Machine Learning for Signal Processing

Expectation Maximization Mixture Models

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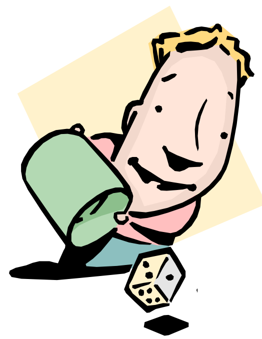
Learning Distributions for Data

- Problem: Given a collection of examples from some data, estimate its distribution
- Solution: Assign a model to the distribution
  - Learn parameters of model from data
- Models can be arbitrarily complex
  - Mixture densities, Hierarchical models.

A Thought Experiment



6 3 1 5 4 1 2 4 …

- A person shoots a loaded dice repeatedly
- You observe the series of outcomes
- **You can form a good idea of how the dice is loaded**
  - Figure out what the probabilities of the various numbers are for dice
- P(number) = count(number)/count(rolls)
- This is a *maximum likelihood* estimate
  - Estimate that makes the observed sequence of numbers most probable

The Multinomial Distribution

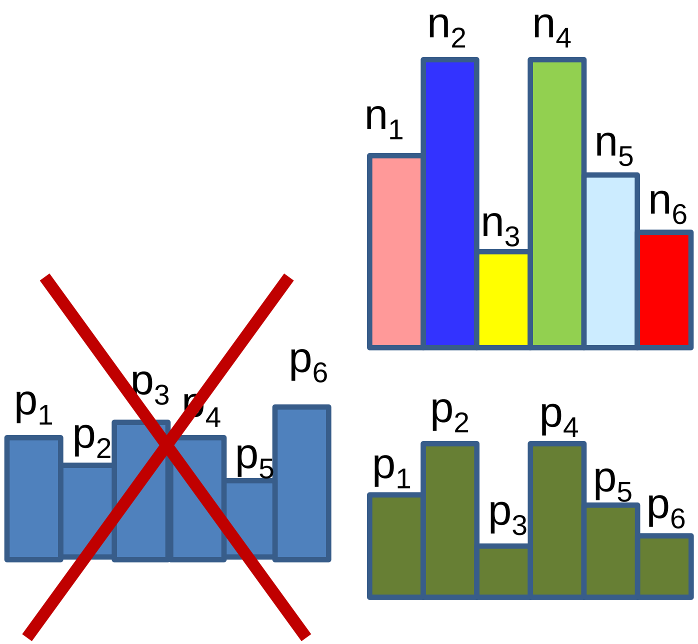
- A probability distribution over a discrete collection of items is a *Multinomial*

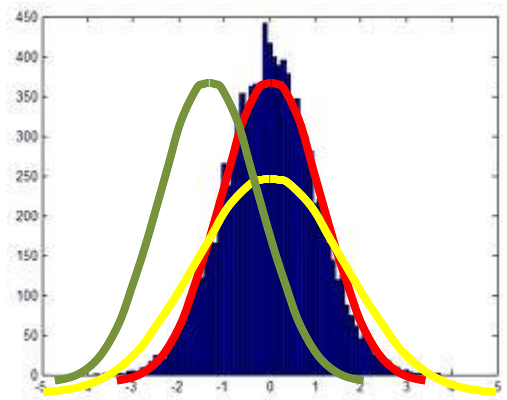
$$P(X : X \text{ belongs to a discrete set}) = P(X)$$

- E.g. the roll of dice
  - X : X in (1,2,3,4,5,6)

- Or the toss of a coin
  - X : X in (head, tails)

Maximum Likelihood Estimation





- Basic principle: Assign a form to the distribution
  - E.g. a multinomial
  - Or a Gaussian
- Find the *distribution* that best fits the histogram of the data

Defining "Best Fit"

- The data are generated by draws from the distribution
  - I.e. the generating process draws from the distribution

- Assumption: The world is a boring place
  - The data you have observed are very typical of the process

- Consequent assumption: The distribution has a high probability of generating the observed data
  - Not necessarily true

- Select the distribution that has the *highest* probability of generating the data
  - Should assign lower probability to less frequent observations and vice versa

Maximum Likelihood Estimation: Multinomial

- Probability of generating (n_1, n_2, n_3, n_4, n_5, n_6)

$$P(n_1, n_2, n_3, n_4, n_5, n_6) = Const \prod_i p_i^{n_i}$$

- Find p_1,p_2,p_3,p_4,p_5,p_6 so that the above is maximized
- Alternately maximize

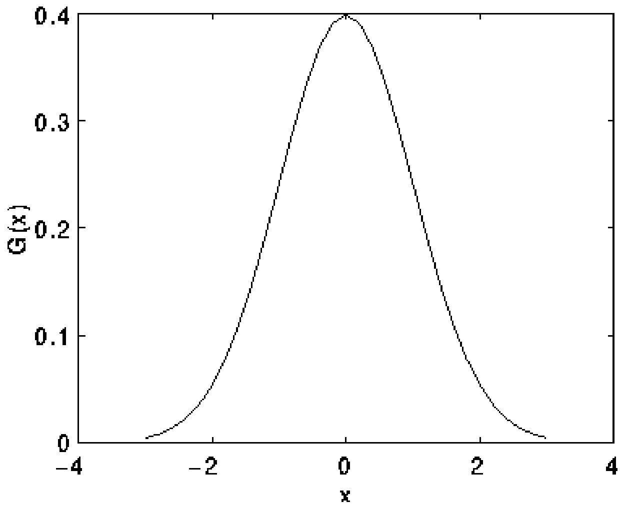
$$\log\left(P(n_1, n_2, n_3, n_4, n_5, n_6)\right) = \log(Const) + \sum_i n_i \log(p_i)$$

