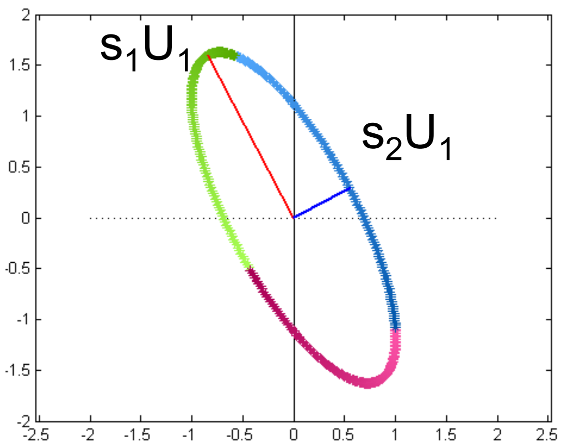




- The major and minor axes of the transformed ellipse define the ellipse
  - They are at right angles
- These are transformations of right-angled vectors on the original circle!

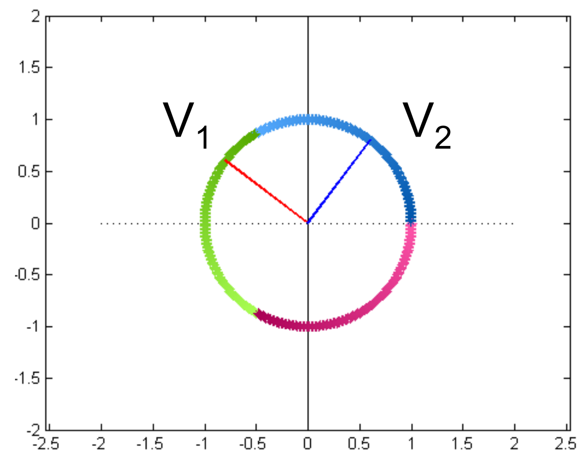
Singular Value Decomposition



$$A = \begin{bmatrix} 1.0 & -0.07 \\ -1.1 & 1.2 \end{bmatrix}$$

$$A = U\,S\,V^T$$

matlab:
[U,S,V] = svd(A)



- U and V are orthonormal matrices
  - Columns are orthonormal vectors
- S is a diagonal matrix

- The *right singular vectors* of V are transformed to the *left singular vectors* in U
  - And scaled by the *singular values* that are the diagonal entries of S

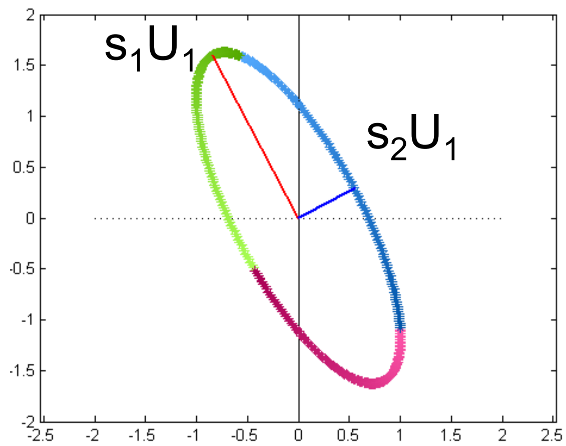
Singular Value Decomposition

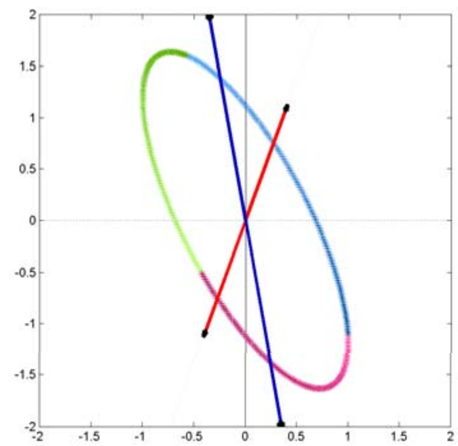
- The left and right singular vectors are not the same
  - If A is not a square matrix, the left and right singular vectors will be of different dimensions

- The singular values are always real

- The largest singular value is the largest amount by which a vector is scaled by A
  - Max (|Ax| / |x|) = s_{max}
- The smallest singular value is the smallest amount by which a vector is scaled by A
  - Min (|Ax| / |x|) = s_{min}
  - This can be 0 (for low-rank or non-square matrices)

