Machine Learning for Signal Processing

Regression and Prediction

Class 14. 17 Oct 2012

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Matrix Identities

$$f(\mathbf{x}) \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_D \end{bmatrix} \qquad df(\mathbf{x}) = \begin{bmatrix} \frac{df}{dx_1} dx_1 \\ \frac{df}{dx_2} dx_2 \\ \ldots \\ \frac{df}{dx_D} dx_D \end{bmatrix}$$

- The derivative of a scalar function w.r.t. a vector is a vector

Matrix Identities

$$f(\mathbf{x}) \qquad \mathbf{x} = \begin{bmatrix} x_{11} & x_{12} & .. & x_{1D} \\ x_{21} & x_{22} & .. & x_{2D} \\ .. & .. & .. & .. \\ x_{D1} & x_{D2} & .. & x_{DD} \end{bmatrix} \qquad df(\mathbf{x}) = \begin{bmatrix} \frac{df}{dx_{11}} dx_{11} & \frac{df}{dx_{12}} dx_{12} & .. & \frac{df}{dx_{1D}} dx_{1D} \\ \frac{df}{dx_{21}} dx_{21} & \frac{df}{dx_{22}} dx_{22} & .. & \frac{df}{dx_{2D}} dx_{2D} \\ .. & .. & .. & .. \\ \frac{df}{dx_{D1}} dx_{D1} & \frac{df}{dx_{D2}} dx_{D2} & .. & \frac{df}{dx_{DD}} dx_{DD} \end{bmatrix}$$

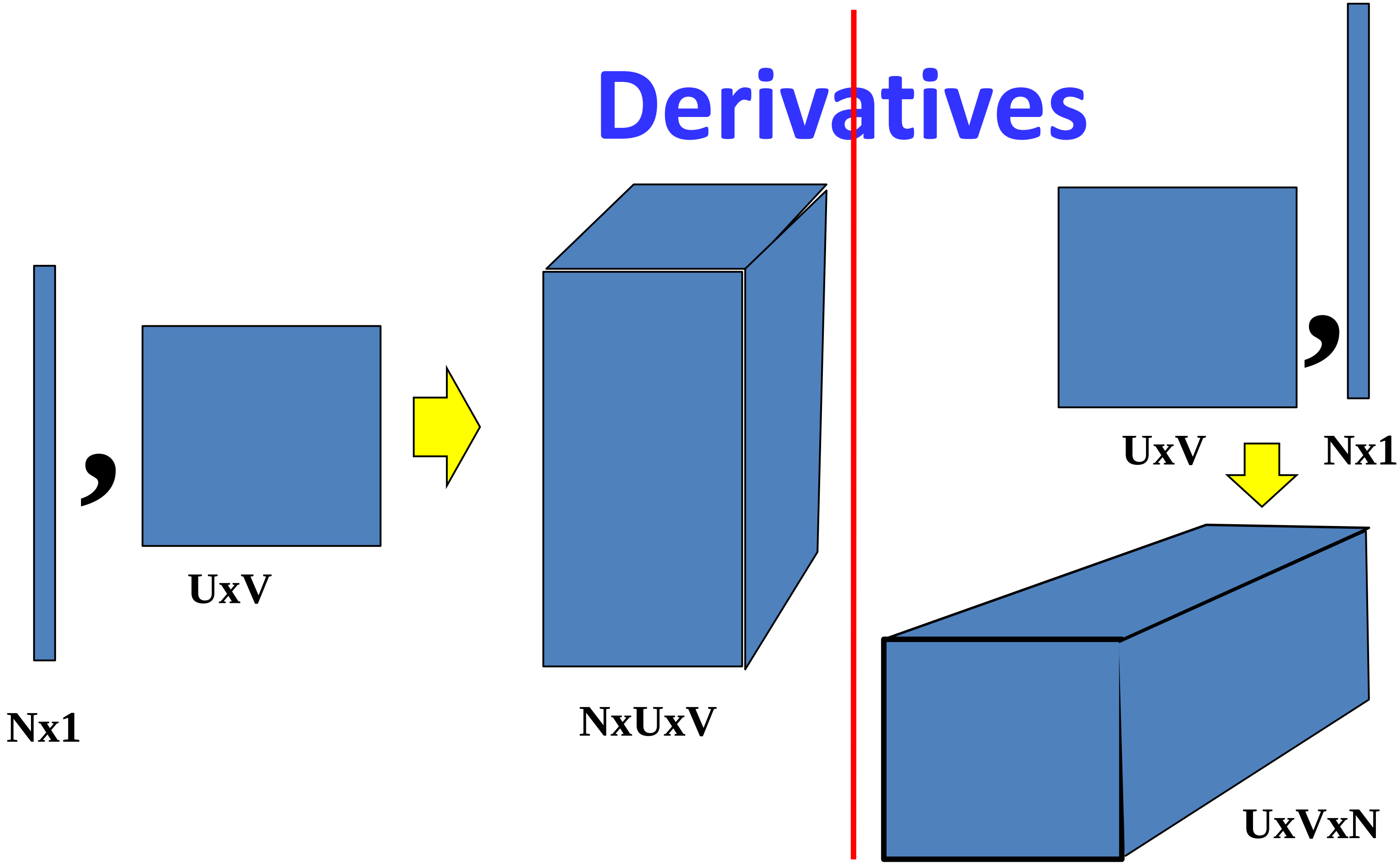
- The derivative of a scalar function w.r.t. a vector is a vector
- The derivative w.r.t. a matrix is a matrix

Matrix Identities

$$\mathbf{F}(\mathbf{x}) \qquad \mathbf{F} = \begin{bmatrix} F_1 \\ F_2 \\ \cdots \\ F_N \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_D \end{bmatrix} \qquad \begin{bmatrix} dF_1 \\ dF_2 \\ \cdots \\ dF_N \end{bmatrix} = \begin{bmatrix} \frac{dF_1}{dx_1}dx_1 & \frac{dF_1}{dx_2}dx_2 & & \frac{dF_1}{dx_D}dx_D \\ \frac{dF_2}{dx_1}dx_1 & \frac{dF_2}{dx_2}dx_2 & .. & \frac{dF_2}{dx_D}dx_D \\ .. & .. & .. & .. \\ \frac{dF_N}{dx_1}dx_1 & \frac{dF_N}{dx_2}dx_2 & & \frac{dF_N}{dx_D}dx_D \end{bmatrix}$$

- The derivative of a vector function w.r.t. a vector is a matrix
  - Note transposition of order

Derivatives



- In general: Differentiating an MxN function by a UxV argument results in an MxNxUxV tensor derivative

Matrix derivative identities

$$d(\mathbf{Xa}) = \mathbf{X}d\mathbf{a} \qquad d(\mathbf{a}^T\mathbf{X}) = \mathbf{X}^T d\mathbf{a}$$

X is a matrix, **a** is a vector.
Solution may also be $\mathbf{X}^T$

$$d(\mathbf{AX}) = (d\mathbf{A})\mathbf{X} \ ; \quad d(\mathbf{XA}) = \mathbf{X}(d\mathbf{A})$$

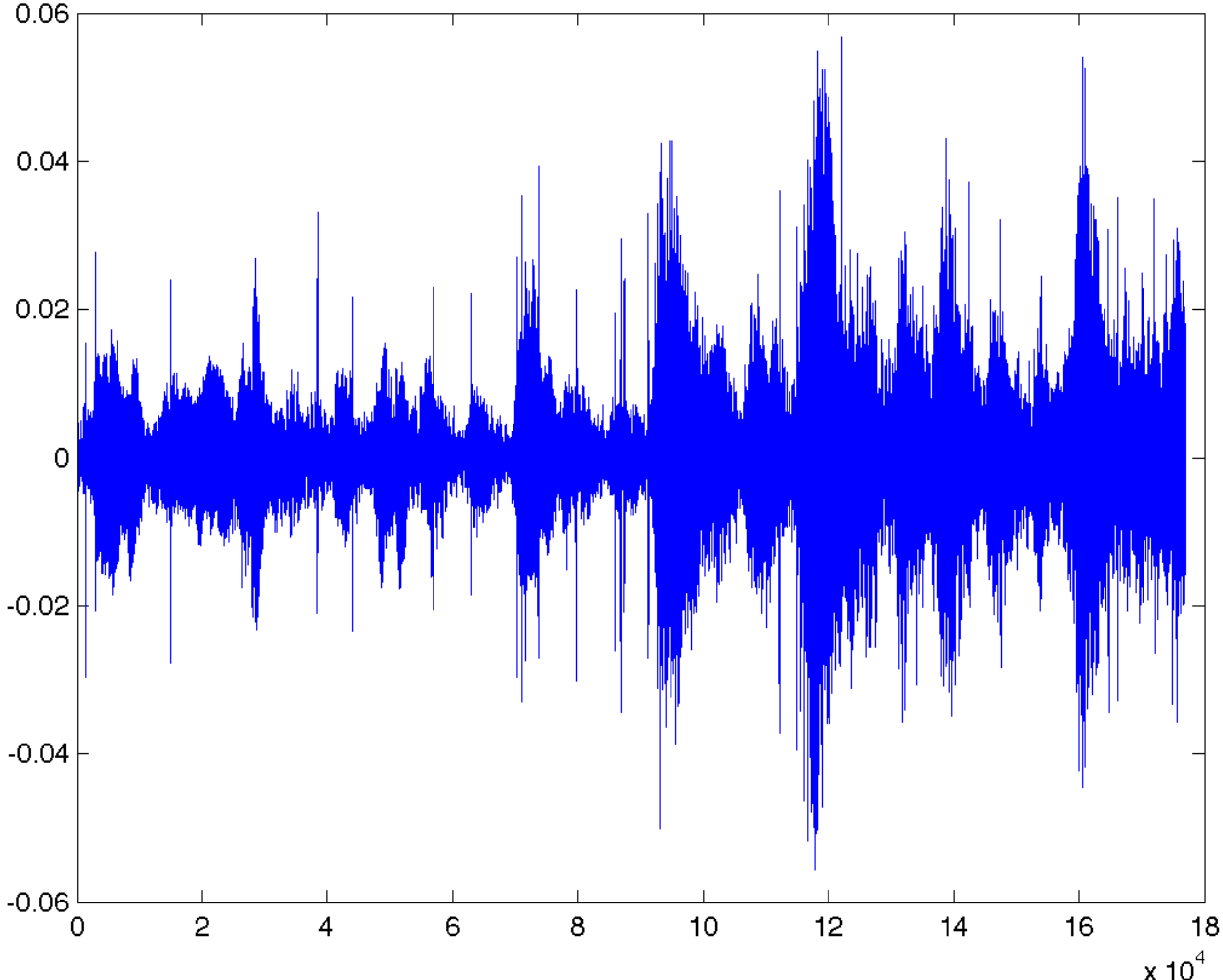
A is a matrix

$$d\left(\mathbf{a}^T\mathbf{Xa}\right) = \mathbf{a}^T\left(\mathbf{X} + \mathbf{X}^T\right)d\mathbf{a}$$

$$d\left(trace\left(\mathbf{A}^T\mathbf{XA}\right)\right) = d\left(trace\left(\mathbf{XAA}^T\right)\right) = d\left(trace\left(\mathbf{AA}^T\mathbf{X}\right)\right) = (\mathbf{X}^T + \mathbf{X})d\mathbf{A}$$

- Some basic linear and quadratic identities

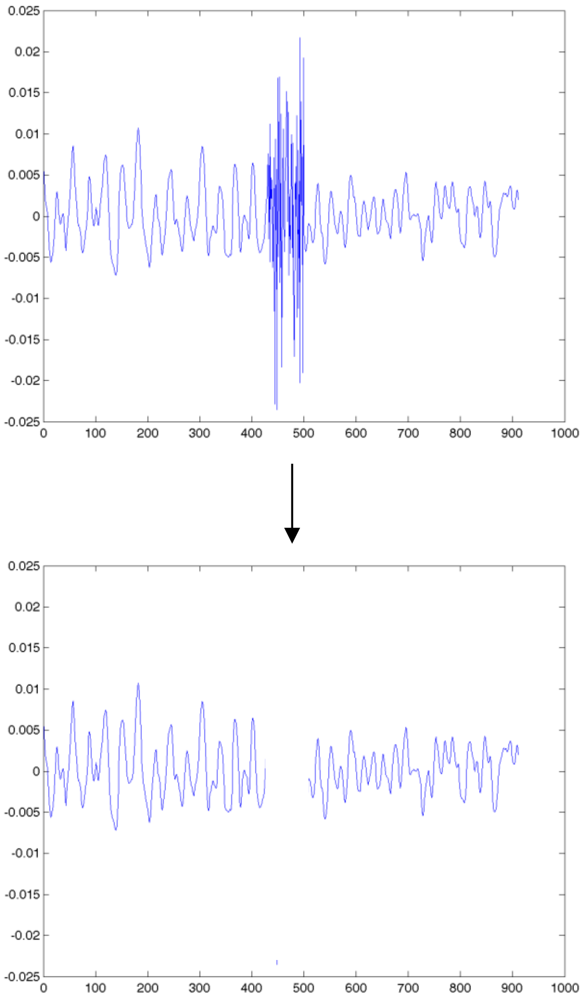
A Common Problem



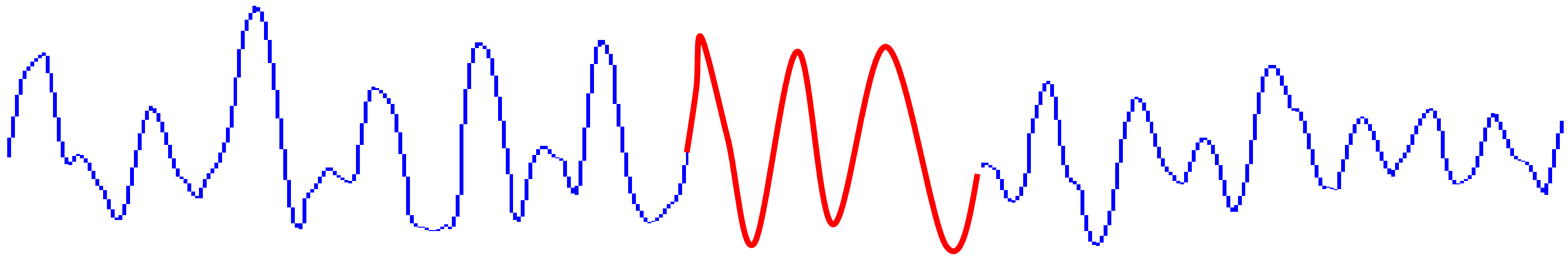
- Can you spot the glitches?

How to fix this problem?

- "Glitches" in audio
  - Must be detected
  - How?
- Then what?
- Glitches must be "fixed"
  - Delete the glitch
    - Results in a "hole"
  - Fill in the hole
  - How?

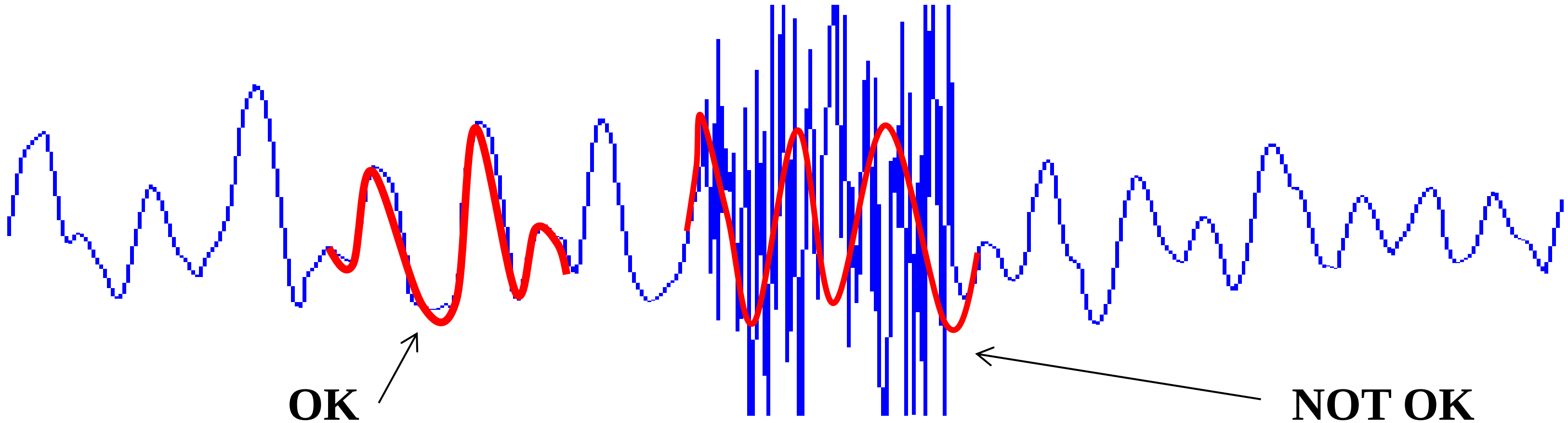


Interpolation..



- "Extend" the curve on the left to "predict" the values in the "blank" region
  - *Forward* prediction
- Extend the blue curve on the right leftwards to predict the blank region
  - *Backward* prediction
- How?
  - Regression analysis..