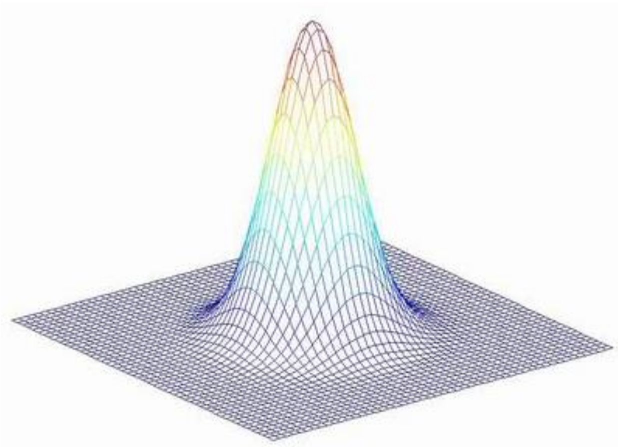
  – Log() is a monotonic function
  – $\mathrm{argmax}_x\, f(x) = \mathrm{argmax}_x \log(f(x))$
- Solving for the probabilities gives us
  – Requires constrained optimization to ensure probabilities sum to 1

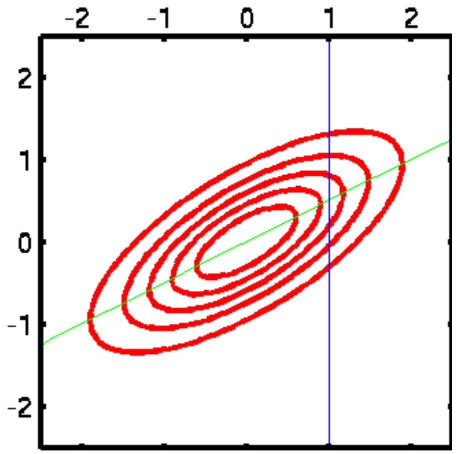
$$p_i = \frac{n_i}{\sum_j n_j}$$

EVENTUALLY ITS JUST COUNTING!

Segue: Gaussians







$$P(X) = N(X;\mu,\Theta) = \frac{1}{\sqrt{(2\pi)^d |\Theta|}} \exp\left(-0.5(X-\mu)^T \Theta^{-1} (X-\mu)\right)$$

- Parameters of a Gaussian:
  - Mean μ, Covariance Θ

Maximum Likelihood: Gaussian

- Given a collection of observations (X_1, X_2,…), estimate mean μ and covariance Θ

$$P(X_1, X_2, \ldots) = \prod_i \frac{1}{\sqrt{(2\pi)^d |\Theta|}} \exp\left(-0.5(X_i - \mu)^T \Theta^{-1} (X_i - \mu)\right)$$

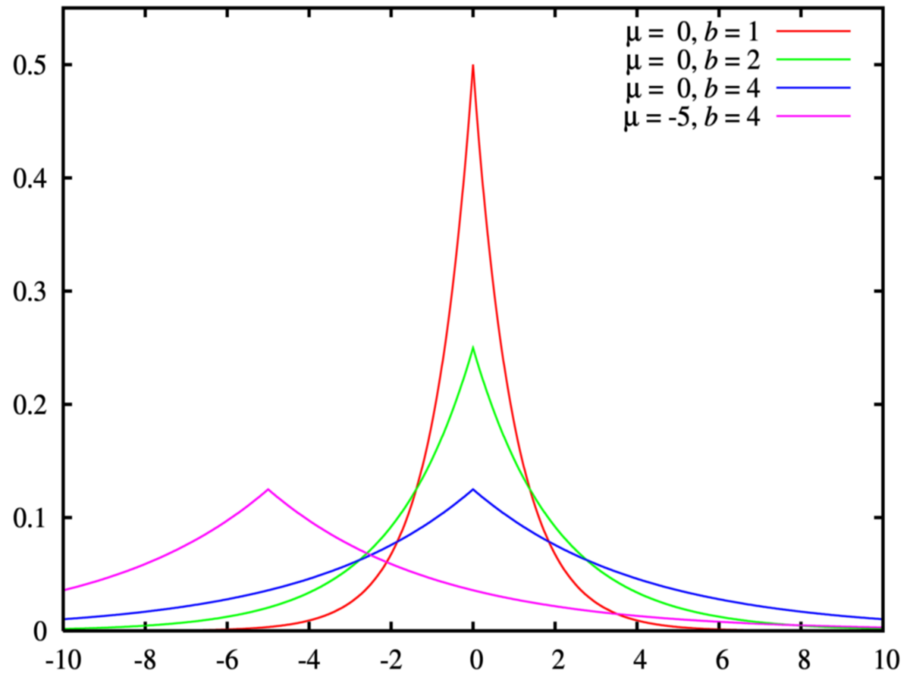
$$\log\left(P(X_1, X_2, \ldots)\right) = C - 0.5 \sum_i \left(\log\left(|\Theta|\right) + (X_i - \mu)^T \Theta^{-1} (X_i - \mu)\right)$$

- Maximizing w.r.t μ and Θ gives us

$$\mu = \frac{1}{N} \sum_i X_i \qquad \Theta = \frac{1}{N} \sum_i (X_i - \mu)(X_i - \mu)^T$$

ITS STILL JUST COUNTING!