The Singular Values

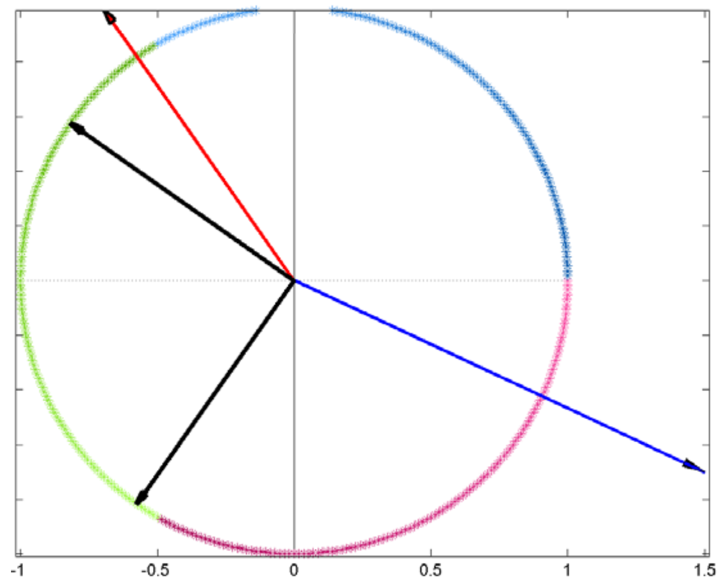




- Square matrices: The product of the singular values is the determinant of the matrix
  - This is also the product of the *eigen* values
  - I.e. there are two different sets of axes whose products give you the area of an ellipse
- For any "broad" rectangular matrix A, the largest singular value of any square submatrix B cannot be larger than the largest singular value of A
  - An analogous rule applies to the smallest singluar value
  - This property is utilized in various problems, such as compressive sensing

Symmetric Matrices

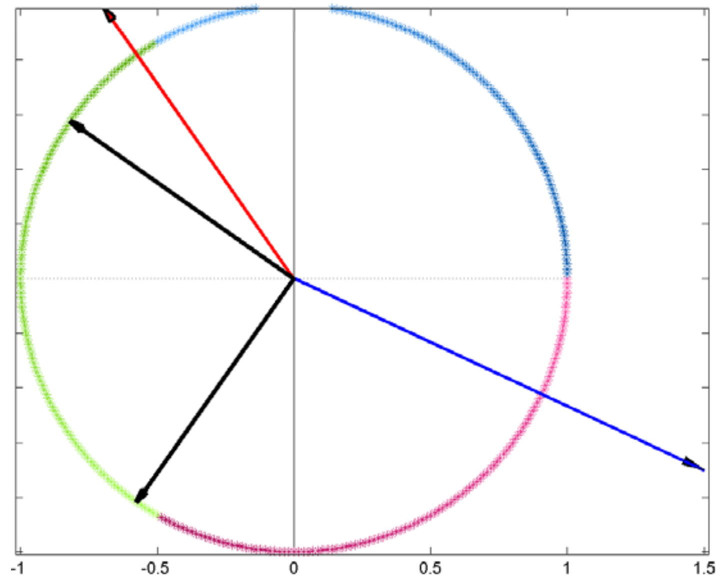
$$\begin{bmatrix} 1.5 & -0.7 \\ -0.7 & 1 \end{bmatrix}$$



- Matrices that do not change on transposition
  - Row and column vectors are identical
- The left and right singular vectors are identical
  - U = V
  - $A = U S U^T$
- They are identical to the *eigen vectors* of the matrix