Detecting the Glitch



- Regression-based reconstruction can be done anywhere
- Reconstructed value will not match actual value
- Large error of reconstruction identifies glitches

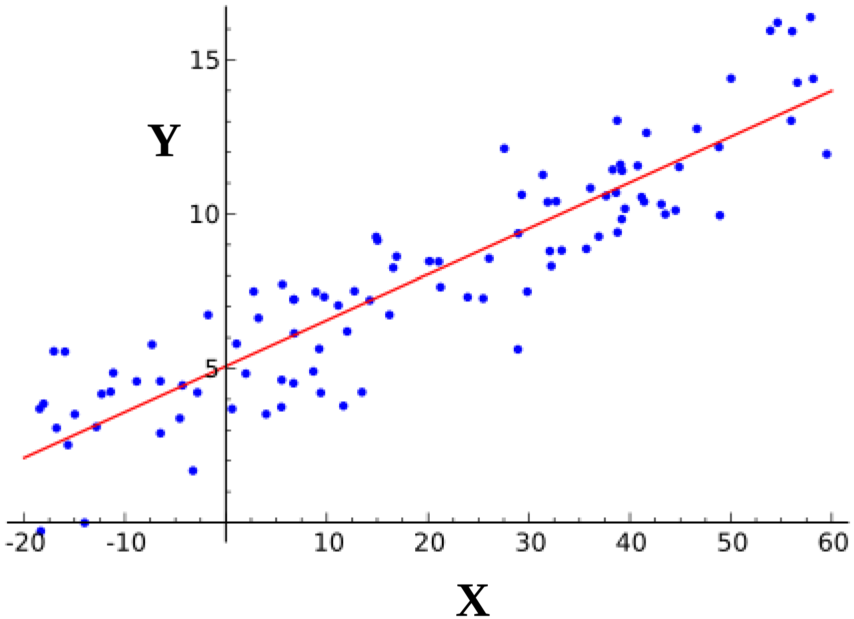
What is a regression

- Analyzing relationship between variables
- Expressed in many forms
- Wikipedia
  - Linear regression, Simple regression, Ordinary least squares, Polynomial regression, General linear model, Generalized linear model, Discrete choice, Logistic regression, Multinomial logit, Mixed logit, Probit, Multinomial probit, ….
- Generally a tool to ***predict*** variables

Regressions for prediction

- $\mathbf{y} = f(\mathbf{x}; \Theta) + e$
- Different possibilities
  - $\mathbf{y}$ is a scalar
    - $\mathbf{y}$ is real
    - $\mathbf{y}$ is categorical (classification)
  - $\mathbf{y}$ is a vector
  - $\mathbf{x}$ is a vector
    - $\mathbf{x}$ is a set of real valued variables
    - $\mathbf{x}$ is a set of categorical variables
    - $\mathbf{x}$ is a combination of the two
  - $f(.)$ is a linear or affine function
  - $f(.)$ is a non-linear function
  - $f(.)$ is a *time-series* model

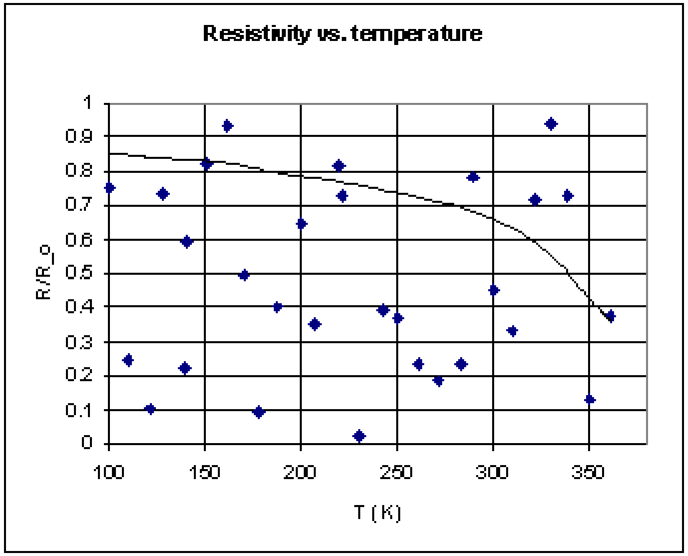
A *linear* regression



- Assumption: relationship between variables is linear
  - A linear *trend* may be found relating **x** and **y**
  - **y** = *dependent* variable
  - **x** = *explanatory* variable
  - Given **x**, **y** can be predicted as an affine function of **x**

An imaginary regression..

- http://pages.cs.wisc.edu/~kovar/hall.html
- Check this shit out (Fig. 1).

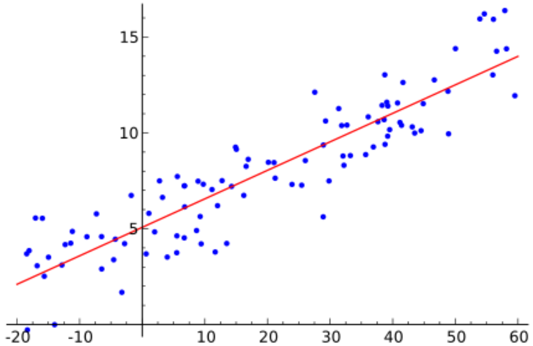


That's bonafide, 100%-real data, my friends. I took it myself over the course of two weeks. And this was not a leisurely two weeks, either; I busted my ass day and night in order to provide you with nothing but the best data possible. Now, let's look a bit more closely at this data, remembering that it is absolutely first-rate. Do you see the exponential dependence? I sure don't. I see a bunch of crap.

Christ, this was such a waste of my time.

Banking on my hopes that whoever grades this will just look at the pictures, I drew an exponential through my noise. I believe the apparent legitimacy is enhanced by the fact that I used a complicated computer program to make the fit. I understand this is the same process by which the top quark was discovered.

Linear Regressions



- $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{b} + \mathbf{e}$
  - $\mathbf{e}$ = prediction error
- Given a "training" set of $\{\mathbf{x}, \mathbf{y}\}$ values: estimate $\mathbf{A}$ and $\mathbf{b}$
  - $\mathbf{y}_1 = \mathbf{A}\mathbf{x}_1 + \mathbf{b} + \mathbf{e}_1$
  - $\mathbf{y}_2 = \mathbf{A}\mathbf{x}_2 + \mathbf{b} + \mathbf{e}_2$
  - $\mathbf{y}_3 = \mathbf{A}\mathbf{x}_3 + \mathbf{b} + \mathbf{e}_3$
  - …
- If $\mathbf{A}$ and $\mathbf{b}$ are well estimated, prediction error will be small

Linear Regression to a scalar

$$y_1 = \boldsymbol{a}^{\mathrm{T}}\mathbf{x}_1 + \mathrm{b} + \mathrm{e}_1$$
$$y_2 = \boldsymbol{a}^{\mathrm{T}}\mathbf{x}_2 + \mathrm{b} + \mathrm{e}_2$$
$$y_3 = \boldsymbol{a}^{\mathrm{T}}\mathbf{x}_3 + \mathrm{b} + \mathrm{e}_3$$

- Define:

$$\mathbf{y} = [y_1\ y_2\ y_3 \ldots] \qquad \mathbf{X} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 & \\ 1 & 1 & 1 & \ldots \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} \mathbf{a} \\ b \end{bmatrix}$$

$$\mathbf{e} = [e_1\ e_2\ e_3 \ldots]$$

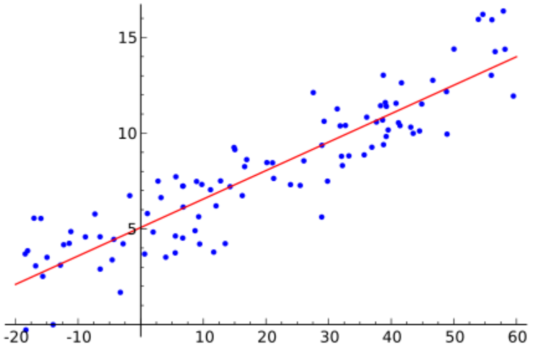
- Rewrite

$$\mathbf{y} = \mathbf{A}^T \mathbf{X} + \mathbf{e}$$

Learning the parameters

$$\mathbf{y} = \mathbf{A}^T \mathbf{X} + \mathbf{e}$$

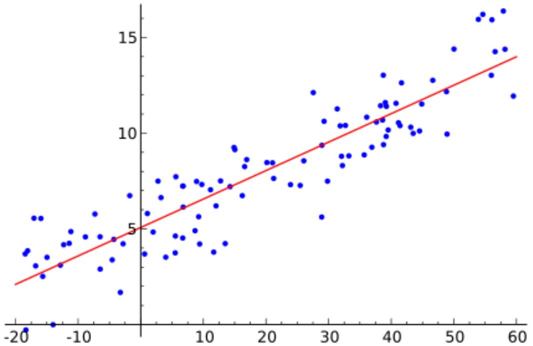
$$\hat{\mathbf{y}} = \mathbf{A}^T \mathbf{X}$$ **Assuming no error**



- Given training data: several $\mathbf{x}, \mathbf{y}$
- Can define a "divergence": $D(\mathbf{y}, \hat{\mathbf{y}})$
  - Measures how much $\hat{\mathbf{y}}$ differs from $\mathbf{y}$
  - Ideally, if the model is accurate this should be small
- Estimate $\mathbf{A}$, $\mathbf{b}$ to minimize $D(\mathbf{y}, \hat{\mathbf{y}})$

The prediction error as divergence

$$y_1 = \mathbf{a}^{\mathrm{T}}\mathbf{x_1} + b + e_1$$
$$y_2 = \mathbf{a}^{\mathrm{T}}\mathbf{x_2} + b + e_2$$
$$y_3 = \mathbf{a}^{\mathrm{T}}\mathbf{x_3} + b + e_3$$



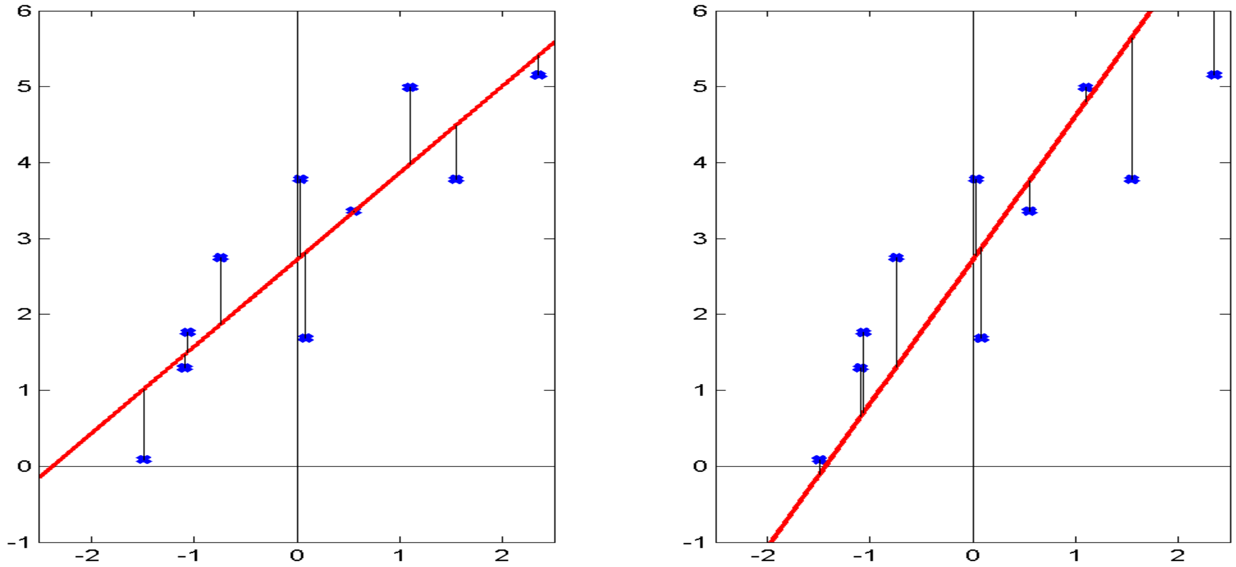
$$\mathbf{y} = \mathbf{A}^T\mathbf{X} + \mathbf{e} \quad = \hat{\mathbf{y}} + \mathbf{e}$$

$$\mathbf{D}(\mathbf{y}, \hat{\mathbf{y}}) = \mathbf{E} = e_1^2 + e_2^2 + e_3^2 + \ldots$$
$$= (y_1 - \mathbf{a}^T\mathbf{x}_1 - b)^2 + (y_2 - \mathbf{a}^T\mathbf{x}_2 - b)^2 + (y_3 - \mathbf{a}^T\mathbf{x}_3 - b)^2 + \ldots$$

$$\mathbf{E} = \left(\mathbf{y} - \mathbf{A}^T\mathbf{X}\right)\left(\mathbf{y} - \mathbf{A}^T\mathbf{X}\right)^T = \left\|\mathbf{y} - \mathbf{A}^T\mathbf{X}\right\|^2$$

- Define divergence as sum of the squared error in predicting **y**

Prediction error as divergence



- $y = \mathbf{a}^{\mathrm{T}}\mathbf{x} + e$
  - e = prediction error
  - Find the "slope" $\mathbf{a}$ such that the total squared length of the error lines is minimized

Solving a linear regression

$$\mathbf{y} = \mathbf{A}^T\mathbf{X} + \mathbf{e}$$

- Minimize squared error

$$\mathbf{E} = \|\mathbf{y} - \mathbf{X}^T\mathbf{A}\|^2 = (\mathbf{y} - \mathbf{A}^T\mathbf{X})(\mathbf{y} - \mathbf{A}^T\mathbf{X})^T$$
$$= \mathbf{y}\mathbf{y}^T + \mathbf{A}^T\mathbf{X}\mathbf{X}^T\mathbf{A} - 2\mathbf{y}\mathbf{X}^T\mathbf{A}$$

- Differentiating w.r.t $\mathbf{A}$ and equating to 0

$$d\mathbf{E} = \left(2\mathbf{A}^T\mathbf{X}\mathbf{X}^T - 2\mathbf{y}\mathbf{X}^T\right)d\mathbf{A} = 0$$

$$\mathbf{A}^T = \mathbf{y}\mathbf{X}^T\left(\mathbf{X}\mathbf{X}^T\right)^{-1} = \mathbf{y}\,pinv(\mathbf{X})$$

$$\mathbf{A} = \left(\mathbf{X}\mathbf{X}^T\right)^{-1}\mathbf{X}\mathbf{y}^T$$

Regression in multiple dimensions

$$\mathbf{y}_1 = \mathbf{A}^T\mathbf{x}_1 + \mathbf{b} + \mathbf{e}_1$$
$$\mathbf{y}_2 = \mathbf{A}^T\mathbf{x}_2 + \mathbf{b} + \mathbf{e}_2$$
$$\mathbf{y}_3 = \mathbf{A}^T\mathbf{x}_3 + \mathbf{b} + \mathbf{e}_3$$

$\mathbf{y}_i$ is a vector

y_{ij} = j^{th} component of vector $\mathbf{y_i}$

$\mathbf{a}_i$ = i^{th} column of $\mathbf{A}$

b_j = j^{th} component of $\mathbf{b}$

- Also called *multiple regression*
- Equivalent of saying:

$$\mathbf{y}_i = \mathbf{A}^T\mathbf{x_i} + \mathbf{b} + \mathbf{e}_i \quad \Rightarrow \quad \begin{aligned} y_{i1} &= \mathbf{a_1}^T\mathbf{x_i} + b_1 + e_{i1} \\ y_{i2} &= \mathbf{a_2}^T\mathbf{x_i} + b_2 + e_{i2} \\ y_{i3} &= \mathbf{a_3}^T\mathbf{x_i} + b_3 + e_{i3} \end{aligned}$$

- Fundamentally no different from N separate single regressions
  - But we can use the relationship between **y**s to our benefit

Multiple Regression

$$\mathbf{Y} = [\mathbf{y}_1\ \mathbf{y}_2\ \mathbf{y}_3 \ldots] \qquad \mathbf{X} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 & \ldots \\ 1 & 1 & 1 & \end{bmatrix} \qquad \hat{\mathbf{A}} = \begin{bmatrix} \mathbf{A} \\ \mathbf{b} \end{bmatrix}$$

$$\mathbf{E} = [\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3 \ldots]$$

Dx1 vector of ones

$$\mathbf{Y} = \hat{\mathbf{A}}^T \mathbf{X} + \mathbf{E}$$

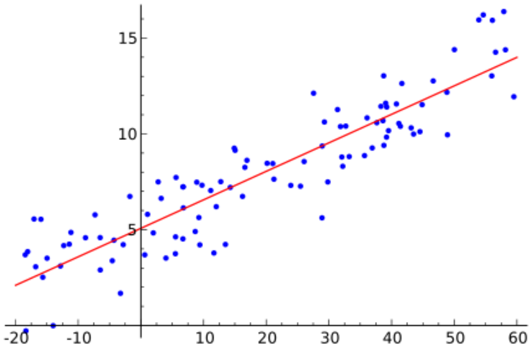
$$DIV = \sum_i \left\| \mathbf{y}_i - \hat{\mathbf{A}}^T \overline{\mathbf{x}}_i \right\|^2 = trace\left((\mathbf{Y} - \hat{\mathbf{A}}^T \mathbf{X})(\mathbf{Y} - \hat{\mathbf{A}}^T \mathbf{X})^T \right)$$

- Differentiating and equating to 0

$$d.Div = -2\left(\mathbf{Y} - \hat{\mathbf{A}}^T \mathbf{X}\right)\mathbf{X}^T d\hat{\mathbf{A}} = 0 \qquad \mathbf{Y}\mathbf{X}^T = \hat{\mathbf{A}}^T \mathbf{X}\mathbf{X}^T$$

$$\hat{\mathbf{A}}^T = \mathbf{Y}\mathbf{X}^T \left(\mathbf{X}\mathbf{X}^T\right)^{-1} = \mathbf{Y} pinv(\mathbf{X}) \qquad \hat{\mathbf{A}} = \left(\mathbf{X}\mathbf{X}^T\right)^{-1} \mathbf{X}\mathbf{Y}^T$$

A Different Perspective



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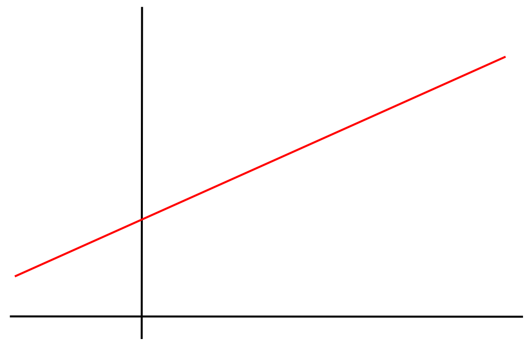

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- $\mathbf{y}$ is a noisy reading of $\mathbf{A}^T\mathbf{x}$

$$\mathbf{y} = \mathbf{A}^T\mathbf{x} + \mathbf{e}$$

- Error $\mathbf{e}$ is Gaussian

$$\mathbf{e} \sim N(0, \sigma^2\mathbf{I})$$

- Estimate $\mathbf{A}$ from $\mathbf{Y} = [\mathbf{y}_1 \ \ \mathbf{y}_2 \ldots \mathbf{y}_N] \quad \mathbf{X} = [\mathbf{x}_1 \ \ \mathbf{x}_2 \ldots \mathbf{x}_N]$

The *Likelihood* of the data

$$\mathbf{y} = \mathbf{A}^T\mathbf{x} + \mathbf{e} \qquad \mathbf{e} \sim N(0, \sigma^2\mathbf{I})$$