Laplacian



$$P(x) = L(x;\mu,b) = \frac{1}{2b}\exp\left(-\frac{|x-\mu|}{b}\right)$$

- Parameters: Median μ, scale b ($b > 0$)
  - μ is also the mean, but is better viewed as the median

Maximum Likelihood: Laplacian

- Given a collection of observations (x_1, x_2,…), estimate mean μ and scale b

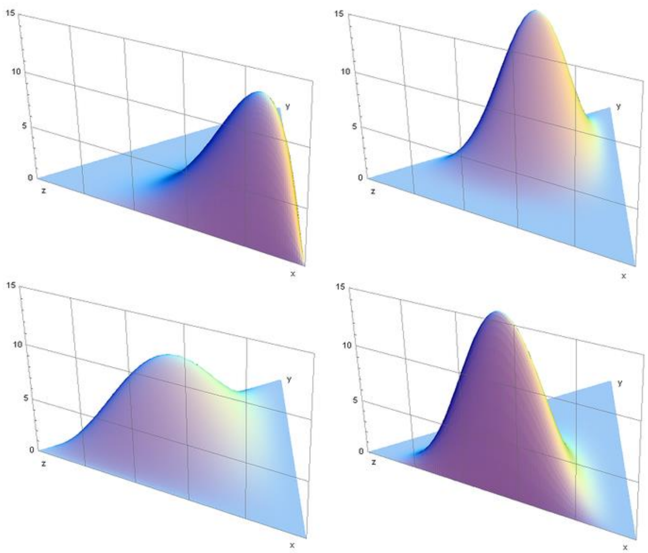
$$\log\left(P(x_1, x_2, \ldots)\right) = C - N\log(b) - \sum_i \frac{|x_i - \mu|}{b}$$

- Maximizing w.r.t μ and b gives us

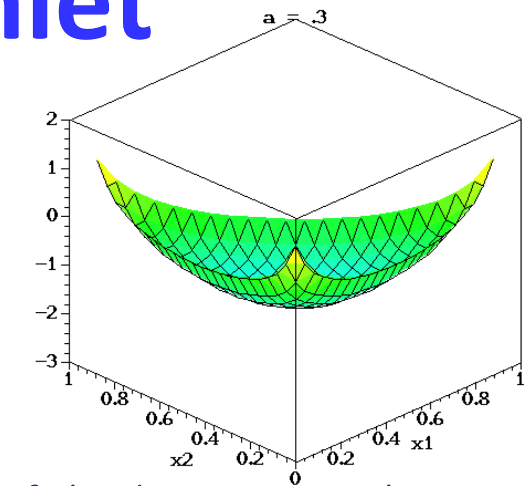
$$\mu = median(\{x_i\}) \quad b = \frac{1}{N}\sum_i |x_i - \mu|$$

Dirichlet

(from wikipedia)



K=3. Clockwise from top left:
α=(6, 2, 2), (3, 7, 5), (6, 2, 6), (2, 3, 4)



log of the density as we change α from α=(0.3, 0.3, 0.3) to (2.0, 2.0, 2.0), keeping all the individual αi's equal to each other.

$$P(X) = D(X;\alpha) = \frac{\prod_i \Gamma(\alpha_i)}{\Gamma\left(\sum_i \alpha_i\right)} \prod_i x_i^{\alpha_i - 1}$$

- Parameters are αs
  - Determine mode and curvature
- Defined only of probability vectors
  - $X = [x_1\ x_2\ ..\ x_K]$, $\Sigma_i\ x_i = 1$, $x_i >= 0$ for all i

Maximum Likelihood: Dirichlet

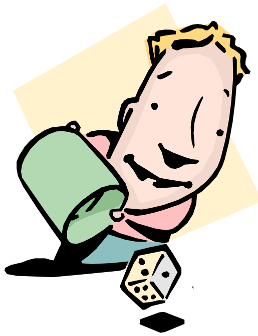
- Given a collection of observations (X_1, X_2,…), estimate α

$$\log\left(P(X_1, X_2, \ldots)\right) = \sum_j \sum_i (\alpha_i - 1)\log(X_{j,i}) + N\sum_i \log\left(\Gamma(\alpha_i)\right) - N\log\left(\Gamma\left(\sum_i \alpha_i\right)\right)$$

- No closed form solution for αs.
  - Needs gradient ascent

- Several distributions have this property: the ML estimate of their parameters have no closed form solution

Continuing the Thought Experiment



6 3 1 5 4 1 2 4 …

4 4 1 6 3 2 1 2 …

- Two persons shoot loaded dice repeatedly
  - The dice are differently loaded for the two of them
- We observe the series of outcomes for both persons