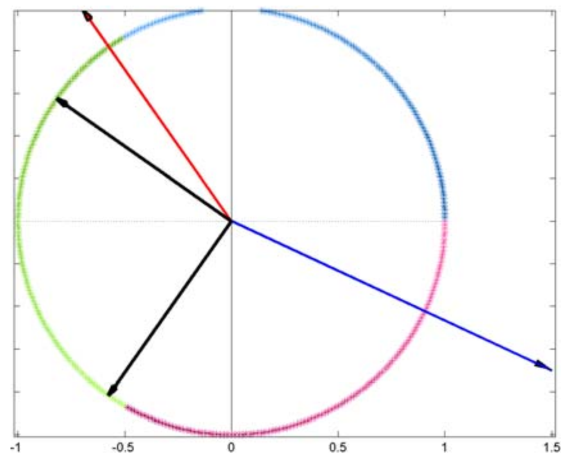
Symmetric Matrices

$$\begin{bmatrix} 1.5 & -0.7 \\ -0.7 & 1 \end{bmatrix}$$

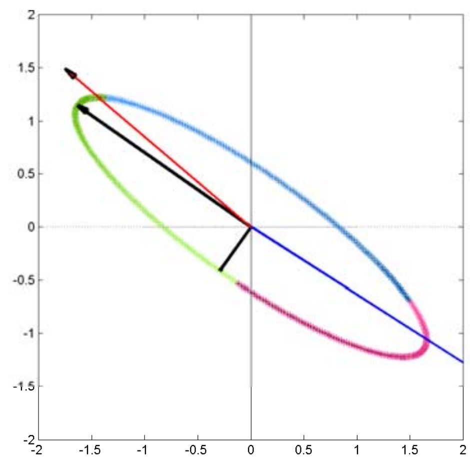


- Matrices that do not change on transposition
  - Row and column vectors are identical
- Symmetric matrix: Eigen vectors and Eigen values are always real
- Eigen vectors are always orthogonal
  - At 90 degrees to one another

Symmetric Matrices



$$\begin{bmatrix} 1.5 & -0.7 \\ -0.7 & 1 \end{bmatrix}$$



- Eigen vectors point in the direction of the major and minor axes of the ellipsoid resulting from the transformation of a spheroid
  - The eigen values are the lengths of the axes

Symmetric matrices

- Eigen vectors V_i are orthonormal
  - $V_i^T V_i = 1$
  - $V_i^T V_j = 0$, i != j

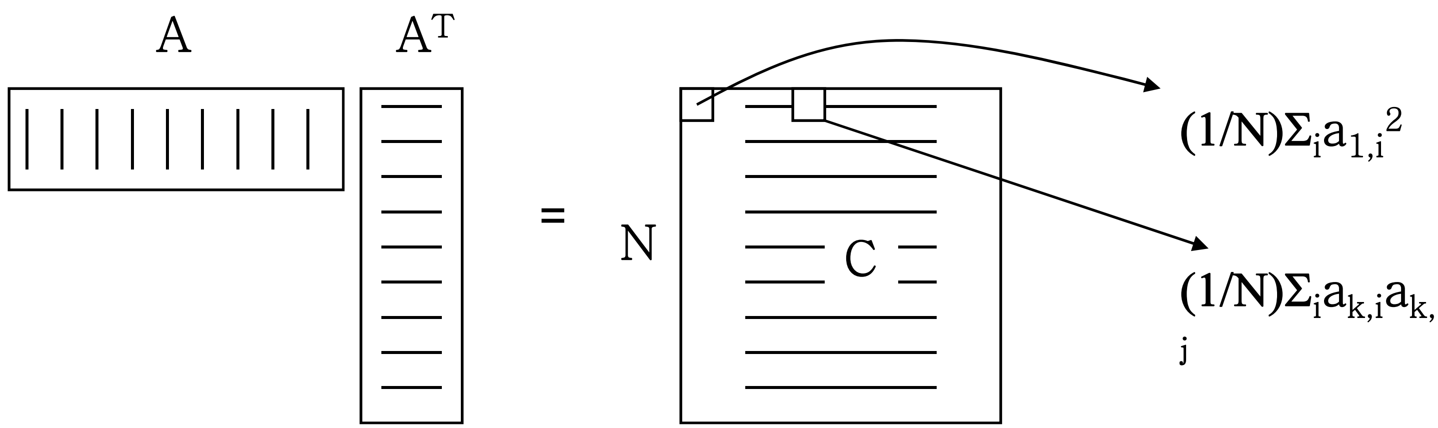
- Listing all eigen vectors in matrix form V
  - $V^T = V^{-1}$
  - $V^T V = I$
  - $V V^T = I$

- $C V_i = \lambda V_i$

- In matrix form : $C V = V L$
  - L is a diagonal matrix with all eigen values

- $C = V L V^T$

The Correlation and Covariance Matrices



- Consider a set of column vectors represented as a DxN matrix M
- The correlation matrix is
  - $C = (1/N)\, MM^T$
    - If the average value (mean) of the vectors in M is 0, C is called the ***covariance*** matrix
    - **covariance = correlation + mean * mean^T**
- Diagonal elements represent average value of the squared value of each dimension
  - Off diagonal elements represent how two components are related
    - How much knowing one lets us guess the value of the other