- Probability of observing a specific **y**, given **x**, for a particular matrix **A**

$$P(\mathbf{y} \mid \mathbf{x}; \mathbf{A}) = N(\mathbf{y}; \mathbf{A}^T\mathbf{x}, \sigma^2\mathbf{I})$$

- Probability of collection: $\mathbf{Y} = [\mathbf{y}_1 \ \mathbf{y}_2 \ldots \mathbf{y}_N]$ $\mathbf{X} = [\mathbf{x}_1 \ \mathbf{x}_2 \ldots \mathbf{x}_N]$

$$P(\mathbf{Y} \mid \mathbf{X}; \mathbf{A}) = \prod_i N(\mathbf{y}_i; \mathbf{A}^T\mathbf{x}_i, \sigma^2\mathbf{I})$$

- Assuming IID for convenience (not necessary)

A Maximum Likelihood Estimate

$$\mathbf{y} = \mathbf{A}^T\mathbf{x} + \mathbf{e} \qquad \mathbf{e} \sim N(0, \sigma^2\mathbf{I}) \qquad \mathbf{Y} = [\mathbf{y}_1 \;\; \mathbf{y}_2 \ldots \mathbf{y}_N] \quad \mathbf{X} = [\mathbf{x}_1 \;\; \mathbf{x}_2 \ldots \mathbf{x}_N]$$

$$P(\mathbf{Y} \mid \mathbf{X}) = \prod_i \frac{1}{\sqrt{(2\pi\sigma^2)^D}} \exp\left(\frac{-1}{2\sigma^2} \left\| \mathbf{y}_i - \mathbf{A}^T\mathbf{x}_i \right\|^2 \right)$$

$$\log P(\mathbf{Y} \mid \mathbf{X}; \mathbf{A}) = C - \sum_i \frac{1}{2\sigma^2} \left\| \mathbf{y}_i - \mathbf{A}^T\mathbf{x}_i \right\|^2$$

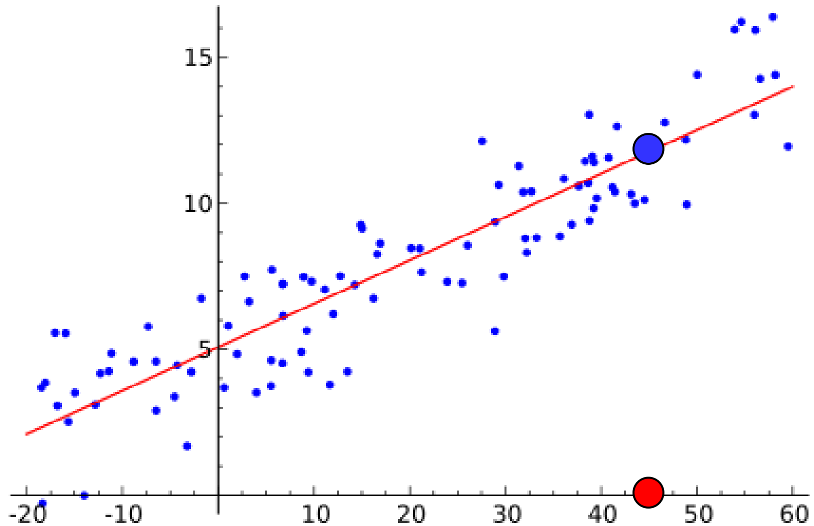
$$= C - \frac{1}{2\sigma^2} trace\left((\mathbf{Y} - \mathbf{A}^T\mathbf{X})(\mathbf{Y} - \mathbf{A}^T\mathbf{X})^T \right)$$

- Maximizing the log probability is identical to minimizing the trace
  - Identical to the least squares solution

$$\mathbf{A}^T = \mathbf{Y}\mathbf{X}^T\left(\mathbf{X}\mathbf{X}^T\right)^{-1} = \mathbf{Y}pinv(\mathbf{X})$$

$$\mathbf{A} = \left(\mathbf{X}\mathbf{X}^T\right)^{-1}\mathbf{X}\mathbf{Y}^T$$

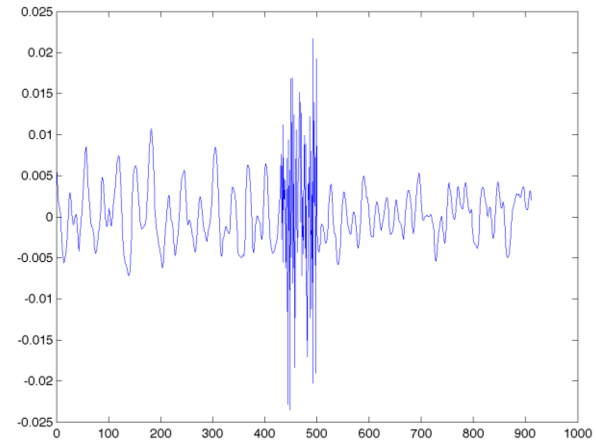
Predicting an output



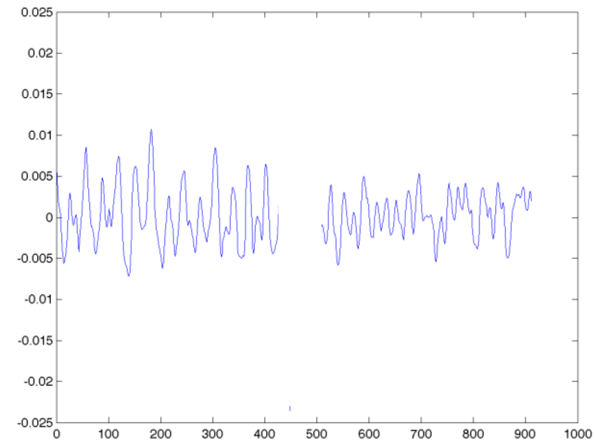
- From a collection of training data, have learned $\mathbf{A}$
- Given $\mathbf{x}$ for a new instance, but not $\mathbf{y}$, what is $\mathbf{y}$?
- Simple solution: $\hat{\mathbf{y}} = \mathbf{A}^T \mathbf{X}$

Applying it to our problem

- Prediction by regression



- Forward regression
- $x_t = a_1 x_{t-1} + a_2 x_{t-2} \ldots a_k x_{t-k} + e$

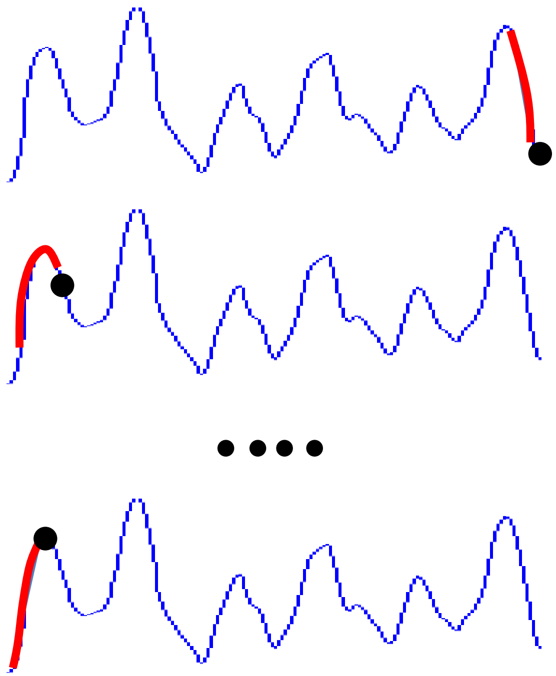


- Backward regression
- $x_t = b_1 x_{t+1} + b_2 x_{t+2} \ldots b_k x_{t+k} +$

Applying it to our problem

- Forward prediction

$$\begin{bmatrix} x_t \\ x_{t-1} \\ .. \\ x_{K+1} \end{bmatrix} = \begin{bmatrix} x_{t-1} & x_{t-2} & .. & x_{t-K} \\ x_{t-2} & x_{t-3} & .. & x_{t-K-1} \\ .. & .. & .. & .. \\ x_K & x_{K-1} & .. & x_1 \end{bmatrix} \mathbf{a}_t + \begin{bmatrix} e_t \\ e_{t-1} \\ .. \\ e_{K+1} \end{bmatrix}$$



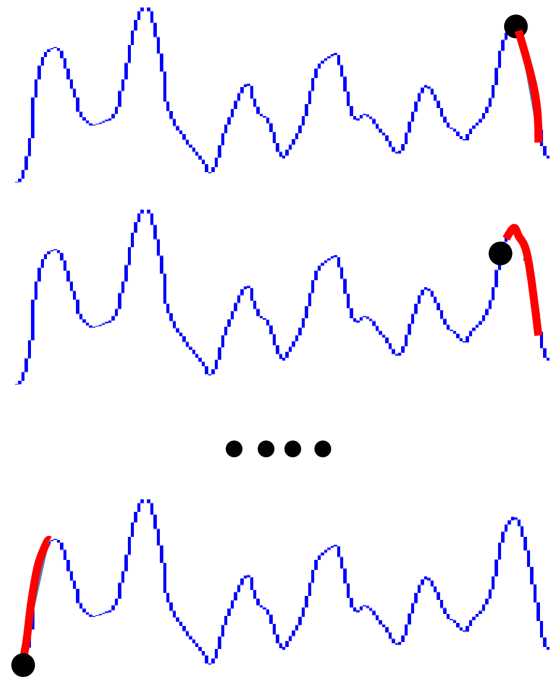
$$\mathbf{x} = \mathbf{X}\mathbf{a}_t + \mathbf{e}$$

$$pinv(\mathbf{X})\mathbf{x} = \mathbf{a}_t$$

Applying it to our problem

- Backward prediction

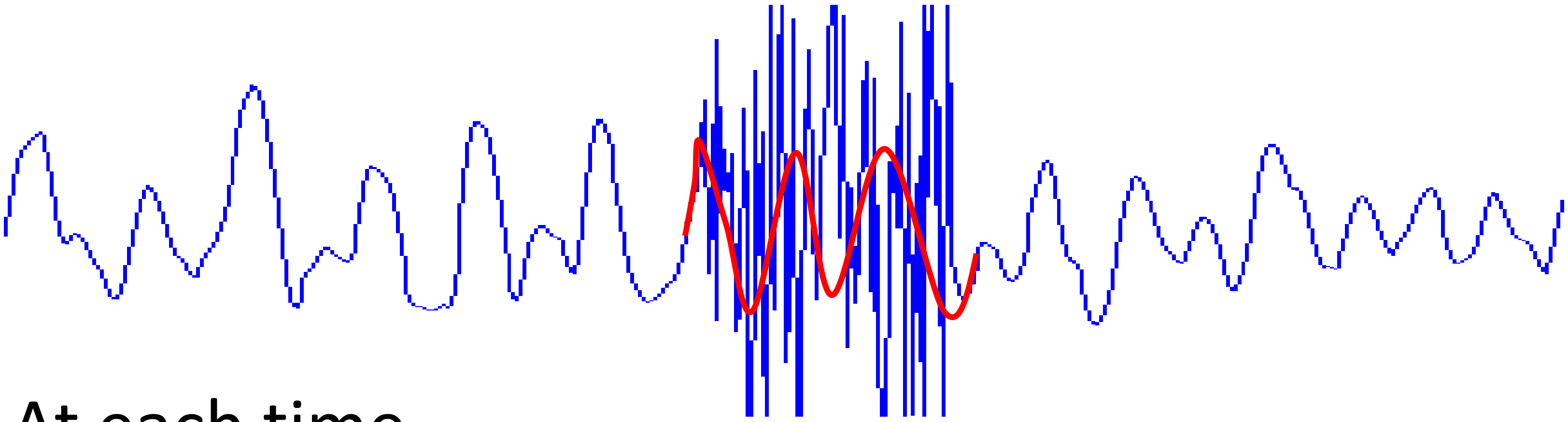
$$\begin{bmatrix} x_{t-K-1} \\ x_{t-K-2} \\ .. \\ x_1 \end{bmatrix} = \begin{bmatrix} x_t & x_{t-1} & .. & x_{t-K} \\ x_{t-1} & x_{t-2} & .. & x_{t-K-1} \\ .. & .. & .. & .. \\ x_{K+1} & x_K & .. & x_2 \end{bmatrix} \mathbf{b}_t + \begin{bmatrix} e_{t-K-1} \\ e_{t-K-2} \\ .. \\ e_1 \end{bmatrix}$$



$$\bar{\mathbf{x}} = \bar{\mathbf{X}}\mathbf{b}_t + \mathbf{e}$$

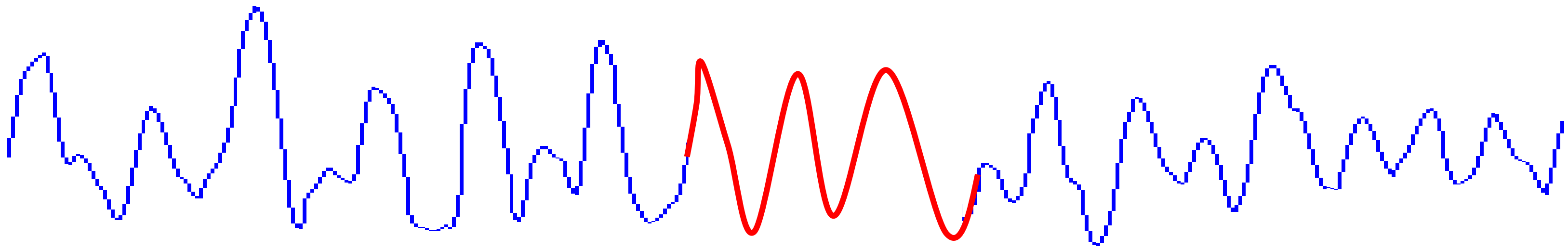
$$pinv(\bar{\mathbf{X}})\bar{\mathbf{x}} = \mathbf{b}_t$$

Finding the burst



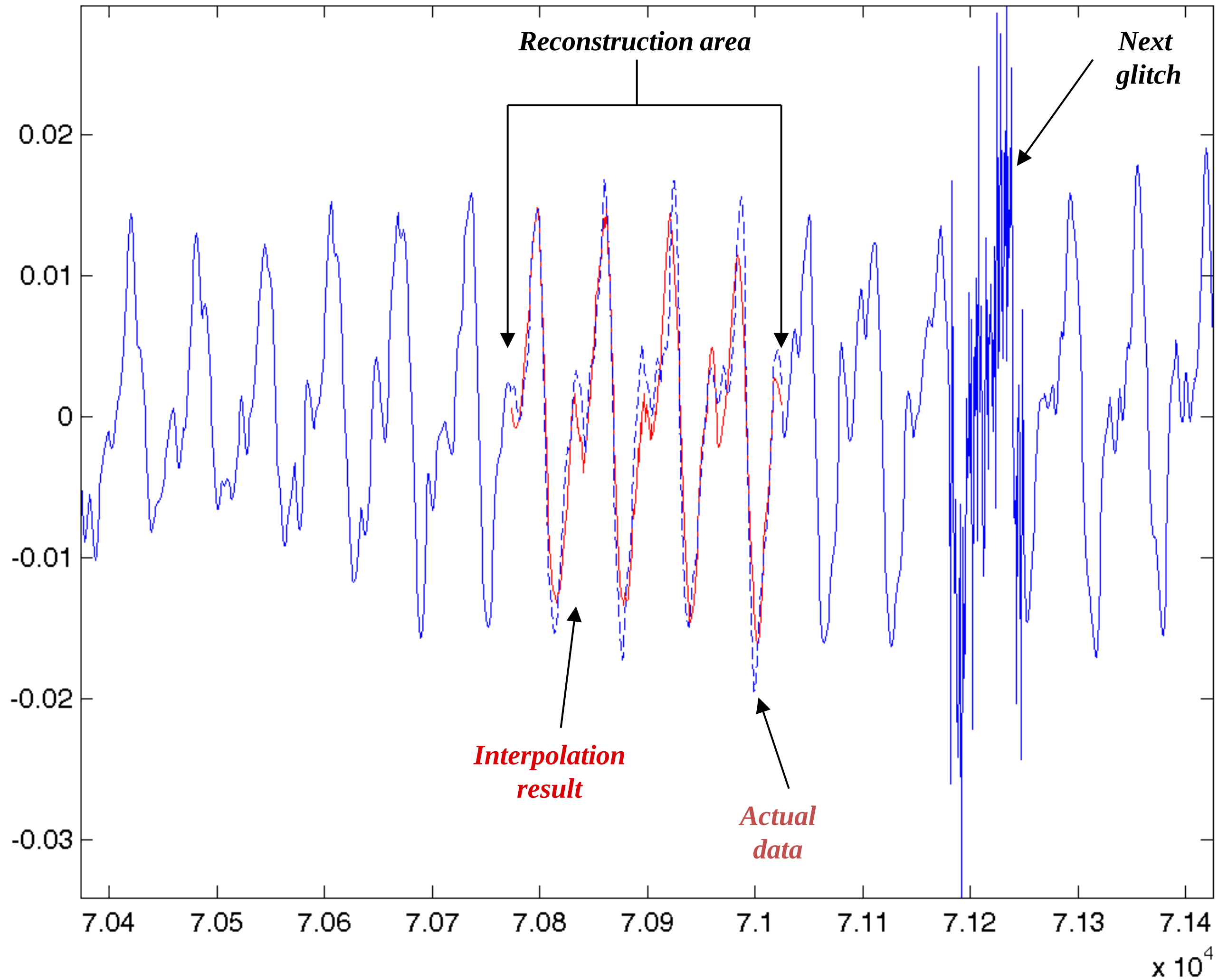
- At each time
  - Learn a "forward" predictor $\mathbf{a}_t$
  - At each time, predict next sample $x_t^{est} = \Sigma_i \, a_{t,k} x_{t-k}$
  - Compute error: $ferr_t = |x_t - x_t^{est}|^2$
  - Learn a "backward" predict and compute backward error
    - $berr_t$
  - Compute average prediction error over window, threshold

Filling the hole



- Learn "forward" predictor at left edge of "hole"
  - For each missing sample
  - At each time, predict next sample $x_t^{est} = \Sigma_i \, a_{t,k} x_{t-k}$
    - Use estimated samples if real samples are not available
- Learn "backward" predictor at left edge of "hole"
  - For each missing sample
  - At each time, predict next sample $x_t^{est} = \Sigma_i \, b_{t,k} x_{t+k}$
    - Use estimated samples if real samples are not available
- Average forward and backward predictions

Reconstruction zoom in



Distorted signal

Recovered signal

Incrementally learning the regression

$$\mathbf{A} = \left(\mathbf{X}\mathbf{X}^T\right)^{-1}\mathbf{X}\mathbf{Y}^T$$

Requires knowledge of *all* (x,y) pairs

- Can we learn **A** incrementally instead?
  - As data comes in?