- **How to determine the probability distributions of the two dice?**

Estimating Probabilities

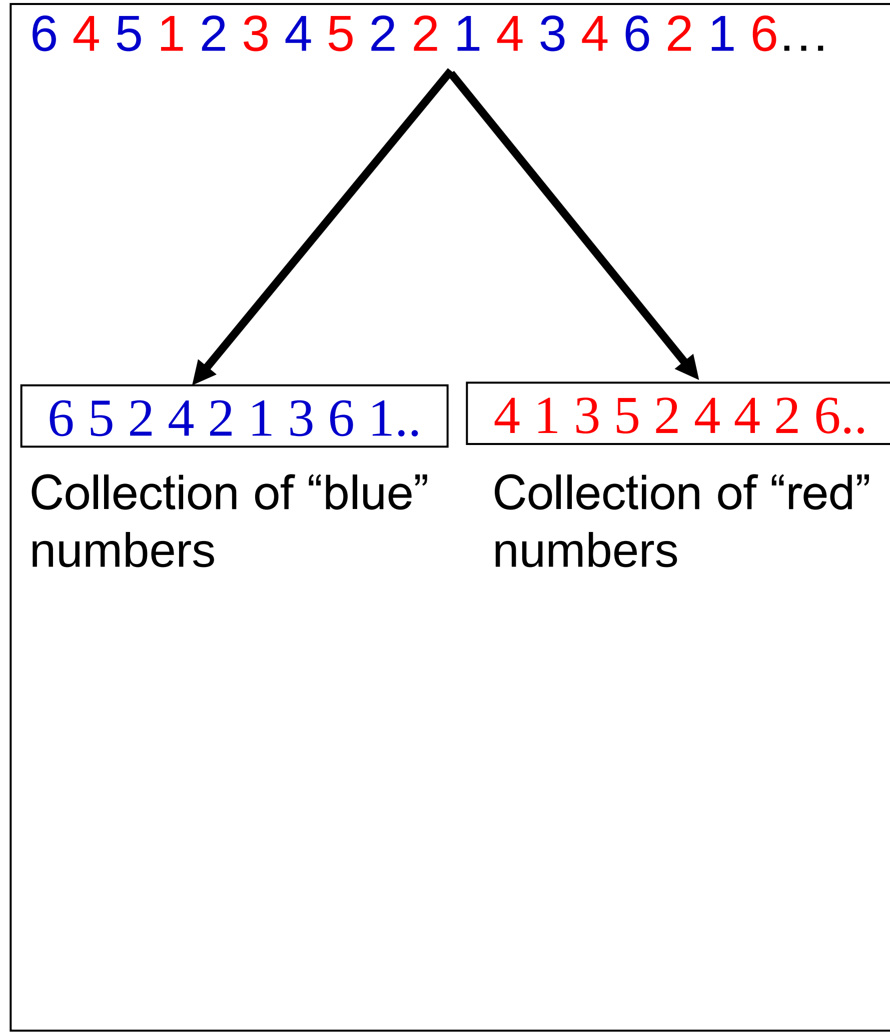
- Observation: The sequence of numbers from the two dice
  - As indicated by the colors, we know who rolled what number

6 4 5 1 2 3 4 5 2 2 1 4 3 4 6 2 1 6…

Estimating Probabilities

- Observation: The sequence of numbers from the two dice
  - As indicated by the colors, we know who rolled what number

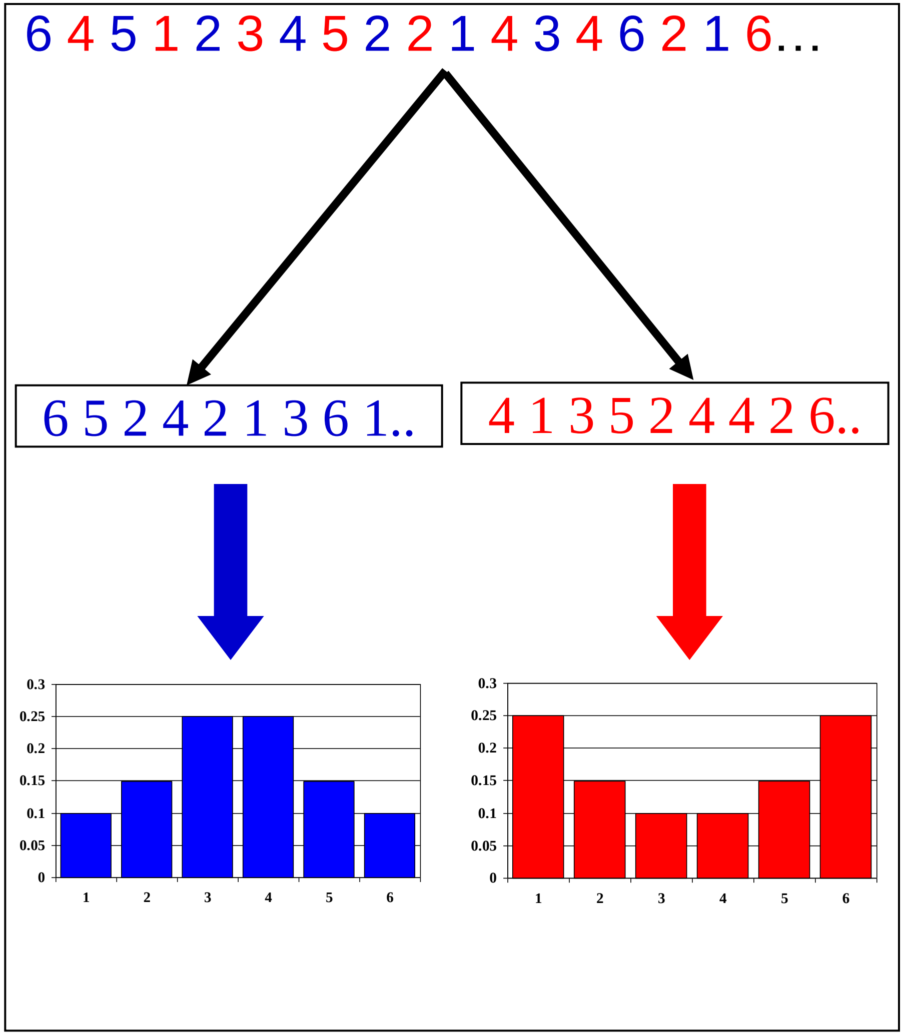
- Segregation: Separate the blue observations from the red



Estimating Probabilities

- Observation: The sequence of numbers from the two dice
  - As indicated by the colors, we know who rolled what number
- Segregation: Separate the blue observations from the red
- From each set compute probabilities for each of the 6 possible outcomes

$$P(number) = \frac{\text{no. of times number was rolled}}{\text{total number of observed rolls}}$$



A Thought Experiment

6 3 1 5 4 1 2 4 … 4 4 1 6 3 2 1 2 …

- Now imagine that you cannot observe the dice yourself
- Instead there is a "caller" who randomly calls out the outcomes
  - 40% of the time he calls out the number from the left shooter, and 60% of the time, the one from the right (and you know this)

- At any time, you do not know which of the two he is calling out
- How do you determine the probability distributions for the two dice?