Correlation / Covariance Matrix

$$C = VLV^T$$

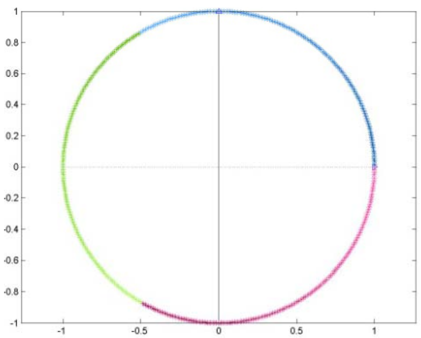
$$Sqrt(C) = V.Sqrt(L).V^T$$

$$Sqrt(C).Sqrt(C) = V.Sqrt(L).V^T V.Sqrt(L).V^T$$

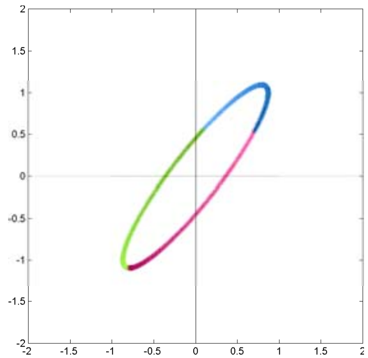
$$= V.Sqrt(L).Sqrt(L)V^T = VLV^T = C$$

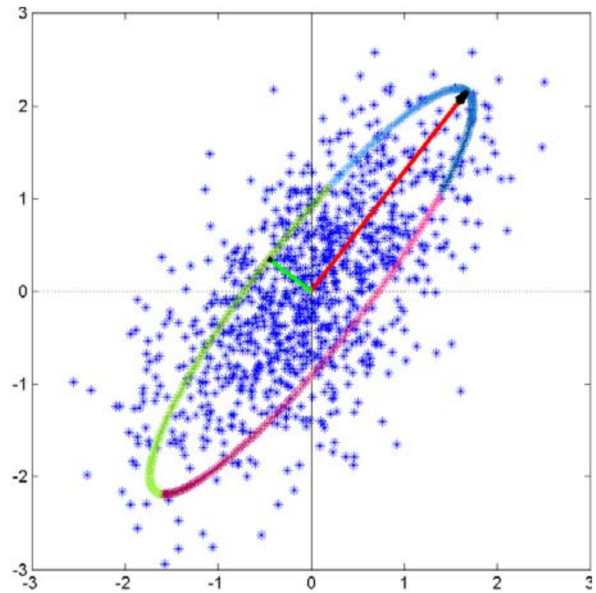
- The correlation / covariance matrix is symmetric
  - Has orthonormal eigen vectors and real, non-negative eigen values
- The *square root* of a correlation or covariance matrix is easily derived from the eigen vectors and eigen values
  - The eigen values of the *square root* of the covariance matrix are the square roots of the eigen values of the covariance matrix
  - These are also the "singular values" of the data set

Square root of the Covariance Matrix



$$\sqrt{C}$$



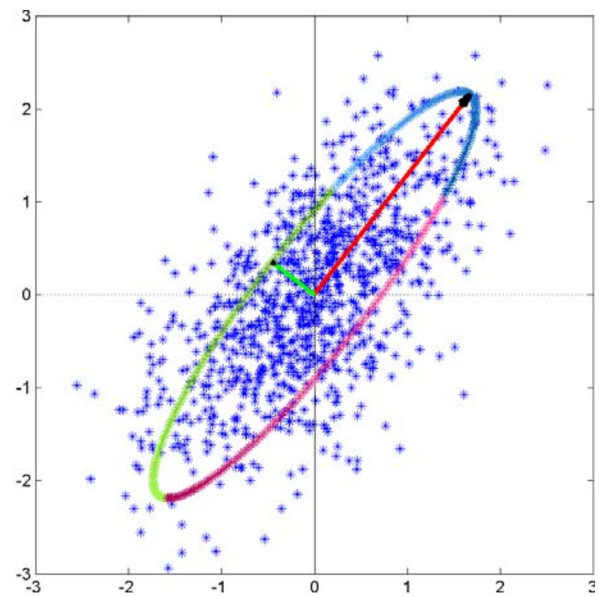


- The square root of the covariance matrix represents the elliptical scatter of the data
- The eigenvectors of the matrix represent the major and minor axes