- The Widrow Hoff rule

Scalar prediction version

$$\mathbf{a}^{t+1} = \mathbf{a}^t + \eta\left(y_t - \hat{y}_t\right)\mathbf{x}_t \qquad \hat{y}_t = \left(\mathbf{a}^t\right)^T \mathbf{x}_t$$

error

- Note the structure
  - Can also be done in batch mode!

Predicting a value

$$\mathbf{A} = \left(\mathbf{X}\mathbf{X}^T\right)^{-1}\mathbf{X}\mathbf{Y}^T$$

$$\hat{\mathbf{y}} = \mathbf{A}^T\mathbf{x} = \mathbf{Y}\mathbf{X}^T\left(\mathbf{X}\mathbf{X}^T\right)^{-1}\mathbf{x}$$

- What are we doing exactly?
  - For the explanation we are assuming no “$\mathbf{b}$” ($\mathbf{X}$ is 0 mean)
  - Explanation generalizes easily even otherwise

$$\mathbf{C} = \mathbf{X}\mathbf{X}^T$$

- Let $\hat{\mathbf{x}} = \mathbf{C}^{-\frac{1}{2}}\mathbf{x}$ and $\hat{\mathbf{X}} = \mathbf{C}^{-\frac{1}{2}}\mathbf{X}$
  - Whitening $\mathbf{x}$
  - $N^{-0.5}\,\mathbf{C}^{-0.5}$ is the *whitening* matrix for $\mathbf{x}$

$$\hat{\mathbf{y}} = \mathbf{Y}\mathbf{X}^T\mathbf{C}^{-\frac{1}{2}}\mathbf{C}^{-\frac{1}{2}}\mathbf{x} = \mathbf{Y}\hat{\mathbf{X}}^T\hat{\mathbf{x}}_i$$

Predicting a value

$$\hat{\mathbf{y}} = \mathbf{Y}\hat{\mathbf{X}}^T\hat{\mathbf{x}} = \sum_i \hat{\mathbf{x}}_i^T \hat{\mathbf{x}} \mathbf{y}_i$$

$$\hat{\mathbf{y}} = \mathbf{Y}\hat{\mathbf{X}}^T\hat{\mathbf{x}} = \frac{1}{N}\begin{bmatrix}\mathbf{y}_1 & \cdots & \mathbf{y}_N\end{bmatrix}\begin{bmatrix}\hat{\mathbf{x}}_1^T \\ \vdots \\ \hat{\mathbf{x}}_N^T\end{bmatrix}\hat{\mathbf{x}} = \sum_i \mathbf{y}_i \left(\hat{\mathbf{x}}_i^T \hat{\mathbf{x}}\right)$$

- What are we doing exactly?

Predicting a value

$$\hat{\mathbf{y}} = \sum_i \mathbf{y}_i \left(\hat{\mathbf{x}}_i^T \hat{\mathbf{x}} \right)$$

- Given training instances $(\mathbf{x}_\mathrm{i}, \mathbf{y}_\mathrm{i})$ for $i = 1..\mathrm{N}$, estimate $\mathbf{y}$ for a new test instance of $\mathbf{x}$ with unknown $\mathbf{y}$:
- $\mathbf{y}$ is simply a *weighted sum of the* $\mathbf{y}_i$ *instances from the training data*
- The weight of any $\mathbf{y}_i$ is simply the inner product between its corresponding $\mathbf{x}_i$ and the new $\mathbf{x}$
  - With due whitening and scaling..

What are we doing: A different perspective

$$\hat{\mathbf{y}} = \mathbf{A}^T\mathbf{x} = \mathbf{Y}\mathbf{X}^T\left(\mathbf{X}\mathbf{X}^T\right)^{-1}\mathbf{x}$$

- Assumes $\mathbf{X}\mathbf{X}^{\mathrm{T}}$ is invertible
- What if it is not
  - Dimensionality of X is greater than number of observations?
  - Underdetermined
- In this case $\mathbf{X}^{\mathrm{T}}\mathbf{X}$ will generally be invertible

$$\mathbf{A} = \mathbf{X}\left(\mathbf{X}^T\mathbf{X}\right)^{-1}\mathbf{Y}^T \qquad \hat{\mathbf{y}} = \mathbf{Y}\left(\mathbf{X}^T\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{x}$$

High-dimensional regression

$$\hat{\mathbf{y}} = \mathbf{Y}\left(\mathbf{X}^T\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{x}$$

- $\mathbf{X}^{\mathrm{T}}\mathbf{X}$ is the "Gram Matrix"

$$\mathbf{G} = \begin{bmatrix} \mathbf{x}_1^T\mathbf{x}_1 & \mathbf{x}_1^T\mathbf{x}_2 & \cdots & \mathbf{x}_1^T\mathbf{x}_N \\ \mathbf{x}_2^T\mathbf{x}_1 & \mathbf{x}_2^T\mathbf{x}_2 & \cdots & \mathbf{x}_2^T\mathbf{x}_N \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{x}_N^T\mathbf{x}_1 & \mathbf{x}_N^T\mathbf{x}_2 & \cdots & \mathbf{x}_N^T\mathbf{x}_N \end{bmatrix}$$

$$\hat{\mathbf{y}} = \mathbf{Y}\mathbf{G}^{-1}\mathbf{X}^T\mathbf{x}$$

High-dimensional regression

$$\hat{\mathbf{y}} = \mathbf{Y}\mathbf{G}^{-1}\mathbf{X}^T\mathbf{x}$$

- Normalize $\mathbf{Y}$ by the inverse of the gram matrix

$$\dddot{\mathbf{Y}} = \mathbf{Y}\mathbf{G}^{-1}$$

- Working our way down..

$$\hat{\mathbf{y}} = \dddot{\mathbf{Y}}\mathbf{X}^T\mathbf{x}$$

$$\hat{\mathbf{y}} = \sum_i \dddot{\mathbf{y}}_i \mathbf{x}_i^T \mathbf{x}$$

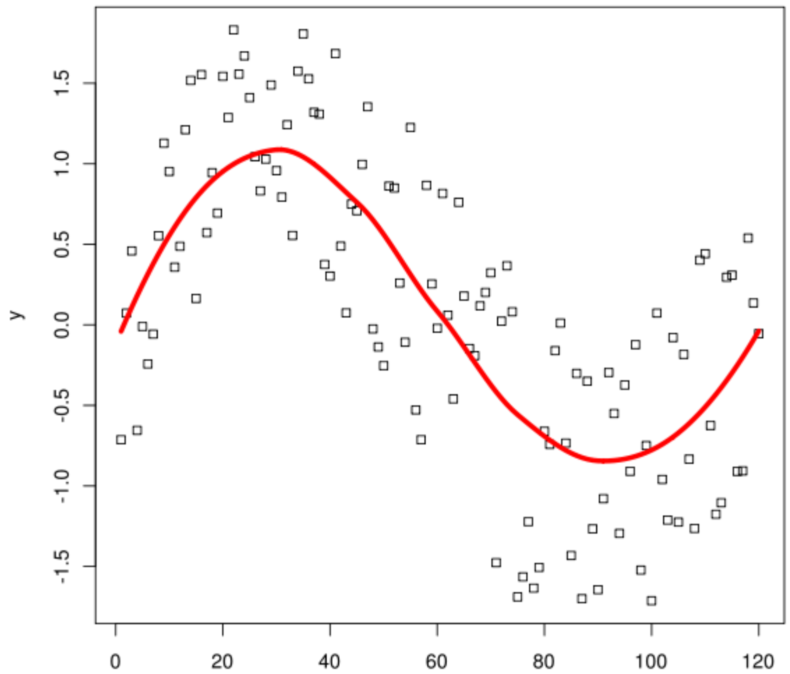
Linear Regression in High-dimensional Spaces

$$\hat{\mathbf{y}} = \sum_i \dddot{\mathbf{y}}_i \mathbf{x}_i^T \mathbf{x}$$

$$\dddot{\mathbf{Y}} = \mathbf{Y}\mathbf{G}^{-1}$$

- Given training instances $(\mathbf{x}_i, \mathbf{y}_i)$ for $i = 1..N$, estimate $\mathbf{y}$ for a new test instance of $\mathbf{x}$ with unknown $\mathbf{y}$:
- $\mathbf{y}$ is simply a *weighted sum of the normalized* $\mathbf{y}_i$ *instances from the training data*
  - The normalization is done via the Gram Matrix
- The weight of any $\mathbf{y}_i$ is simply the inner product between its corresponding $\mathbf{x}_i$ and the new $\mathbf{x}$

Relationships are not always linear



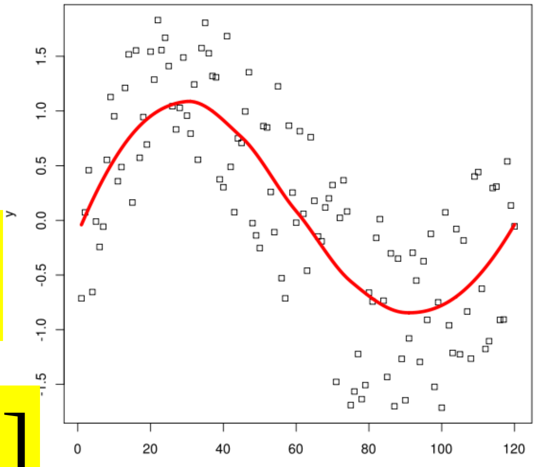
- How do we model these?
- Multiple solutions

Non-linear regression

- y = A$\boldsymbol{\varphi}$(**x**)+e

$$\mathbf{x} \rightarrow \boldsymbol{\varphi}(\mathbf{x}) = [\phi_1(\mathbf{x}) \;\; \phi_2(\mathbf{x}) \ldots \phi_N(\mathbf{x})]$$

$$\mathbf{X} \rightarrow \boldsymbol{\Phi}(\mathbf{X}) = [\boldsymbol{\varphi}(\mathbf{x}_1) \;\; \boldsymbol{\varphi}(\mathbf{x}_2) \ldots \boldsymbol{\varphi}(\mathbf{x}_K)]$$



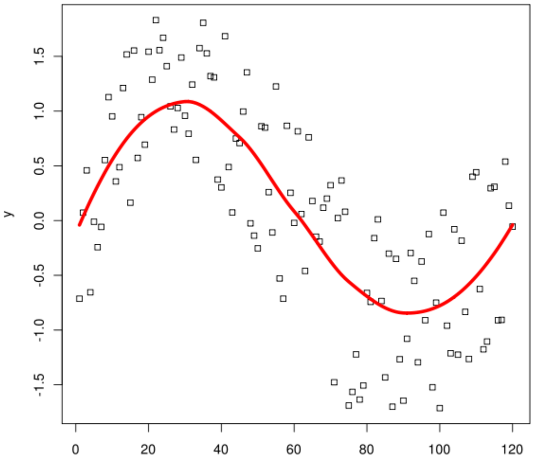
- $\mathbf{Y} = \mathbf{A}\boldsymbol{\Phi}(\mathbf{X}) + \mathbf{e}$
- Replace **X** with **Φ**(**X**) in earlier equations for solution

$$\mathbf{A} = \left(\boldsymbol{\Phi}(\mathbf{X})\boldsymbol{\Phi}(\mathbf{X})^T\right)^{-1}\boldsymbol{\Phi}(\mathbf{X})\mathbf{Y}^T$$

Problem

- $\mathbf{Y} = \mathbf{A}\Phi(\mathbf{X})+\mathbf{e}$
- Replace $\mathbf{X}$ with $\Phi(\mathbf{X})$ in earlier equations for solution

$$\mathbf{A} = \left(\Phi(\mathbf{X})\Phi(\mathbf{X})^T\right)^{-1}\Phi(\mathbf{X})\mathbf{Y}^T$$



- $\Phi(\mathbf{X})$ may be in a very high-dimensional space
- The high-dimensional space (or the transform $\Phi(\mathbf{X})$) may be unknown..

The regression is in high dimensions

- *Linear regression*: $$\hat{\mathbf{y}} = \sum_i \dddot{\mathbf{y}}_i \mathbf{x}_i^T \mathbf{x}$$ $$\dddot{\mathbf{Y}} = \mathbf{Y}\mathbf{G}^{-1}$$

- High-dimensional regression

$$\mathbf{G} = \begin{bmatrix} \Phi(\mathbf{x}_1)^T\Phi(\mathbf{x}_1) & \Phi(\mathbf{x}_2)^T\Phi(\mathbf{x}_2) & \cdots & \Phi(\mathbf{x}_1)^T\Phi(\mathbf{x}_N) \\ \Phi(\mathbf{x}_2)^T\Phi(\mathbf{x}_1) & \Phi(\mathbf{x}_2)^T\Phi(\mathbf{x}_2) & \cdots & \Phi(\mathbf{x}_2)^T\Phi(\mathbf{x}_N) \\ \vdots & \vdots & \ddots & \vdots \\ \Phi(\mathbf{x}_1)^T\Phi(\mathbf{x}_1) & \Phi(\mathbf{x}_N)^T\Phi(\mathbf{x}_2) & \cdots & \Phi(\mathbf{x}_N)^T\Phi(\mathbf{x}_N) \end{bmatrix}$$

$$\dddot{\mathbf{Y}} = \mathbf{Y}\mathbf{G}^{-1}$$

$$\hat{\mathbf{y}} = \sum_i \dddot{\mathbf{y}}_i \Phi(\mathbf{x}_i)^T \Phi(\mathbf{x})$$

Doing it with Kernels

- *High-dimensional regression with Kernels*:

$$K(\mathbf{x},\mathbf{y}) = \Phi(\mathbf{x})^T \Phi(\mathbf{y})$$

$$\mathbf{G} = \begin{bmatrix} K(\mathbf{x}_1,\mathbf{x}_1) & K(\mathbf{x}_1,\mathbf{x}_1) & \cdots & K(\mathbf{x}_1,\mathbf{x}_N) \\ K(\mathbf{x}_2,\mathbf{x}_1) & K(\mathbf{x}_2,\mathbf{x}_2) & \cdots & K(\mathbf{x}_2,\mathbf{x}_N) \\ \vdots & \vdots & \ddots & \vdots \\ K(\mathbf{x}_N,\mathbf{x}_1) & K(\mathbf{x}_N,\mathbf{x}_2) & \cdots & K(\mathbf{x}_N,\mathbf{x}_N) \end{bmatrix}$$

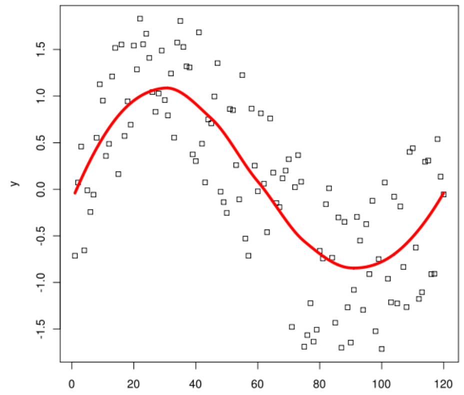
- Regression in Kernel Hilbert Space..

$$\dddot{\mathbf{Y}} = \mathbf{Y}\mathbf{G}^{-1}$$

$$\hat{\mathbf{y}} = \sum_i \dddot{\mathbf{y}}_i K(\mathbf{x}_i,\mathbf{x})$$

A different way of finding nonlinear relationships: Locally linear regression

- Previous discussion: Regression parameters are optimized over the entire training set



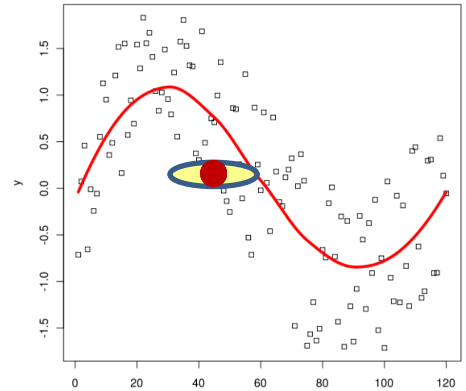
- Minimize

$$\mathbf{E} = \sum_{all\ i} \left\| \mathbf{y}_i - \mathbf{A}^T \mathbf{x}_i - \mathbf{b} \right\|^2$$

- Single global regression is estimated and applied to all future $\mathbf{x}$
- Alternative: *Local regression*
- *Learn a regression that is specific to* $\mathbf{x}$

Being non-committal: Local Regression

- Estimate the regression to be applied to any $\mathbf{x}$ using training instances near $\mathbf{x}$



$$\mathbf{E} = \sum_{\mathbf{x}_j \in neighborhood(\mathbf{x})} \left\| \mathbf{y}_i - \mathbf{A}^T \mathbf{x}_i - \mathbf{b} \right\|^2$$

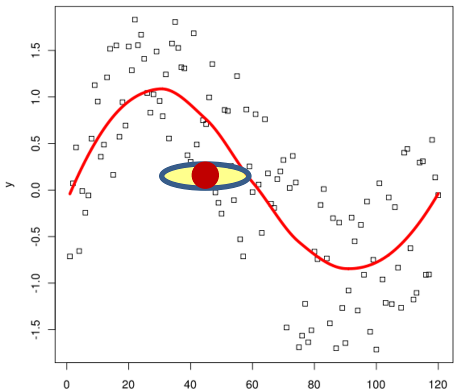
- The resultant regression has the form

$$\mathbf{y} = \sum_{\mathbf{x}_j \in neighborhood(\mathbf{x})} d(\mathbf{x}, \mathbf{x}_j)\mathbf{y}_j + \mathbf{e}$$

  – Note : this regression is specific to $\mathbf{x}$
    - A separate regression must be learned for every $\mathbf{x}$

Local Regression

$$\mathbf{y} = \sum_{\mathbf{x}_j \in neighborhood(\mathbf{x})} d(\mathbf{x}, \mathbf{x}_j)\mathbf{y}_j + \mathbf{e}$$

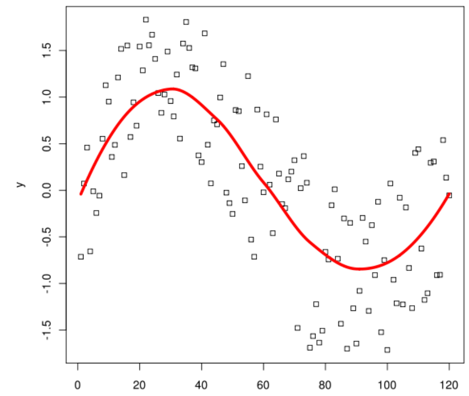


- But what is d()?
  - For linear regression d() is an inner product
- More generic form: Choose d() as a function of the *distance between* $\mathbf{x}$ *and* $\mathbf{x}_j$
- If d() falls off rapidly with $|\mathbf{x}$ *and* $\mathbf{x}_j|$ the "neighbhorhood" requirement can be relaxed

$$\mathbf{y} = \sum_{all} d(\mathbf{x}, \mathbf{x}_j)\mathbf{y}_j + \mathbf{e}$$

Kernel Regression: d() = K()

$$\hat{\mathbf{y}} = \frac{\sum_i K_h(\mathbf{x} - \mathbf{x}_i)\mathbf{y}_i}{\sum_i K_h(\mathbf{x} - \mathbf{x}_i)}$$



- Typical Kernel functions: Gaussian, Laplacian, other density functions
  - Must fall off rapidly with increasing distance between $\mathbf{x}$ and $\mathbf{x}_j$
- Regression is *local* to every $\mathbf{x}$: Local regression
- Actually a non-parametric MAP estimator of $\mathbf{y}$
  - But first.. MAP estimators..