A Thought Experiment

6 3 1 5 4 1 2 4 …

4 4 1 6 3 2 1 2 …

- How do you now determine the probability distributions for the two sets of dice …

- .. If you do not even know what fraction of time the blue numbers are called, and what fraction are red?

A Mixture Multinomial

- The caller will call out a number X in any given callout IF
  - He selects “RED”, and the Red die rolls the number X
  - OR
  - He selects “BLUE” and the Blue die rolls the number X
- P(X) = P(Red)P(X|Red) + P(Blue)P(X|Blue)
  - E.g. P(6) = P(Red)P(6|Red) + P(Blue)P(6|Blue)
- A distribution that *combines* (or *mixes*) multiple multinomials is a *mixture* multinomial

$$P(X) = \sum_{Z} P(Z) P(X \mid Z)$$

Mixture weights — Component multinomials

Mixture Distributions

Mixture Gaussian

$$P(X) = \sum_{Z} P(Z) P(X \mid Z)$$

Mixture weights — Component distributions

$$P(X) = \sum_{Z} P(Z) N(X; \mu_z, \Theta_z)$$

Mixture of Gaussians and Laplacians

$$P(X) = \sum_{Z} P(Z) N(X; \mu_z, \Theta_z) + \sum_{Z} P(Z) \prod_{i} L(X_i; \mu_z, b_{z,i})$$

- Mixture distributions mix several component distributions
  - Component distributions may be of varied type
- Mixing weights must sum to 1.0
- Component distributions integrate to 1.0
- Mixture distribution integrates to 1.0

Maximum Likelihood Estimation

- For our problem:
  - Z = color of dice

$$P(X) = \sum_Z P(Z)P(X \mid Z)$$

$$P(n_1, n_2, n_3, n_4, n_5, n_6) = Const \prod_X P(X)^{n_X} = Const \prod_X \left(\sum_Z P(Z)P(X \mid Z) \right)^{n_X}$$

- Maximum likelihood solution: Maximize

$$\log(P(n_1, n_2, n_3, n_4, n_5, n_6)) = \log(Const) + \sum_X n_X \log\left(\sum_Z P(Z)P(X \mid Z) \right)$$

- No closed form solution (summation inside log)!
  - In general ML estimates for mixtures do not have a closed form
  - USE EM!

Expectation Maximization

- It is possible to estimate all parameters in this setup using the Expectation Maximization (or EM) algorithm

- First described in a landmark paper by Dempster, Laird and Rubin
  - Maximum Likelihood Estimation from incomplete data, via the EM Algorithm, Journal of the Royal Statistical Society, Series B, 1977

- Much work on the algorithm since then

- The principles behind the algorithm existed for several years prior to the landmark paper, however.

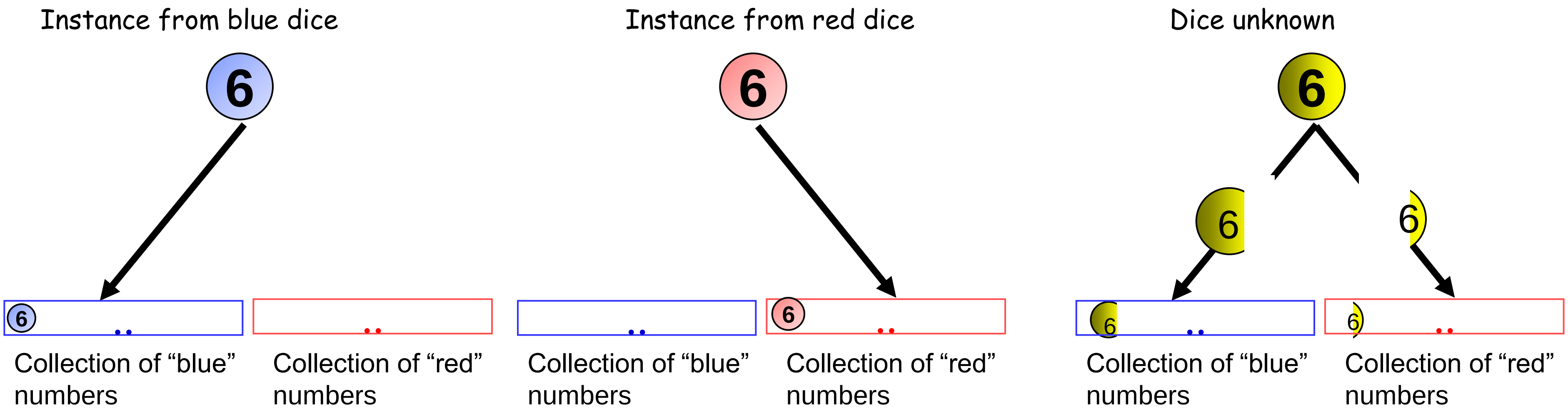
Expectation Maximization

- Iterative solution
- Get some initial estimates for all parameters
  - Dice shooter example: This includes probability distributions for dice AND the probability with which the caller selects the dice
- Two steps that are iterated:
  - ***Expectation Step:*** Estimate statistically, the values of ***unseen*** variables
  - ***Maximization Step:*** Using the estimated values of the unseen variables as truth, estimates of the model parameters