The Covariance Matrix

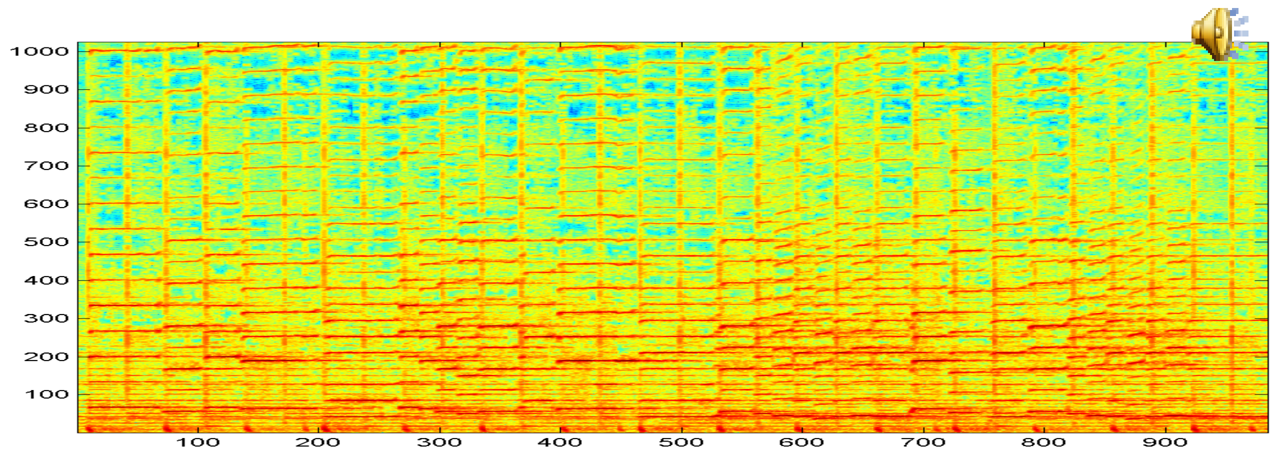
Any vector $V = a_{V,1}$ * eigenvec1 + $a_{V,2}$ *eigenvec2 + ..

$\Sigma_V a_{V,i}$ = eigenvalue(i)



- Projections along the N eigen vectors with the largest eigen values represent the N greatest "energy-carrying" components of the matrix
- Conversely, N "bases" that result in the least square error are the N best eigen vectors

An audio example



- The spectrogram has 974 vectors of dimension 1025
- The covariance matrix is size 1025 x 1025
- There are 1025 eigenvectors

Eigen Reduction

$$M = spectrogram$$ 1025x1000

$$C = M.M^T$$ 1025x1025

V = 1025x1025

$$[V, L] = eig(C)$$

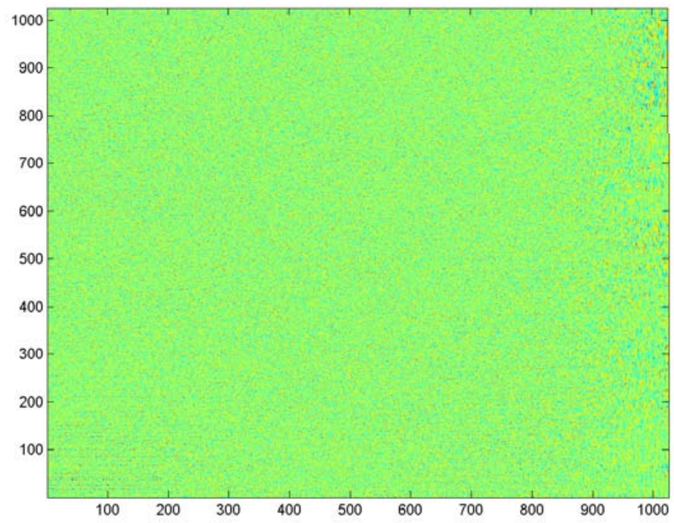
$$V_{reduced} = [V_1 \quad . \quad . \quad V_{25}]$$ 1025x25