Map Estimators

- MAP (*Maximum A Posteriori*): Find a "best guess" for $\mathbf{y}$ (statistically), given known $\mathbf{x}$

$$\mathbf{y} = argmax_{\,Y}\, P(\mathrm{Y}|\mathbf{x})$$

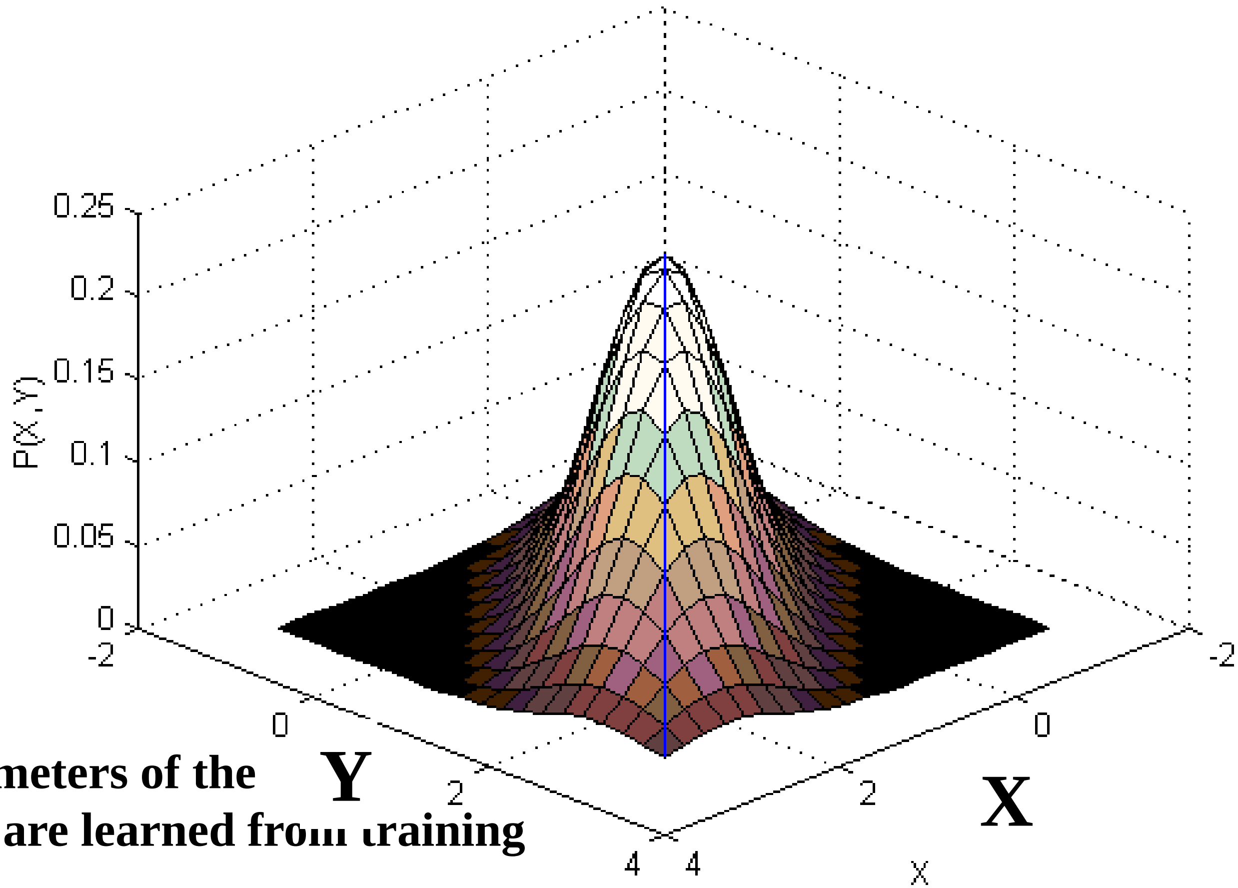
- ML (*Maximum Likelihood*): Find that value of $\mathbf{y}$ for which the statistical best guess of $\mathbf{x}$ would have been the observed $\mathbf{x}$

$$\mathbf{y} = argmax_{\,Y}\, P(\mathbf{x}|\mathrm{Y})$$

- MAP is simpler to visualize

MAP estimation: Gaussian PDF

Assume X and Y are jointly Gaussian



The parameters of the Gaussian are learned from training data

Learning the parameters of the Gaussian

$$\mathbf{z} = \begin{bmatrix} \mathbf{y} \\ \mathbf{x} \end{bmatrix}$$

$$\mu_{\mathbf{z}} = \frac{1}{N}\sum_{i=1}^{N} \mathbf{z}_i$$

$$C_{\mathbf{z}} = \frac{1}{N}\sum_{i=1}^{N} (\mathbf{z}_i - \mu_{\mathbf{z}})(\mathbf{z}_i - \mu_{\mathbf{z}})^T$$

$$\mu_{\mathbf{z}} = \begin{bmatrix} \mu_{\mathbf{y}} \\ \mu_{\mathbf{x}} \end{bmatrix}$$

$$C_{\mathbf{z}} = \begin{bmatrix} C_{XX} & C_{XY} \\ C_{YX} & C_{YY} \end{bmatrix}$$

Learning the parameters of the Gaussian

$$\mathbf{z} = \begin{bmatrix} \mathbf{y} \\ \mathbf{x} \end{bmatrix}$$

$$\mu_{\mathbf{z}} = \frac{1}{N}\sum_{i=1}^{N}\mathbf{z}_i$$

$$C_{\mathbf{z}} = \frac{1}{N}\sum_{i=1}^{N}(\mathbf{z}_i - \mu_{\mathbf{z}})(\mathbf{z}_i - \mu_{\mathbf{z}})^T$$

$$\mu_{\mathbf{z}} = \begin{bmatrix} \mu_{\mathbf{y}} \\ \mu_{\mathbf{x}} \end{bmatrix}$$

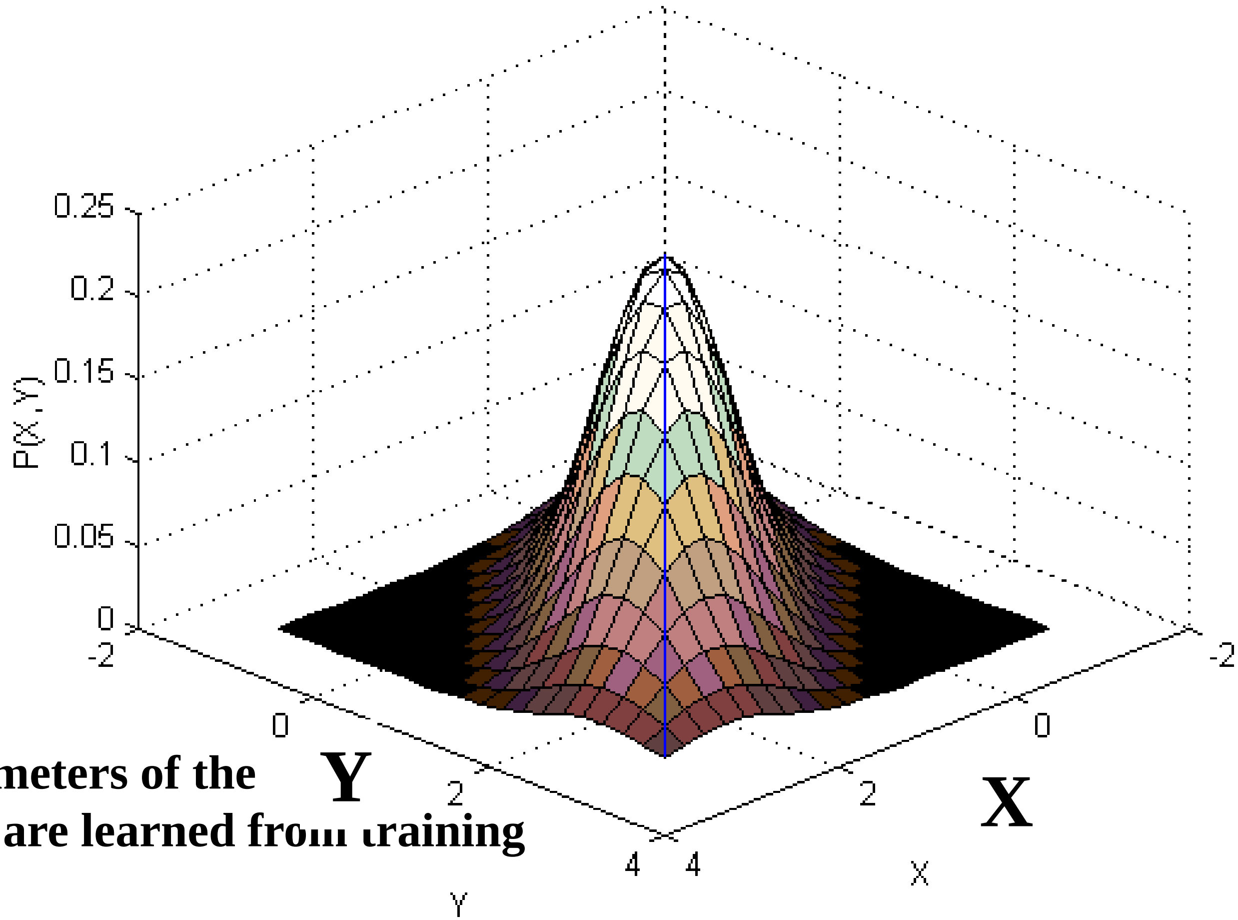
$$C_{\mathbf{z}} = \begin{bmatrix} C_{XX} & C_{XY} \\ C_{YX} & C_{YY} \end{bmatrix}$$

$$\mu_{\mathbf{x}} = \frac{1}{N}\sum_{i=1}^{N}\mathbf{x}_i$$

$$C_{XY} = \frac{1}{N}\sum_{i=1}^{N}(\mathbf{x}_i - \mu_{\mathbf{x}})(\mathbf{y}_i - \mu_{\mathbf{y}})^T$$