EM: The auxiliary function

- EM iteratively optimizes the following auxiliary function
- $Q(\theta, \theta') = \Sigma_Z P(Z|X,\theta') \log(P(Z,X \mid \theta))$
  - Z are the unseen variables
  - Assuming Z is discrete (may not be)
- θ' are the parameter estimates from the previous iteration
- θ are the estimates to be obtained in the current iteration

Expectation Maximization as counting



- Hidden variable: *Z*
  - Dice: The identity of the dice whose number has been called out
- If we knew *Z* for every observation, we could estimate all terms
  - By adding the observation to the right bin
- Unfortunately, we do not know *Z* – it is hidden from us!
- Solution: FRAGMENT THE OBSERVATION

Fragmenting the Observation

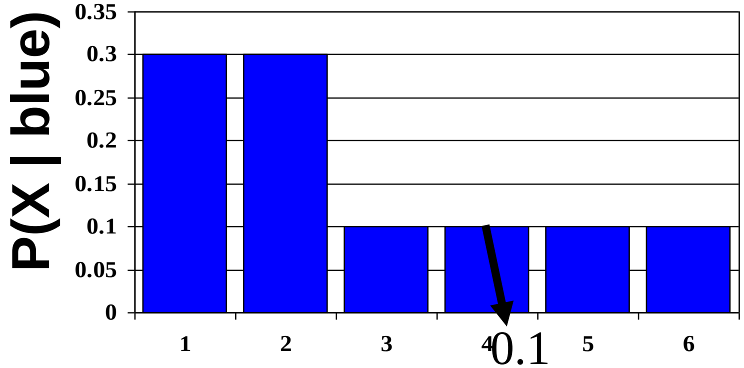
- EM is an iterative algorithm
  - At each time there is a *current* estimate of parameters
- The "size" of the fragments is proportional to the *a posteriori probability* of the component distributions
  - The *a posteriori* probabilities of the various values of *Z* are computed using Bayes' rule:

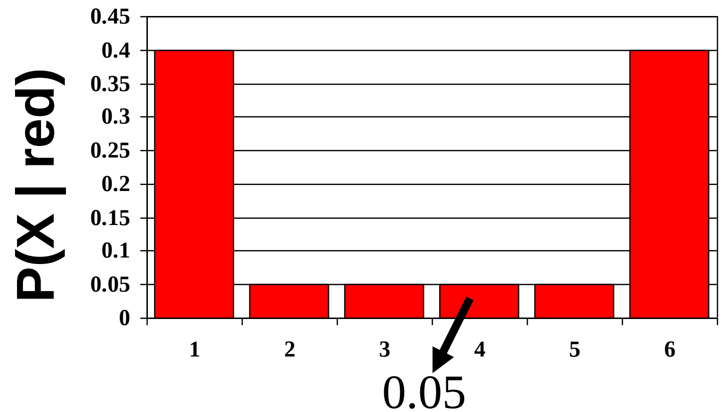
$$P(Z \mid X) = \frac{P(X \mid Z)P(Z)}{P(X)} = CP(X \mid Z)P(Z)$$

- Every dice gets a fragment of size P(dice | number)

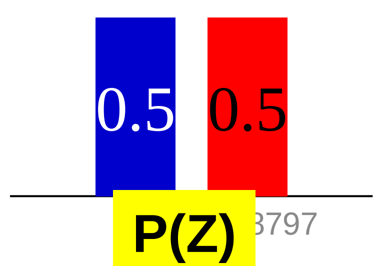
Expectation Maximization

- Hypothetical Dice Shooter Example:
- We obtain an initial estimate for the probability distribution of the two sets of dice (somehow):





- We obtain an initial estimate for the probability with which the caller calls out the two shooters (somehow)



Expectation Maximization

- Hypothetical Dice Shooter Example:
- Initial estimate:
  - P(blue) = P(red) = 0.5
  - P(4 | blue) = 0.1, for P(4 | red) = 0.05
- Caller has just called out 4
- Posterior probability of colors:

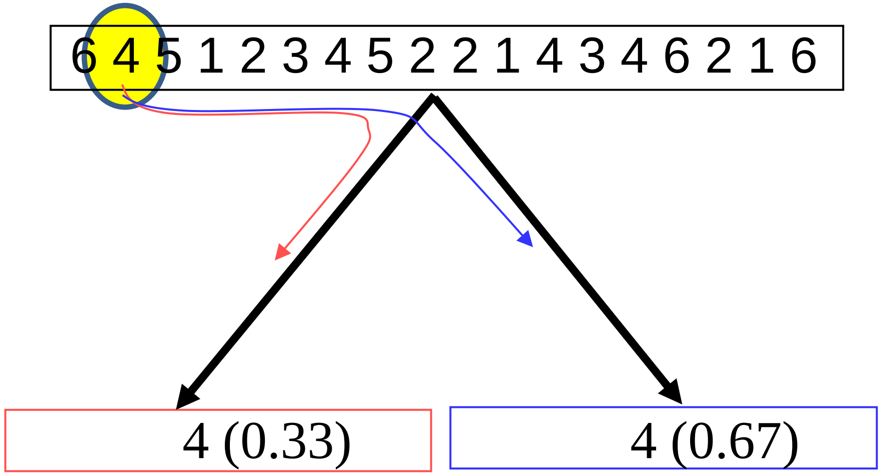
$$P(red \mid X = 4) = CP(X = 4 \mid Z = red)P(Z = red) = C \times 0.05 \times 0.5 = C\,0.025$$

$$P(blue \mid X = 4) = CP(X = 4 \mid Z = blue)P(Z = blue) = C \times 0.1 \times 0.5 = C\,0.05$$