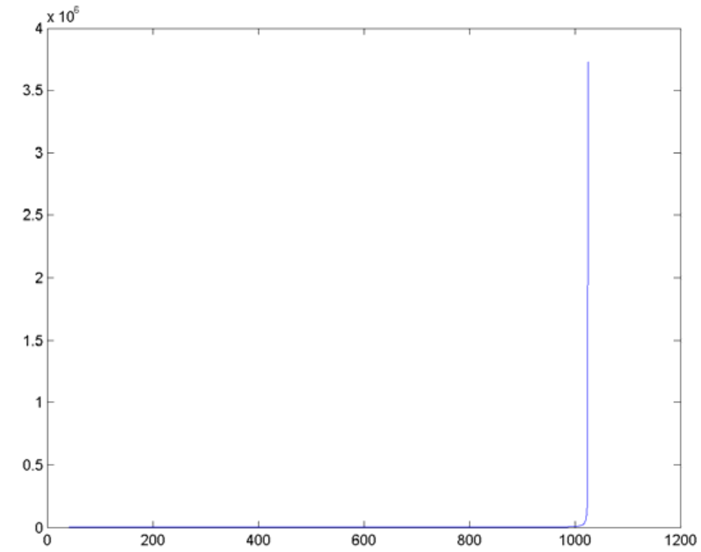
$$M_{lowdim} = Pinv(V_{reduced})M$$ 25x1000

$$M_{reconstructed} = V_{reduced} M_{lowdim}$$ 1025x1000

- Compute the Covariance/Correlation
- Compute Eigen vectors and values
- Create matrix from the 25 Eigen vectors corresponding to 25 highest Eigen values
- Compute the weights of the 25 eigenvectors
- To reconstruct the spectrogram – compute the projection on the 25 eigen vectors

Eigenvalues and Eigenvectors





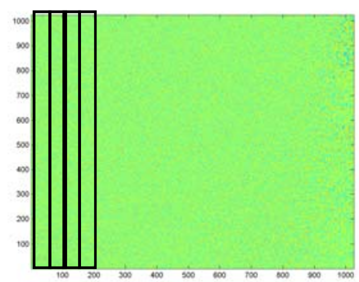
- Left panel: Matrix with 1025 eigen vectors
- Right panel: Corresponding eigen values
  - Most eigen values are close to zero
    - The corresponding eigenvectors are "unimportant"

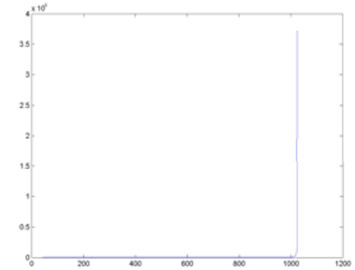
$$M = spectrogram$$

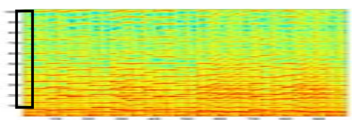
$$C = M.M^T$$

$$[V, L] = eig(C)$$

Eigenvalues and Eigenvectors





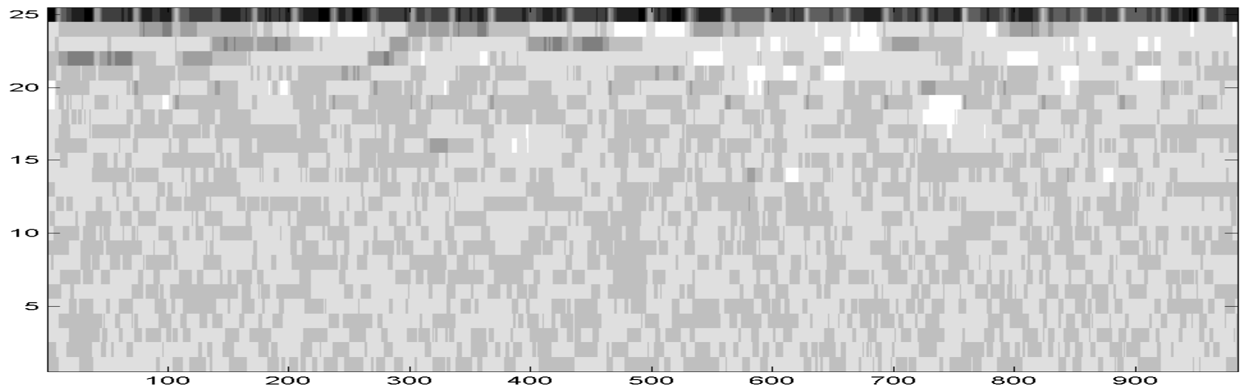


Vec = a1 *eigenvec1 + a2 * eigenvec2 + a3 * eigenvec3 …

- The vectors in the spectrogram are linear combinations of all 1025 eigen vectors
- The eigen vectors with low eigen values contribute very little
  - The average value of a_i is proportional to the square root of the eigenvalue
  - Ignoring these will not affect the composition of the spectrogram