MAP estimation: Gaussian PDF

Assume X and Y are jointly Gaussian

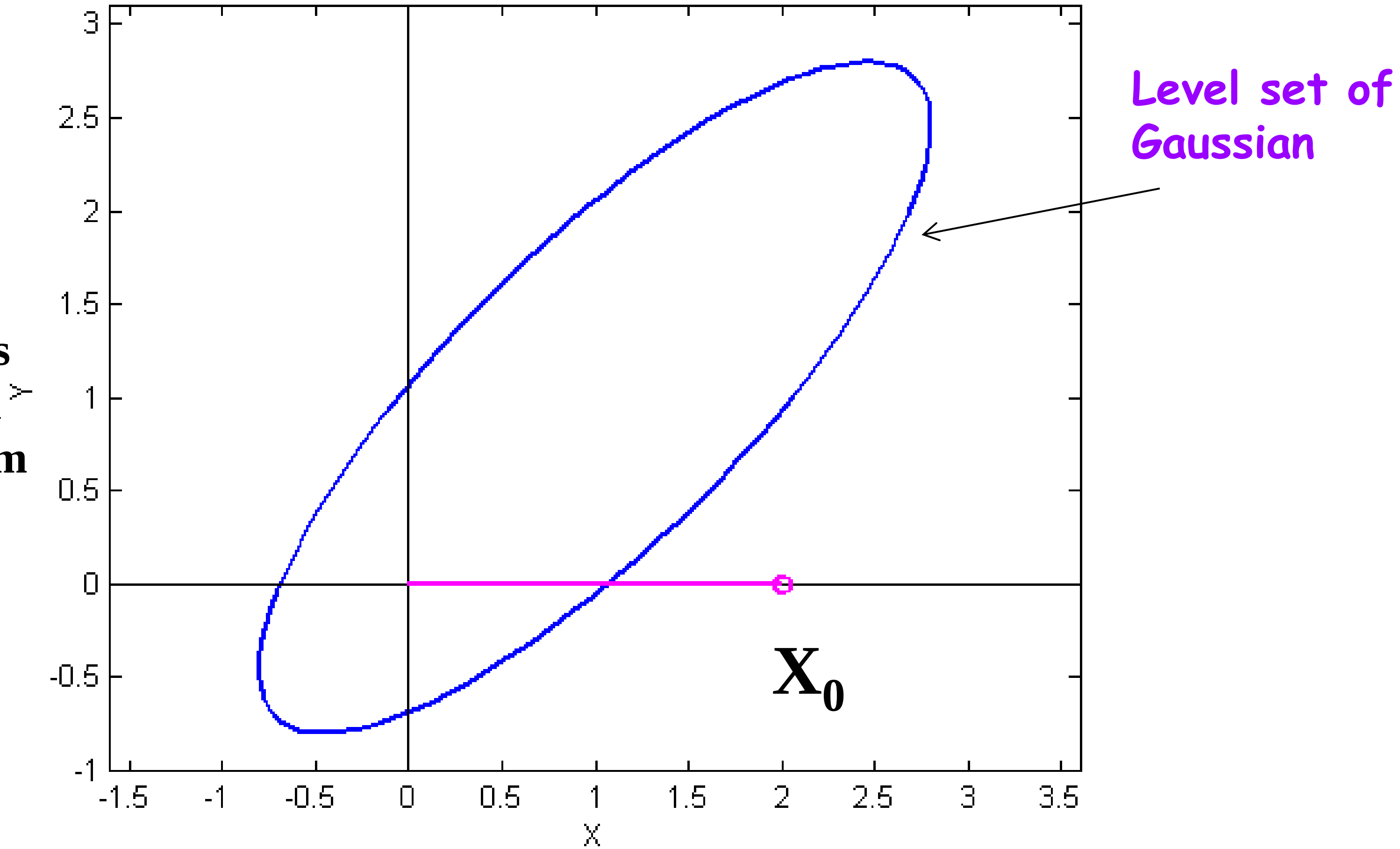


The parameters of the Gaussian are learned from training data

MAP Estimator for Gaussian RV

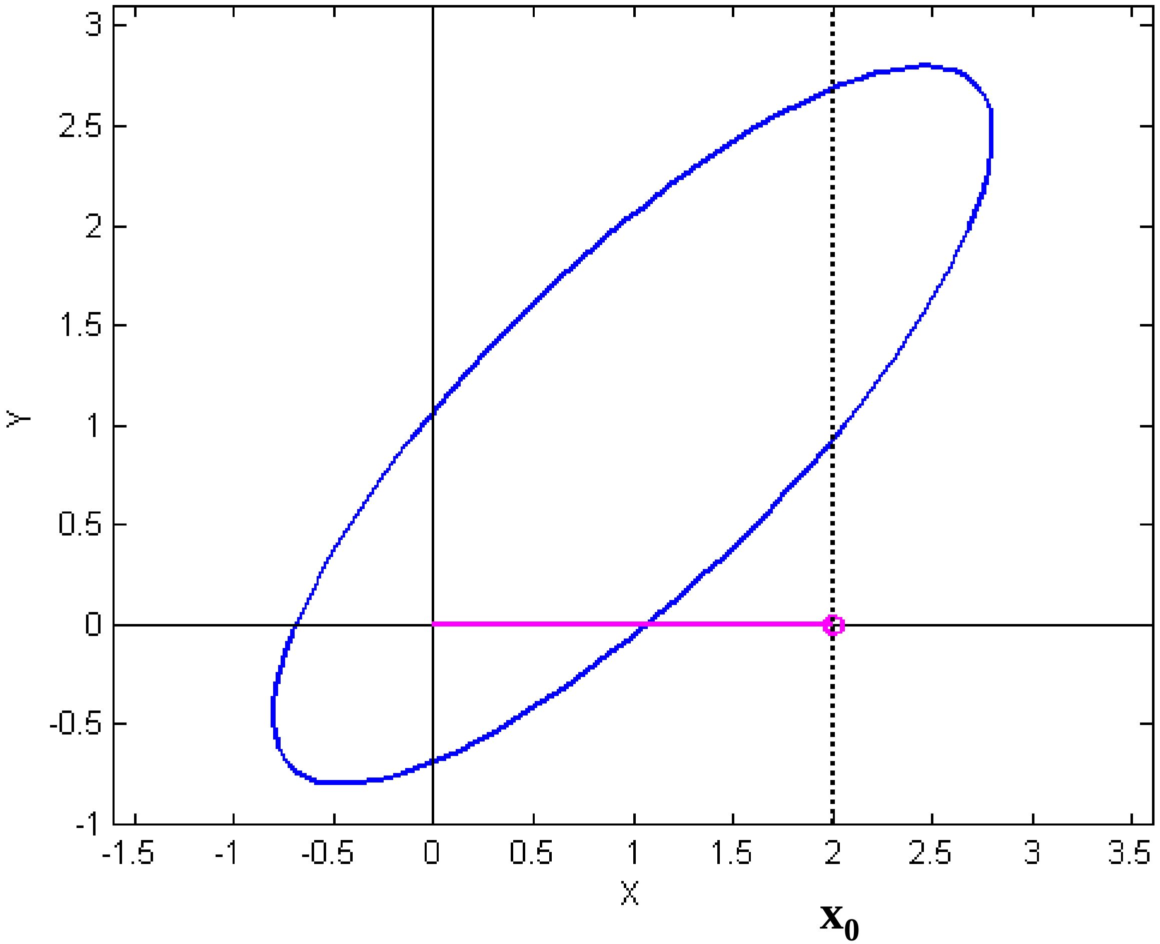
Assume X and Y are jointly Gaussian

The parameters of the Gaussian are learned from training data

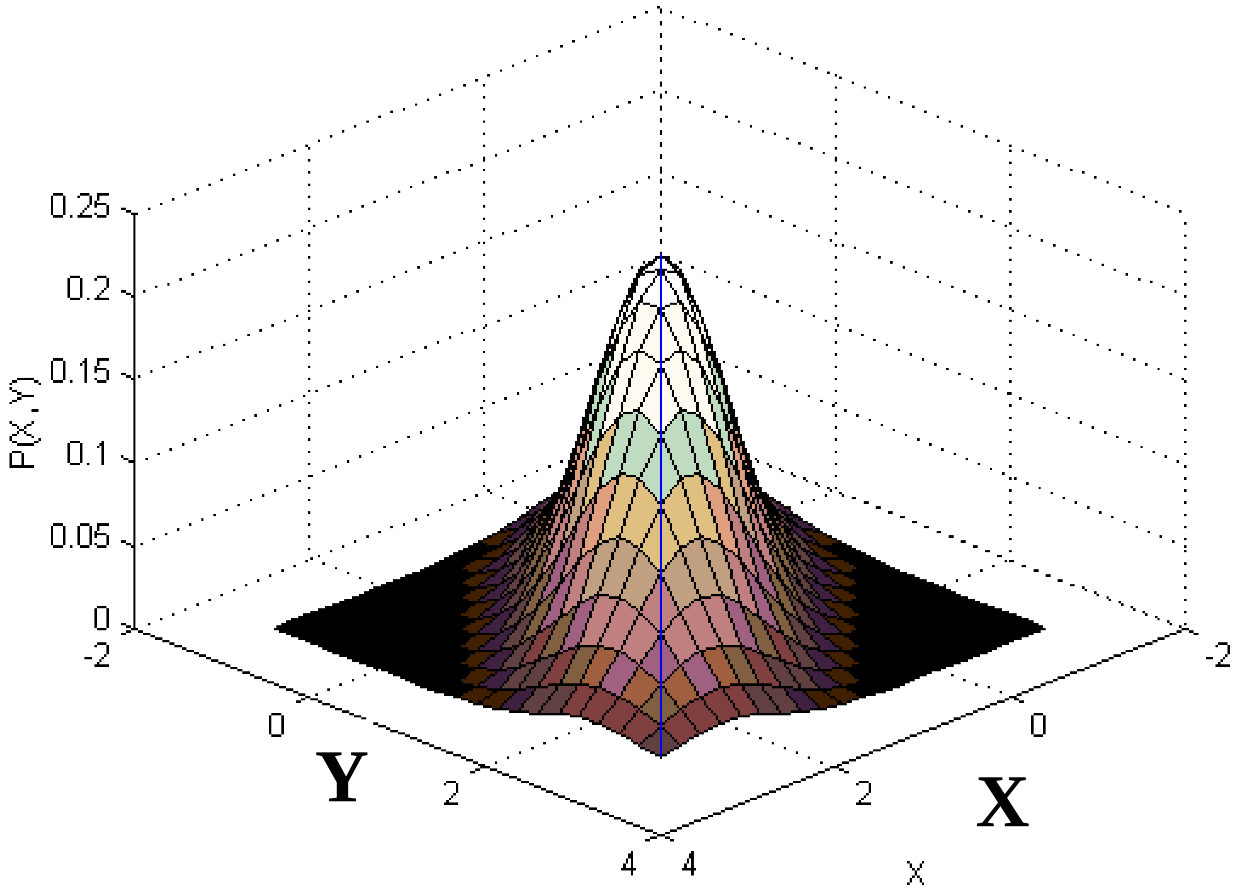


Now we are given an X, but no Y
What is Y?

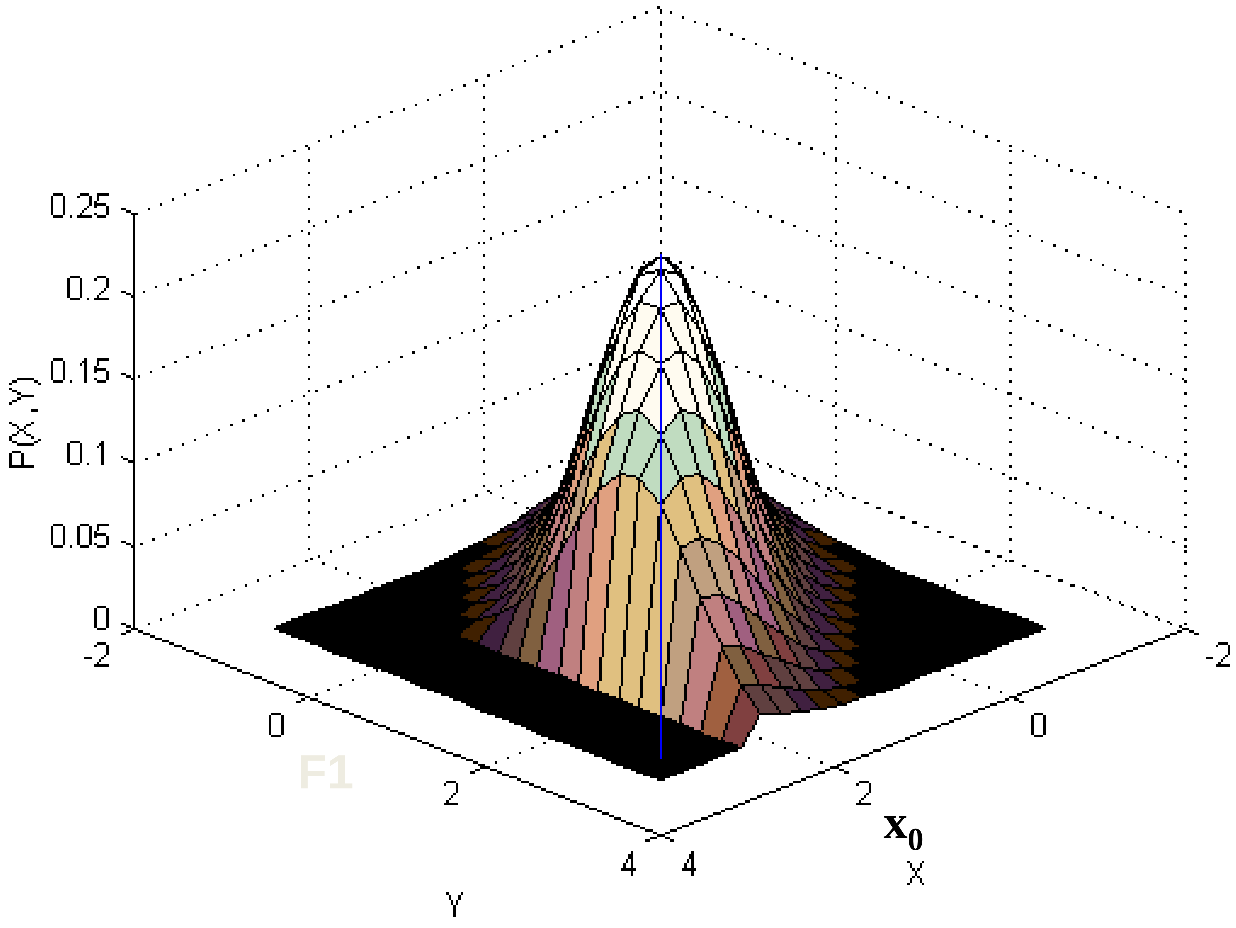
MAP estimator for Gaussian RV



MAP estimation: Gaussian PDF

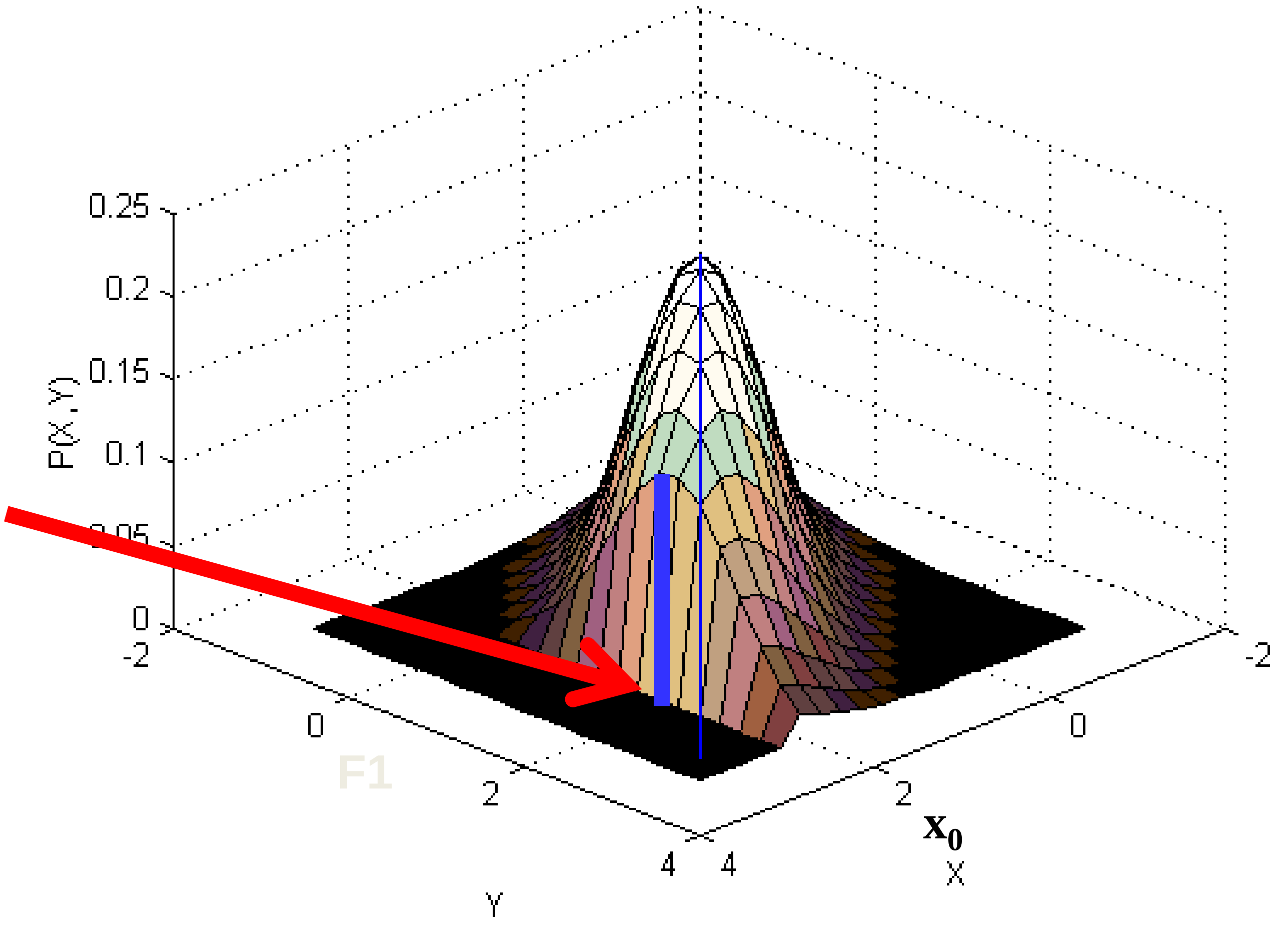


MAP estimation: The Gaussian at a particular value of X



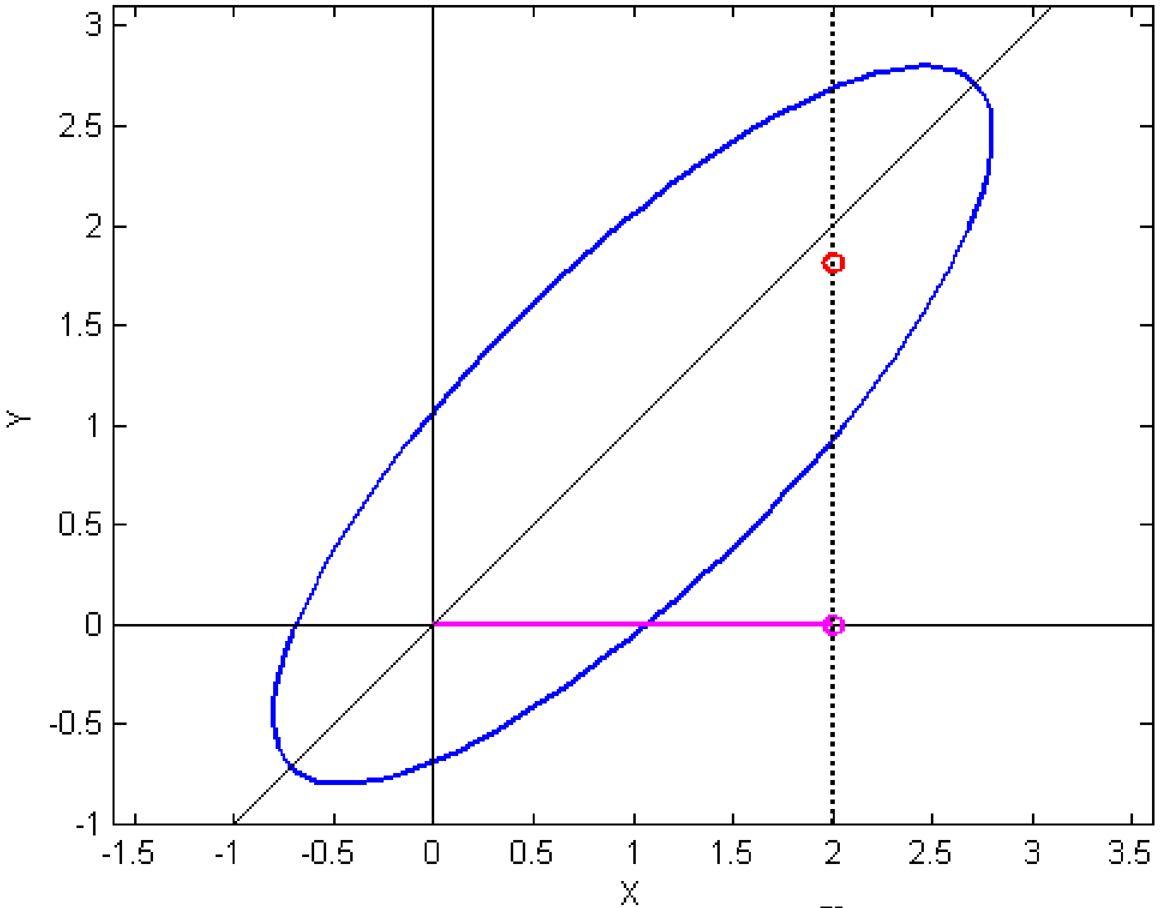
MAP estimation: The Gaussian at a particular value of X

Most likely value



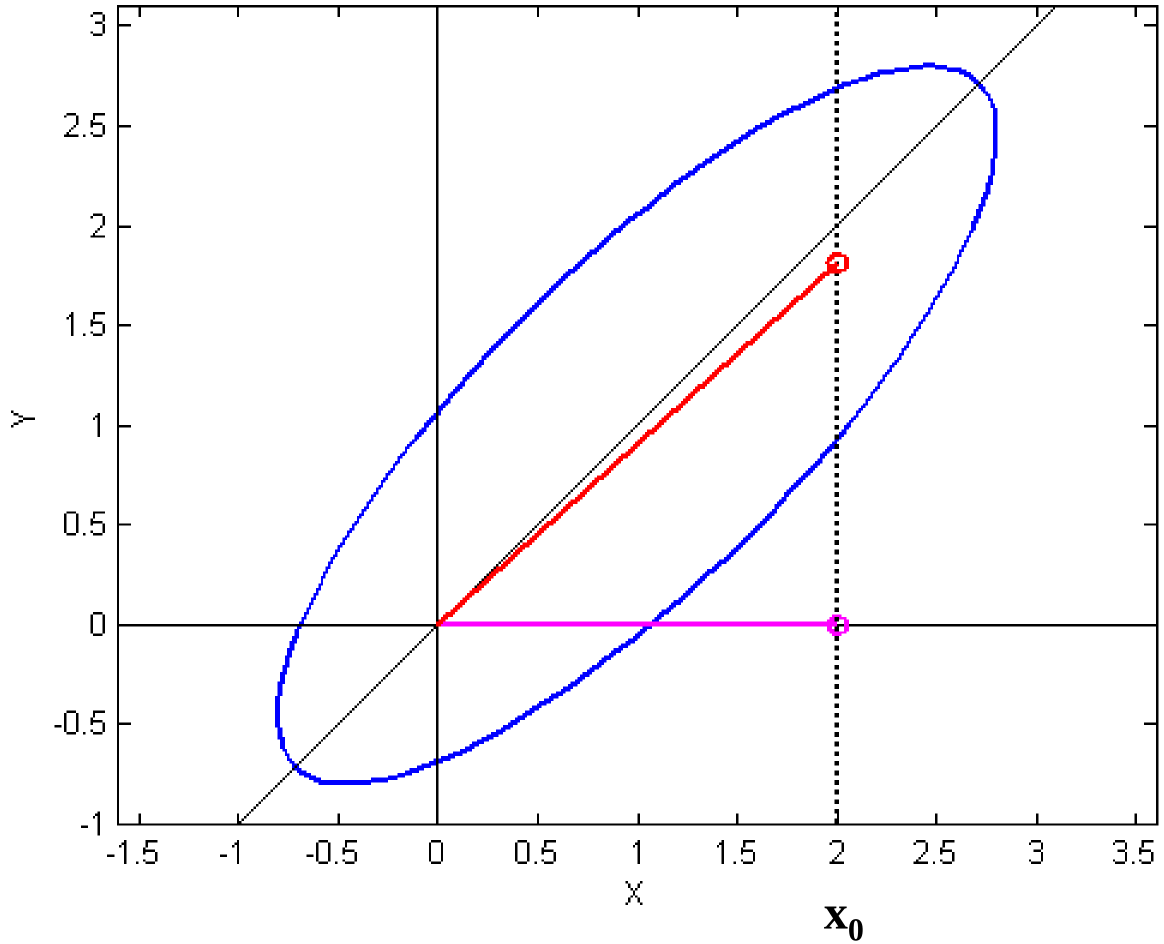
MAP Estimation of a Gaussian RV

$Y = \text{argmax}_y\, P(y|\, X)$???