$$P(red \mid X = 4) = \frac{C\,0.025}{C\,0.025 + C\,0.05}$$

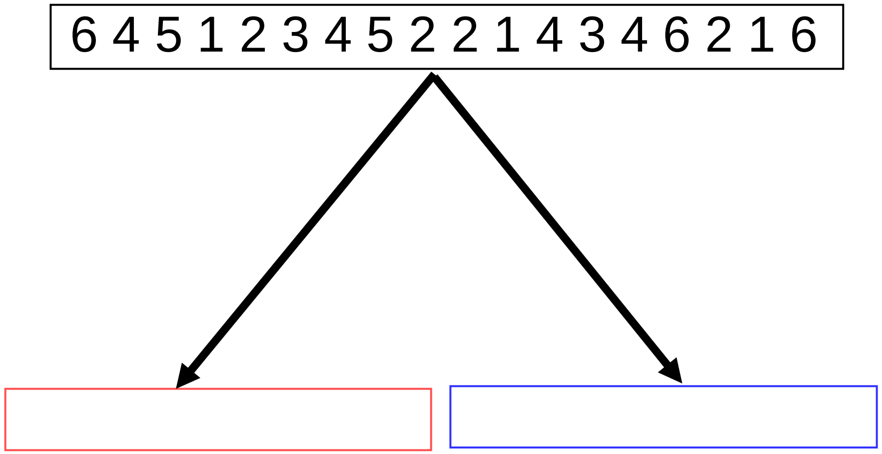
$$P(red \mid X = 4) = 0.33 \qquad P(blue \mid X = 4) = 0.67$$

Expectation Maximization



Expectation Maximization

- Every observed roll of the dice contributes to both “Red” and “Blue”



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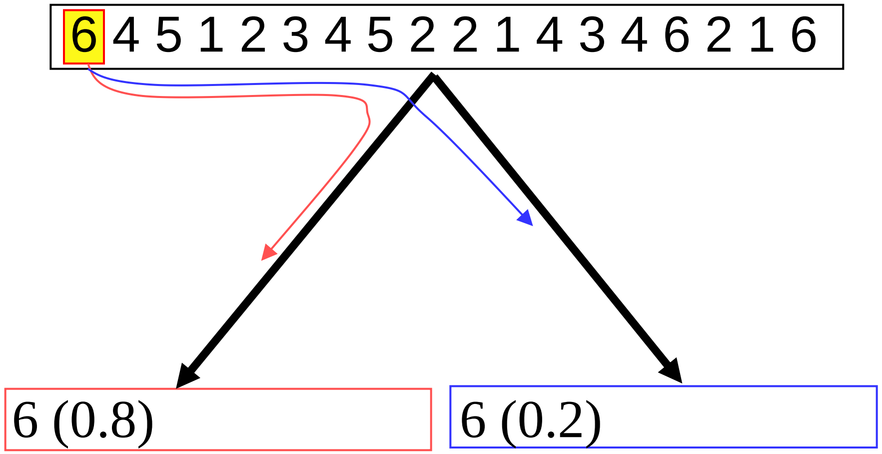

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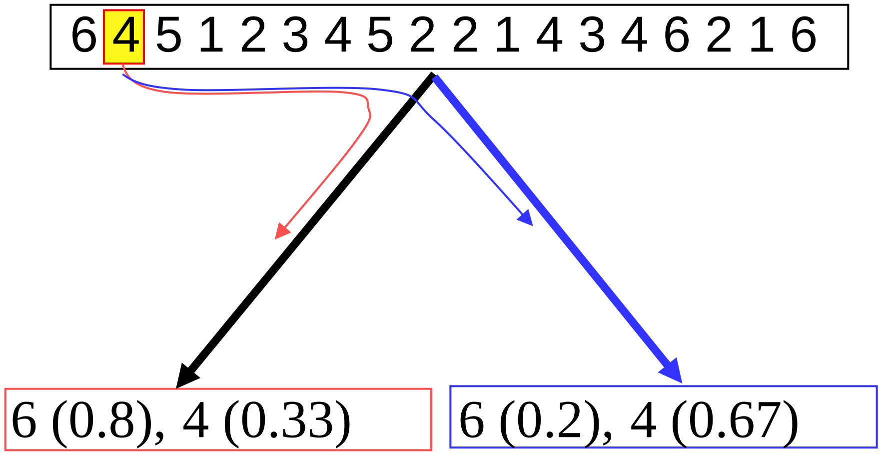

Expectation Maximization

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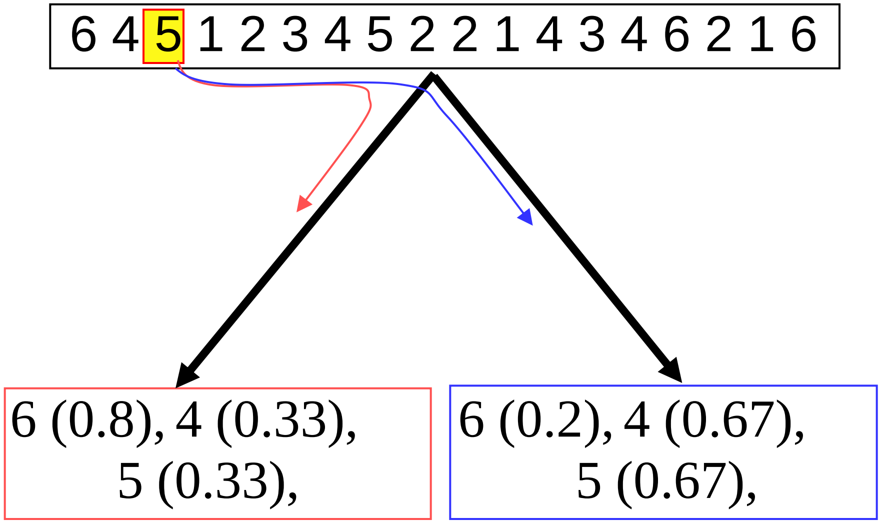