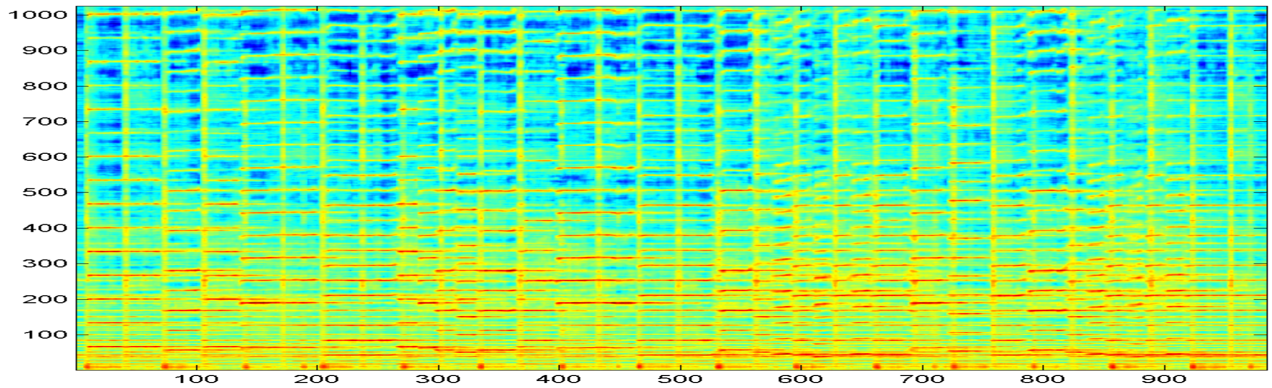
An audio example

$$V_{reduced} = [V_1 \quad . \quad . \quad V_{25}]$$
$$M_{lowdim} = Pinv(V_{reduced})M$$



- The same spectrogram projected down to the 25 eigen vectors with the highest eigen values
  - Only the 25-dimensional weights are shown
    - The weights with which the 25 eigen vectors must be added to compose a least squares approximation to the spectrogram

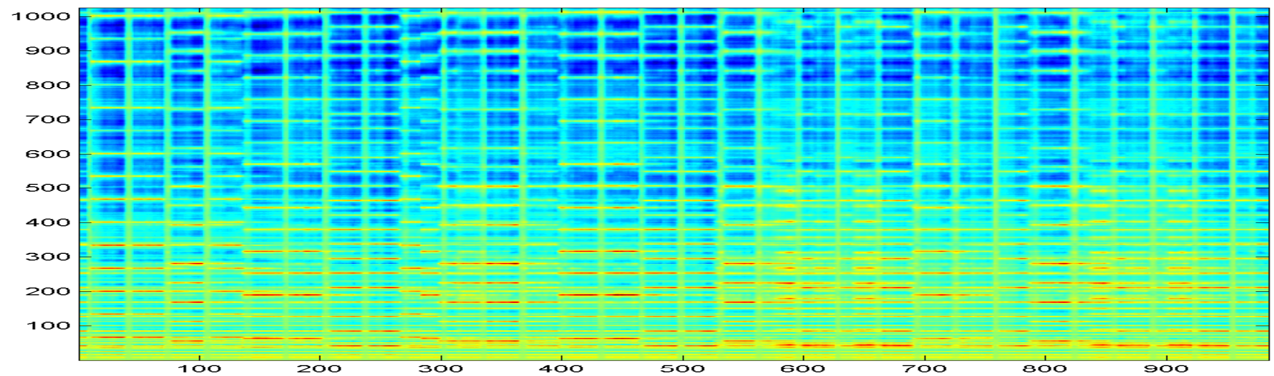
An audio example



$$M_{reconstructed} = V_{reduced} M_{lowdim}$$

- The same spectrogram constructed from only the 25 eigen vectors with the highest eigen values
  - Looks similar
    - With 100 eigenvectors, it would be indistinguishable from the original
  - Sounds pretty close
  - But now sufficient to store 25 numbers per vector (instead of 1024)