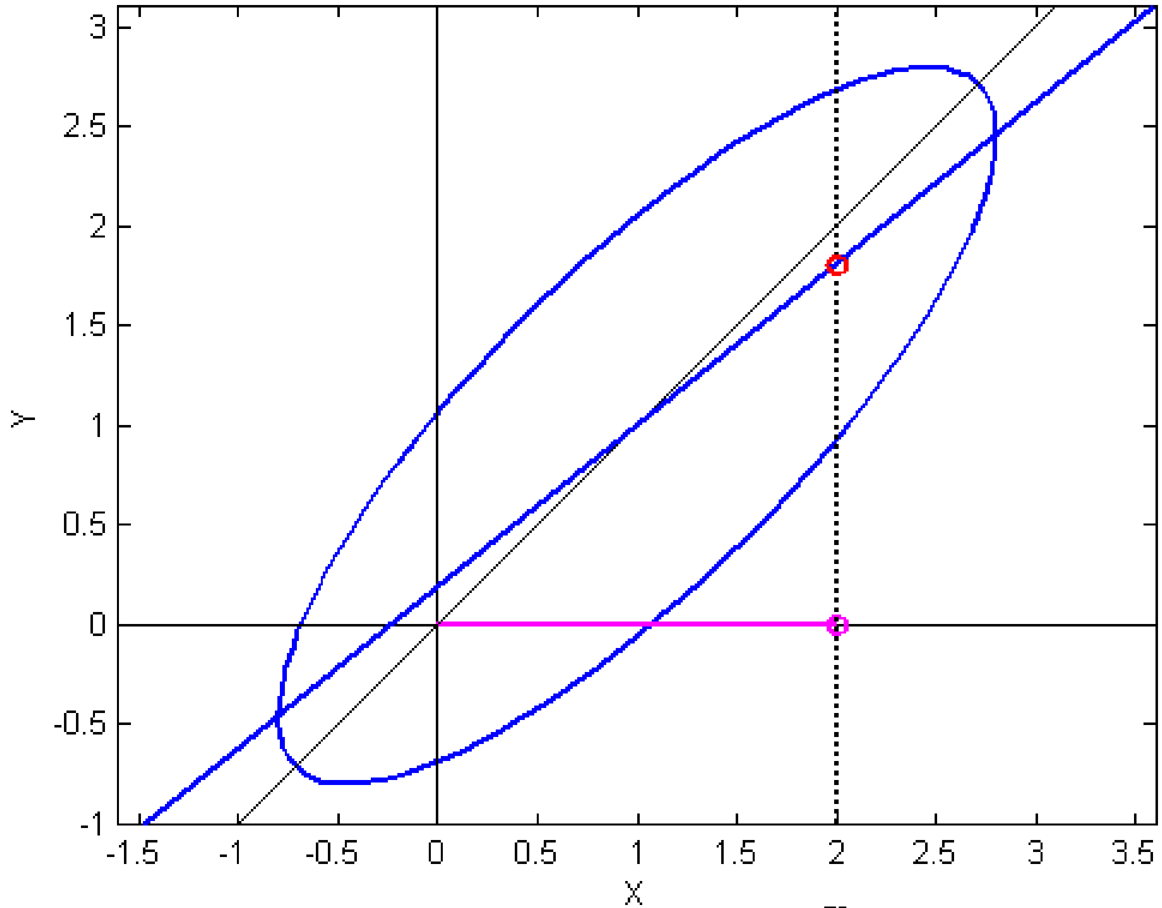
$\mathbf{x}_0$

MAP Estimation of a Gaussian RV



MAP Estimation of a Gaussian RV

$$Y = argmax_y\, P(y|\, X)$$

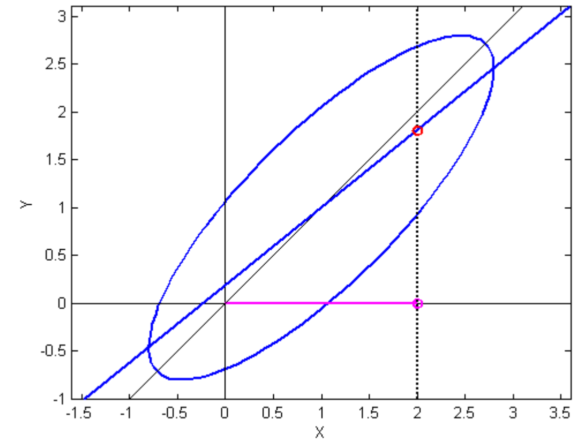


$\mathbf{x_0}$

So what is this value?

- Clearly a line
- Equation of Line:

$$\hat{y} = \mu_Y + C_{YX} C_{XX}^{-1} (x - \mu_x)$$



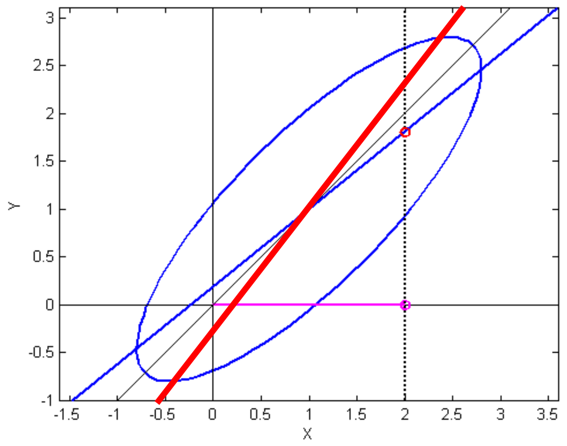
- Scalar version given; vector version is identical

$$\hat{\mathbf{y}} = \mu_Y + C_{YX} C_{XX}^{-1} (\mathbf{x} - \mu_{\mathbf{x}})$$

- Derivation? Later in the program a bit
  - Note the similarity to regression

This is a *multiple* regression

$$\hat{\mathbf{y}} = \mu_Y + C_{YX} C_{XX}^{-1} \left(\mathbf{x} - \mu_{\mathbf{x}} \right)$$



- This is the MAP estimate of $\mathbf{y}$
  - $\mathbf{y} = argmax_{\,Y}\, P(\mathrm{Y}|\mathbf{x})$
- What about the ML estimate of $\mathbf{y}$
  - $argmax_{\,Y}\, P(\mathbf{x}|\mathrm{Y})$
- Note: Neither of these may be the *regression* line!
  - MAP estimation of *y* is the regression on Y for Gaussian RVs
  - *But this is not the MAP estimation of the regression parameter*

Its also a *minimum-mean-squared error* estimate

- General principle of MMSE estimation:
  - $\mathbf{y}$ is unknown, $\mathbf{x}$ is known
  - Must estimate it such that the *expected* squared error is minimized

$$Err = E[\|\mathbf{y} - \hat{\mathbf{y}}\|^2 \mid \mathbf{x}]$$

  - Minimize above term

Its also a *minimum-mean-squared error* estimate

- Minimize error:

$$Err = E[\|\mathbf{y} - \hat{\mathbf{y}}\|^2 \mid \mathbf{x}] = E[(\mathbf{y} - \hat{\mathbf{y}})^T (\mathbf{y} - \hat{\mathbf{y}}) \mid \mathbf{x}]$$

$$Err = E[\mathbf{y}^T\mathbf{y} + \hat{\mathbf{y}}^T\hat{\mathbf{y}} - 2\hat{\mathbf{y}}^T\mathbf{y} \mid \mathbf{x}] = E[\mathbf{y}^T\mathbf{y} \mid \mathbf{x}] + \hat{\mathbf{y}}^T\hat{\mathbf{y}} - 2\hat{\mathbf{y}}^T E[\mathbf{y} \mid \mathbf{x}]$$

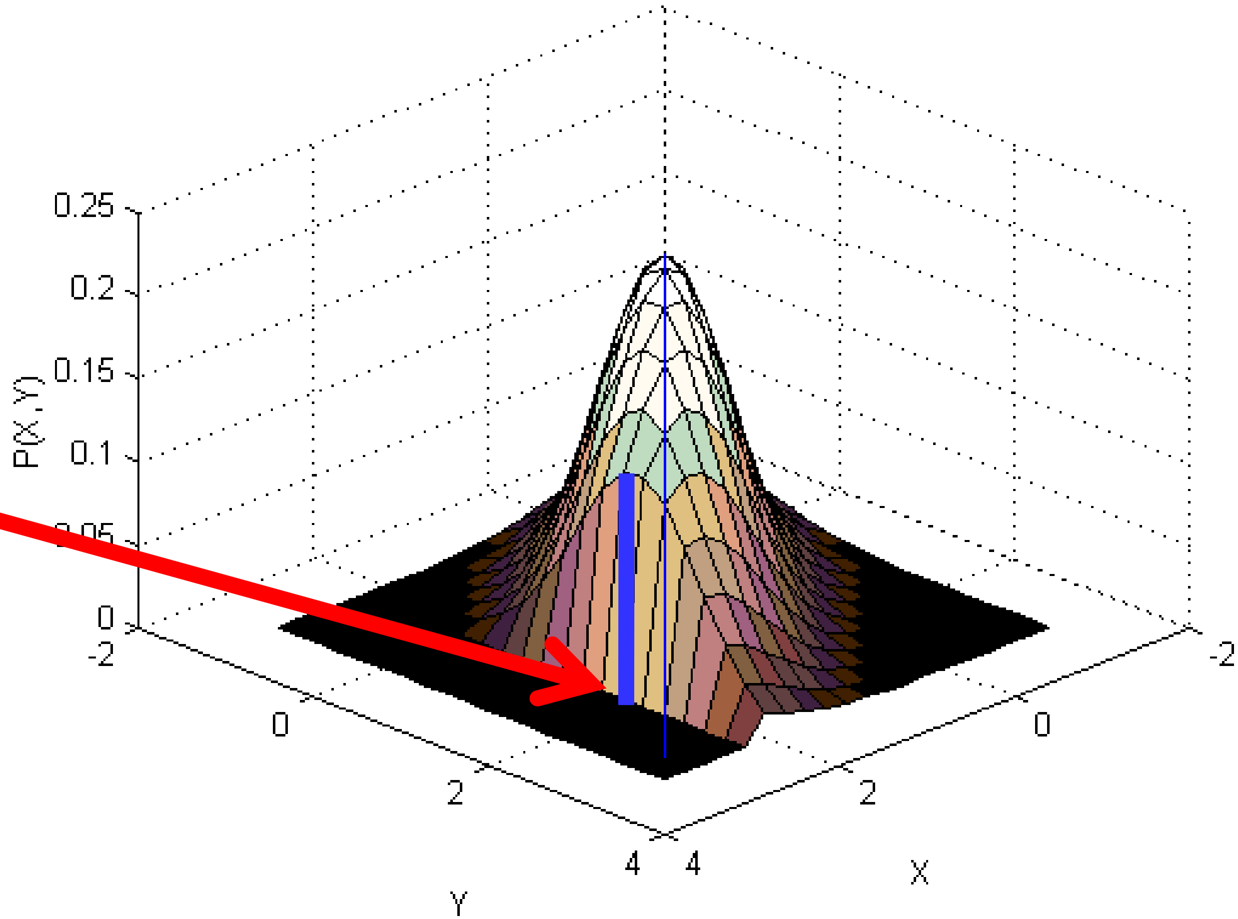
- Differentiating and equating to 0:

$$d.Err = 2\hat{\mathbf{y}}^T d\hat{\mathbf{y}} - 2E[\mathbf{y} \mid \mathbf{x}]^T d\hat{\mathbf{y}} = 0$$

$$\boxed{\hat{\mathbf{y}} = E[\mathbf{y} \mid \mathbf{x}]}$$

The MMSE estimate is the mean of the distribution

For the Gaussian: MAP = MMSE



Most likely value

is also

The MEAN value

- Would be true of any symmetric distribution

MMSE estimates for mixture distributions

$$\boxed{P(\mathbf{y} \mid \mathbf{x}) = \sum_k P(k) P(\mathbf{y} \mid k, \mathbf{x})}$$

- Let $P(\mathbf{y}|\mathbf{x})$ be a mixture density
- The MMSE estimate of $\mathbf{y}$ is given by

$$\boxed{E[\mathbf{y} \mid \mathbf{x}] = \int \mathbf{y} \sum_k P(k) P(\mathbf{y} \mid k, \mathbf{x}) d\mathbf{y}} \quad \boxed{= \sum_k P(k) \int \mathbf{y} P(\mathbf{y} \mid k, \mathbf{x}) d\mathbf{y}}$$

$$\boxed{= \sum_k P(k) E[\mathbf{y} \mid k, \mathbf{x}]}$$

- Just a weighted combination of the MMSE estimates from the component distributions

MMSE estimates from a Gaussian mixture

- Let P($\mathbf{x}$,$\mathbf{y}$) be a Gaussian Mixture

$$\mathbf{z} = \begin{bmatrix} \mathbf{y} \\ \mathbf{x} \end{bmatrix}$$

$$P(\mathbf{x},\mathbf{y}) = P(\mathbf{z}) = \sum_k P(k) N(\mathbf{z}; \mu_k, \Sigma_k)$$

- P($\mathbf{y}$|$\mathbf{x}$) is also a Gaussian mixture

$$P(\mathbf{y} \mid \mathbf{x}) = \frac{P(\mathbf{x},\mathbf{y})}{P(\mathbf{x})} = \frac{\sum_k P(k,\mathbf{x},\mathbf{y})}{P(\mathbf{x})} = \frac{\sum_k P(\mathbf{x}) P(k \mid \mathbf{x}) P(\mathbf{y} \mid \mathbf{x}, k)}{P(\mathbf{x})}$$

$$P(\mathbf{y} \mid \mathbf{x}) = \sum_k P(k \mid \mathbf{x}) P(\mathbf{y} \mid \mathbf{x}, k)$$

MMSE estimates from a Gaussian mixture

- Let P(**y**|**x**) is a Gaussian Mixture

$$P(\mathbf{y} \mid \mathbf{x}) = \sum_k P(k \mid \mathbf{x}) P(\mathbf{y} \mid \mathbf{x}, k)$$

$$P(\mathbf{y}, \mathbf{x}, k) = N([\mathbf{y}; \mathbf{x}]; [\mu_{k,\mathbf{y}}; \mu_{k,\mathbf{x}}], \begin{bmatrix} C_{k,\mathbf{yy}} & C_{k,\mathbf{yx}} \\ C_{k,\mathbf{xy}} & C_{k,\mathbf{xx}} \end{bmatrix})$$

$$P(\mathbf{y} \mid \mathbf{x}, k) = N(\mathbf{y}; \mu_{k,\mathbf{y}} + C_{k,\mathbf{yx}} C_{k,\mathbf{xx}}^{-1} (\mathbf{x} - \mu_{k,\mathbf{x}}), \Theta)$$

$$P(\mathbf{y} \mid \mathbf{x}) = \sum_k P(k \mid \mathbf{x}) N(\mathbf{y}; \mu_{k,\mathbf{y}} + C_{k,\mathbf{yx}} C_{k,\mathbf{xx}}^{-1} (\mathbf{x} - \mu_{k,\mathbf{x}}), \Theta)$$

MMSE estimates from a Gaussian mixture

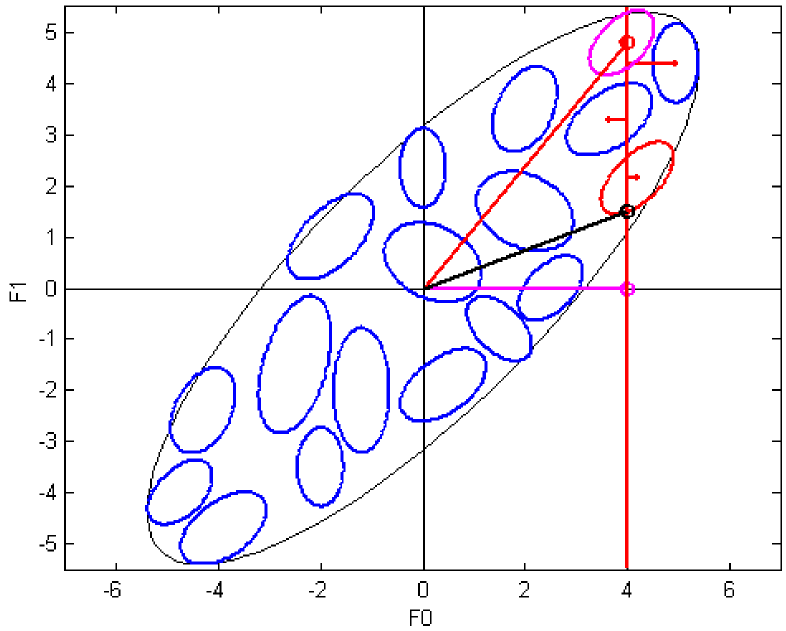
$$P(\mathbf{y}|\mathbf{x}) = \sum_k P(k|\mathbf{x}) N(\mathbf{y}; \mu_{k,\mathbf{y}} + C_{k,\mathbf{yx}} C_{k,\mathbf{xx}}^{-1}(\mathbf{x} - \mu_{k,\mathbf{x}}), \Theta)$$

- $P(\mathbf{y}|\mathbf{x})$ is a mixture Gaussian density
- $E[\mathbf{y}|\mathbf{x}]$ is also a mixture

$$E[\mathbf{y}|\mathbf{x}] = \sum_k P(k|\mathbf{x}) E[\mathbf{y}|k,\mathbf{x}]$$

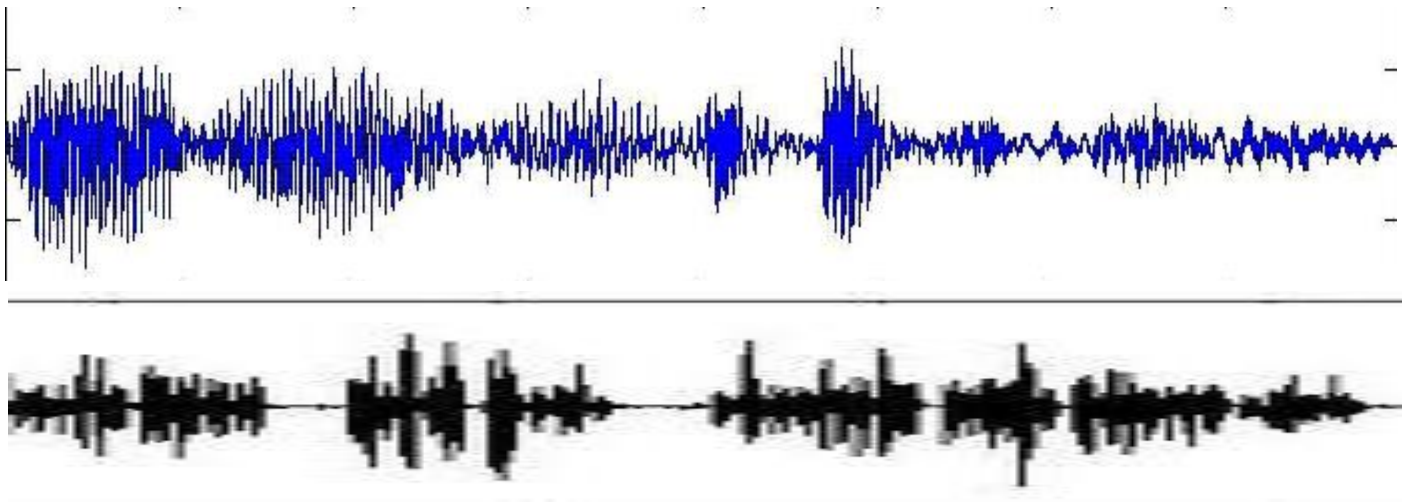
$$E[\mathbf{y}|\mathbf{x}] = \sum_k P(k|\mathbf{x}) \left(\mu_{k,\mathbf{y}} + C_{k,\mathbf{yx}} C_{k,\mathbf{xx}}^{-1}(\mathbf{x} - \mu_{k,\mathbf{x}})\right)$$

MMSE estimates from a Gaussian mixture



- A mixture of estimates from individual Gaussians

Voice Morphing



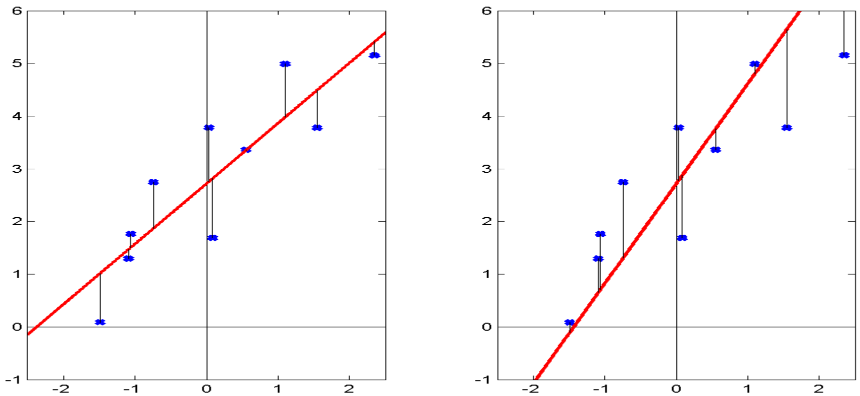
- *Align training recordings from both speakers*
  - Cepstral vector sequence
- Learn a GMM on joint vectors
- Given speech from one speaker, find MMSE estimate of the other
- *Synthesize from cepstra*

MMSE with GMM: Voice Transformation

- Festvox GMM transformation suite (Toda)

	awb	bdl	jmk	slt
awb				
bdl				
jmk				
slt				

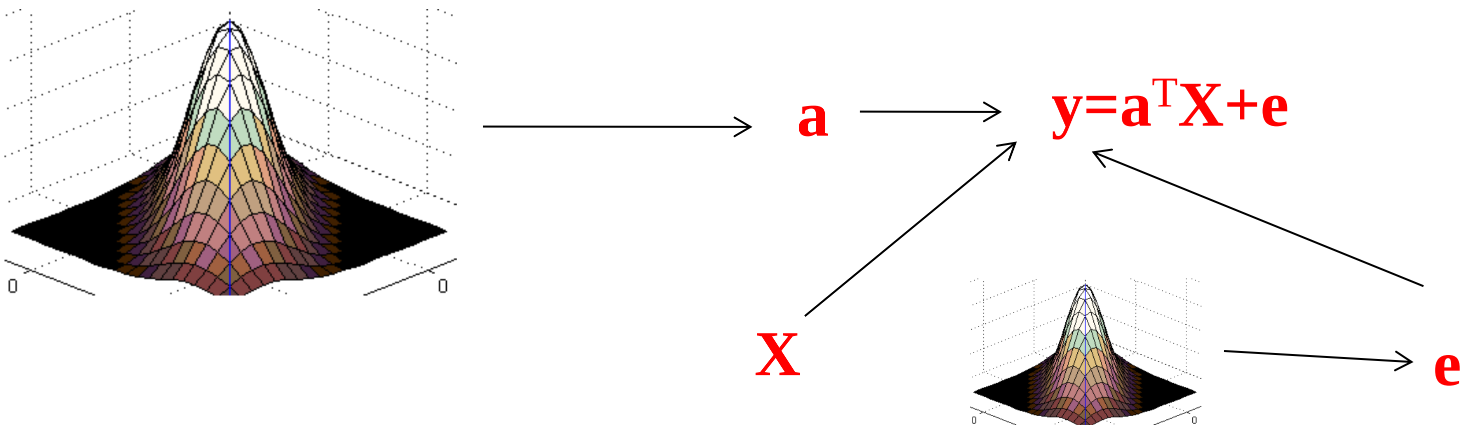
A problem with regressions



$$\mathbf{A} = \left(\mathbf{X}\mathbf{X}^T\right)^{-1}\mathbf{X}\mathbf{Y}^T$$

- ML fit is sensitive
  - Error is squared
  - Small variations in data → large variations in weights
  - Outliers affect it adversely
- Unstable
  - If dimension of $\mathbf{X}$ >= no. of instances
    - ($\mathbf{X}\mathbf{X}^T$) is not invertible

MAP estimation of weights



- Assume weights drawn from a Gaussian
  - $P(\mathbf{a}) = N(0, \sigma^2 I)$
- Max. Likelihood estimate

$$\hat{\mathbf{a}} = \arg\max_{\mathbf{a}} \log P(\mathbf{y} \mid \mathbf{X}; \mathbf{a})$$

- Maximum *a posteriori* estimate

$$\hat{\mathbf{a}} = \arg\max_{\mathbf{a}} \log P(\mathbf{a} \mid \mathbf{y}, \mathbf{X}) = \arg\max_{\mathbf{a}} \log P(\mathbf{y} \mid \mathbf{X}, \mathbf{a}) P(\mathbf{a})$$

MAP estimation of weights

$$\hat{\mathbf{a}} = \arg\max_{\mathbf{A}} \log P(\mathbf{a} \mid \mathbf{y}, \mathbf{X}) = \arg\max_{\mathbf{A}} \log P(\mathbf{y} \mid \mathbf{X}, \mathbf{a}) P(\mathbf{a})$$

- P(**a**) = N(0, σ^2I)
- Log P(**a**) = C – log σ – 0.5σ^{-2} ||**a**||2

$$\log P(\mathbf{y} \mid \mathbf{X}, \mathbf{a}) = C - \frac{1}{2\sigma^2}(\mathbf{y} - \mathbf{a}^T\mathbf{X})^T(\mathbf{y} - \mathbf{a}^T\mathbf{X})^T$$

$$\hat{\mathbf{a}} = \arg\max_{\mathbf{A}} C' - \log\sigma - \frac{1}{2\sigma^2}(\mathbf{y} - \mathbf{a}^T\mathbf{X})^T(\mathbf{y} - \mathbf{a}^T\mathbf{X})^T - 0.5\sigma^2\mathbf{a}^T\mathbf{a}$$

- Similar to ML estimate with an additional term

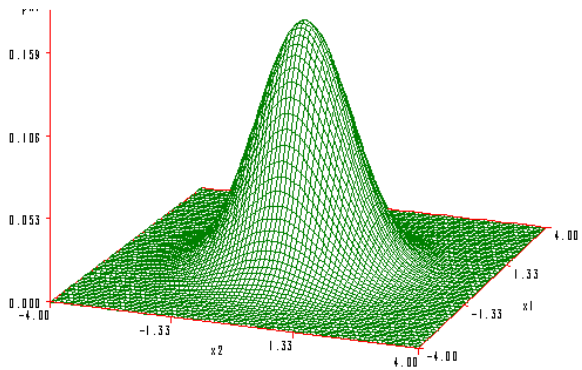
MAP estimate of weights

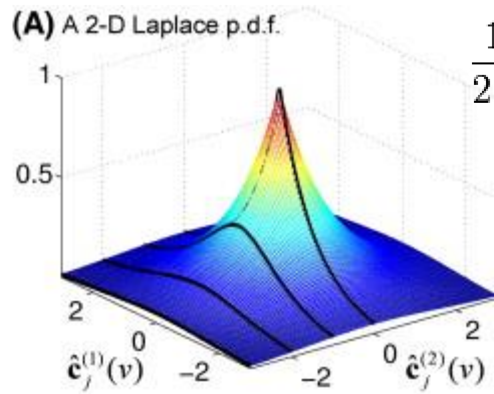
$$dL = \left(2\mathbf{a}^T \mathbf{X}\mathbf{X}^T + 2\mathbf{y}\mathbf{X}^T + 2\sigma\mathbf{I}\right)d\mathbf{a} = 0$$

$$\boxed{\mathbf{a} = \left(\mathbf{X}\mathbf{X}^T + \sigma\mathbf{I}\right)^{-1} \mathbf{X}\mathbf{Y}^T}$$

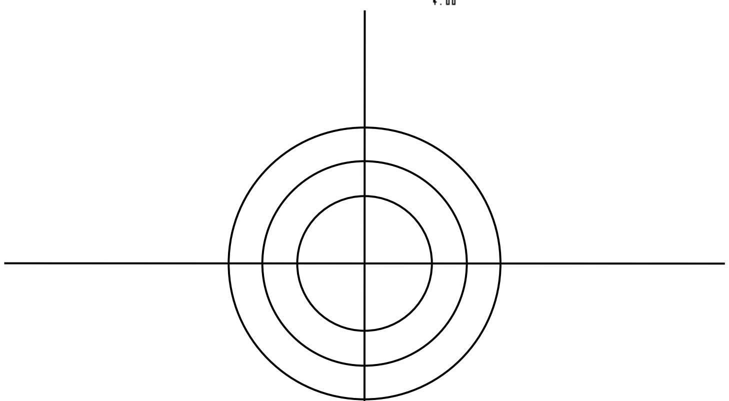
- Equivalent to *diagonal loading* of correlation matrix
  - Improves condition number of correlation matrix
    - Can be inverted with greater stability
  - Will not affect the estimation from well-conditioned data
  - Also called Tikhonov Regularization
    - Dual form: Ridge regression
- **MAP estimate of *weights***
  - **Not to be confused with MAP estimate of Y**

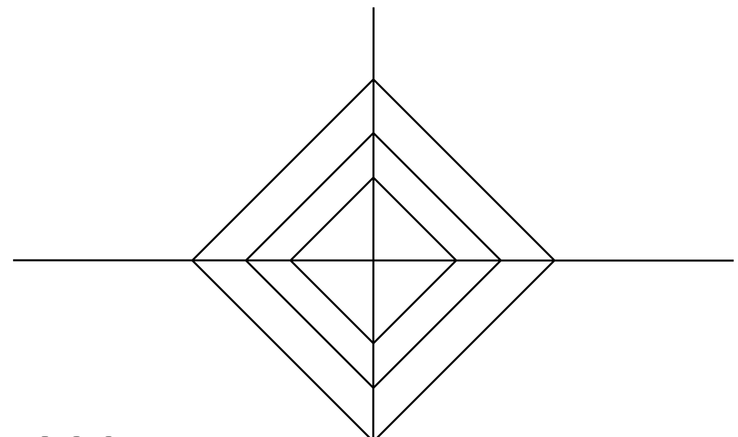
MAP estimate priors





$$\frac{1}{2b}\exp\left(-\frac{|x-\mu|}{b}\right)$$





- Left: Gaussian Prior on W
- Right: Laplacian Prior

MAP estimation of weights with laplacian prior

- Assume weights drawn from a Laplacian
  - $P(\mathbf{a}) = \lambda^{-1}\exp(-\lambda^{-1}|\mathbf{a}|_1)$
- Maximum *a posteriori* estimate

$$\hat{\mathbf{a}} = \arg\max_{\mathbf{A}} C' - (\mathbf{y} - \mathbf{a}^T\mathbf{X})^T(\mathbf{y} - \mathbf{a}^T\mathbf{X})^T - \lambda^{-1}|\mathbf{a}|_1$$

- No closed form solution
  - Quadratic programming solution required
    - Non-trivial

MAP estimation of weights with laplacian prior

- Assume weights drawn from a Laplacian
  - $P(\mathbf{a}) = \lambda^{-1}\exp(-\lambda^{-1}|\mathbf{a}|_1)$
- Maximum *a posteriori* estimate

$$\hat{\mathbf{a}} = \arg\max_{\mathbf{A}} C' - (\mathbf{y} - \mathbf{a}^T\mathbf{X})^T(\mathbf{y} - \mathbf{a}^T\mathbf{X})^T - \lambda^{-1}|\mathbf{a}|_1$$

- Identical to L_1 regularized least-squares estimation

L_1-regularized LSE

$$\hat{\mathbf{a}} = \arg\max_{\mathbf{A}} C' - (\mathbf{y} - \mathbf{a}^T\mathbf{X})^T(\mathbf{y} - \mathbf{a}^T\mathbf{X})^T - \lambda^{-1}|\mathbf{a}|_1$$

- No closed form solution
  - Quadratic programming solutions required

- Dual formulation

$$\hat{\mathbf{a}} = \arg\max_{\mathbf{A}} C' - (\mathbf{y} - \mathbf{a}^T\mathbf{X})^T(\mathbf{y} - \mathbf{a}^T\mathbf{X})^T \quad \textbf{subject to} \quad |\mathbf{a}|_1 \le t$$

- "LASSO" – Least absolute shrinkage and selection operator

LASSO Algorithms

- Various convex optimization algorithms
- LARS: Least angle regression
- Pathwise coordinate descent..
- Matlab code available from web

Regularized least squares

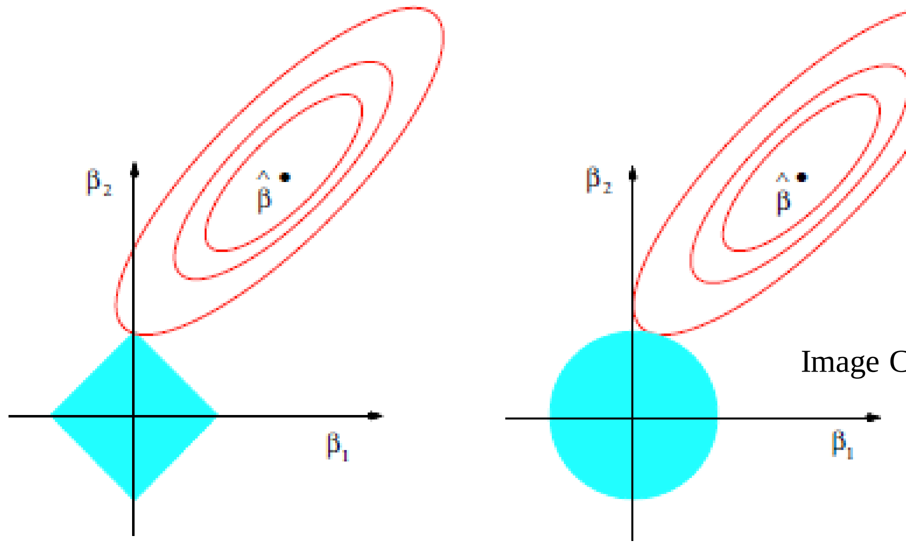


Image Credit: Tibshirani

- Regularization results in selection of suboptimal (in least-squares sense) solution
  - One of the loci outside center
- Tikhonov regularization selects ***shortest*** solution
- L_1 regularization selects ***sparsest*** solution

LASSO and Compressive Sensing

$$\mathbf{Y} = \mathbf{X} \quad \mathbf{a}$$

- Given $\mathbf{Y}$ and $\mathbf{X}$, estimate sparse $\mathbf{W}$
- LASSO:
  - $\mathbf{X}$ = explanatory variable
  - $\mathbf{Y}$ = dependent variable
  - $\mathbf{a}$ = weights of regression
- CS:
  - $\mathbf{X}$ = measurement matrix
  - $\mathbf{Y}$ = measurement
  - $\mathbf{a}$ = data

An interesting problem: Predicting War!

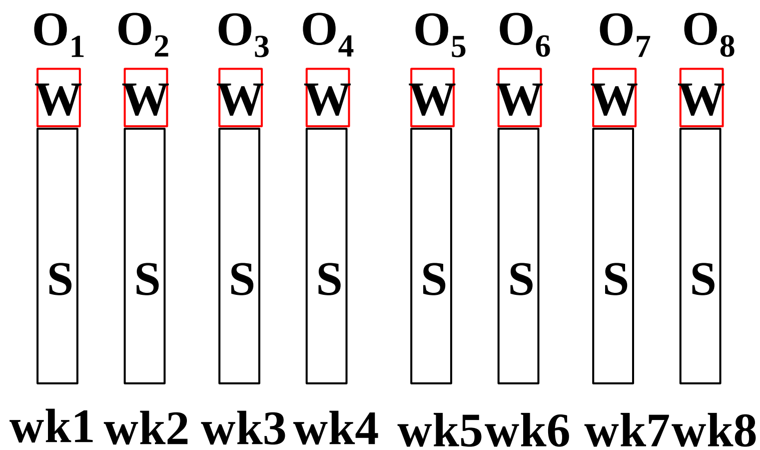
- Economists measure a number of social indicators for countries weekly
  - Happiness index
  - Hunger index
  - Freedom index
  - Twitter records
  - …
- Question: Will there be a revolution or war next week?

An interesting problem: Predicting War!

- Issues:
  - Dissatisfaction builds up – not an instantaneous phenomenon
    - Usually
  - War / rebellion build up much faster
    - Often in hours

- Important to predict
  - Preparedness for security
  - Economic impact

Predicting War



Given
- Sequence of economic indicators for each week
- Sequence of unrest markers for each week
  - At the end of each week we know if war happened or not that week

- Predict probability of unrest next week
  - This could be a new unrest or persistence of a current one

Predicting Time Series

- Need *time-series models*

- HMMs – later in the course