With only 5 eigenvectors



- The same spectrogram constructed from only the 5 eigen vectors with the highest eigen values
  - Highly recognizable

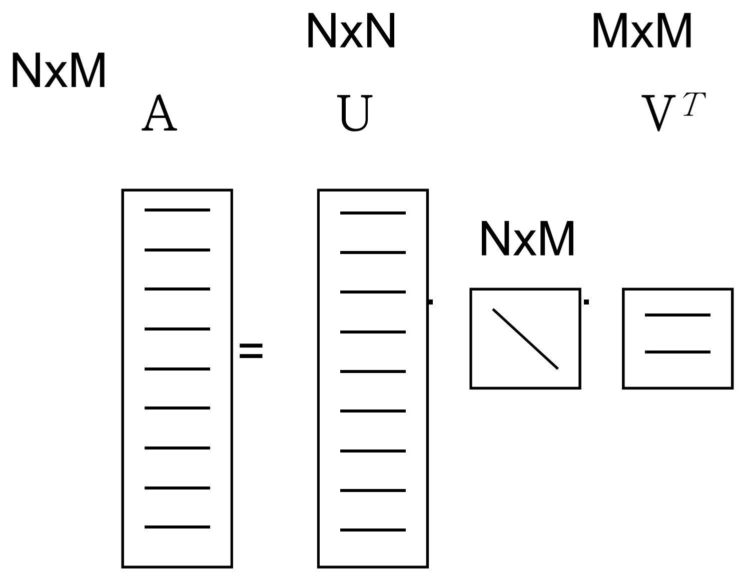
Eigenvectors, Eigenvalues and Covariances

- The eigenvectors and eigenvalues (singular values) derived from the correlation matrix are important
- Do we need to actually compute the correlation matrix?
  - No
- Direct computation using Singular Value Decomposition

SVD vs. Eigen decomposition

- Singluar value decomposition is analogous to the eigen decomposition of the correlation matrix of the data
- The "right" singluar vectors are the eigen vectors of the correlation matrix
  - Show the directions of greatest importance
- The corresponding singular values are the square roots of the eigen values of the correlation matrix
  - Show the importance of the eigen vector

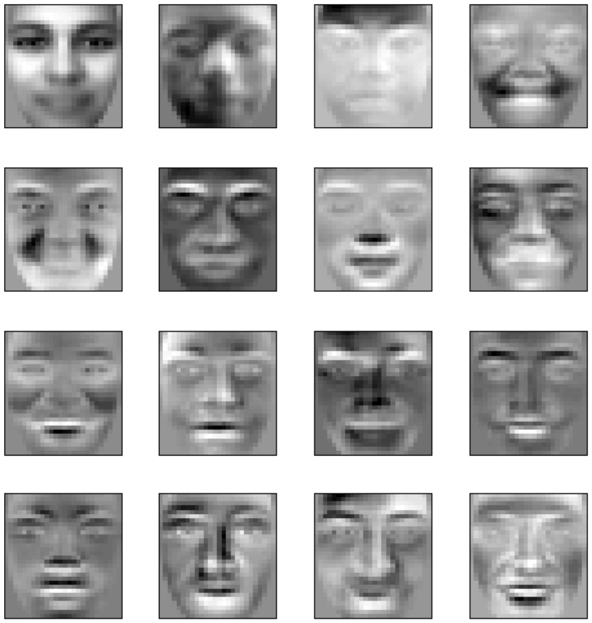
Thin SVD, compact SVD, reduced SVD



- Thin SVD: Only compute the first N columns of U
  - All that is required if N < M
- Compact SVD: Only the left and right eigen vectors corresponding to non-zero singular values are computed
- Reduced SVD: Only compute the columns of U corresponding to the K highest singular values

Why bother with eigens/SVD

- Can provide a unique insight into data
  - Strong statistical grounding
  - Can display complex interactions between the data
  - Can uncover irrelevant parts of the data we can throw out
- Can provide *basis functions*
  - A set of elements to compactly describe our data
  - Indispensable for performing compression and classification
- Used over and over and still perform amazingly well



Eigenfaces
Using a linear transform of the above "eigenvectors" we can compose various faces