Expectation Maximization

- Every observed roll of the dice contributes to both "Red" and "Blue"

6 4 5 1 2 3 4 5 2 2 1 4 3 4 6 2 1 6

6 (0.8), 4 (0.33), 5 (0.33), 1 (0.57), 2 (0.14), 3 (0.33), 4 (0.33), 5 (0.33), 2 (0.14), 2 (0.14), 1 (0.57), 4 (0.33), 3 (0.33), 4 (0.33), 6 (0.8), 2 (0.14), 1 (0.57), 6 (0.8)

6 (0.2), 4 (0.67), 5 (0.67), 1 (0.43), 2 (0.86), 3 (0.67), 4 (0.67), 5 (0.67), 2 (0.86), 2 (0.86), 1 (0.43), 4 (0.67), 3 (0.67), 4 (0.67), 6 (0.2), 2 (0.86), 1 (0.43), 6 (0.2)

Expectation Maximization

- Every observed roll of the dice contributes to both “Red” and “Blue”
- Total count for “Red” is the sum of all the posterior probabilities in the red column
  - 7.31
- Total count for “Blue” is the sum of all the posterior probabilities in the blue column
  - 10.69
  - Note: 10.69 + 7.31 = 18 = the total number of instances

Called	P(red|X)	P(blue|X)
6	.8	.2
4	.33	.67
5	.33	.67
1	.57	.43
2	.14	.86
3	.33	.67
4	.33	.67
5	.33	.67
2	.14	.86
2	.14	.86
1	.57	.43
4	.33	.67
3	.33	.67
4	.33	.67
6	.8	.2
2	.14	.86
1	.57	.43
6	.8	.2
	7.31	10.69

Expectation Maximization

- Total count for “Red” : 7.31
- Red:
  - Total count for 1: 1.71

Called	P(red|X)	P(blue|X)
6	.8	.2
4	.33	.67
5	.33	.67
1	.57	.43
2	.14	.86
3	.33	.67
4	.33	.67
5	.33	.67
2	.14	.86
2	.14	.86
1	.57	.43
4	.33	.67
3	.33	.67
4	.33	.67
6	.8	.2
2	.14	.86
1	.57	.43
6	.8	.2
	7.31	10.69

Expectation Maximization

- Total count for “Red” : 7.31
- Red:
  - Total count for 1: 1.71
  - Total count for 2: 0.56

Called	P(red|X)	P(blue|X)
6	.8	.2
4	.33	.67
5	.33	.67
1	.57	.43
2	.14	.86
3	.33	.67
4	.33	.67
5	.33	.67
2	.14	.86
2	.14	.86
1	.57	.43
4	.33	.67
3	.33	.67
4	.33	.67
6	.8	.2
2	.14	.86
1	.57	.43
6	.8	.2
	7.31	10.69

Expectation Maximization

- Total count for “Red” : 7.31
- Red:
  - Total count for 1: 1.71
  - Total count for 2: 0.56
  - Total count for 3: 0.66

Called	P(red|X)	P(blue|X)
6	.8	.2
4	.33	.67
5	.33	.67
1	.57	.43
2	.14	.86
3	.33	.67
4	.33	.67
5	.33	.67
2	.14	.86
2	.14	.86
1	.57	.43
4	.33	.67
3	.33	.67
4	.33	.67
6	.8	.2
2	.14	.86
1	.57	.43
6	.8	.2
	7.31	10.69

Expectation Maximization

- Total count for “Red” : 7.31
- Red:
  - Total count for 1: 1.71
  - Total count for 2: 0.56
  - Total count for 3: 0.66
  - Total count for 4: 1.32

Called	P(red|X)	P(blue|X)
6	.8	.2
4	.33	.67
5	.33	.67
1	.57	.43
2	.14	.86
3	.33	.67
4	.33	.67
5	.33	.67
2	.14	.86
2	.14	.86
1	.57	.43
4	.33	.67
3	.33	.67
4	.33	.67
6	.8	.2
2	.14	.86
1	.57	.43
6	.8	.2
	7.31	10.69

Expectation Maximization

- Total count for "Red" : 7.31
- Red:
  - Total count for 1: 1.71
  - Total count for 2: 0.56
  - Total count for 3: 0.66
  - Total count for 4: 1.32
  - Total count for 5: 0.66

Called	P(red|X)	P(blue|X)
6	.8	.2
4	.33	.67
5	.33	.67
1	.57	.43
2	.14	.86
3	.33	.67
4	.33	.67
5	.33	.67
2	.14	.86
2	.14	.86
1	.57	.43
4	.33	.67
3	.33	.67
4	.33	.67
6	.8	.2
2	.14	.86
1	.57	.43
6	.8	.2
	7.31	10.69

Expectation Maximization

- Total count for “Red” : 7.31
- Red:
  - Total count for 1: 1.71
  - Total count for 2: 0.56
  - Total count for 3: 0.66
  - Total count for 4: 1.32
  - Total count for 5: 0.66
  - Total count for 6: 2.4

Called	P(red|X)	P(blue|X)
6	.8	.2
4	.33	.67
5	.33	.67
1	.57	.43
2	.14	.86
3	.33	.67
4	.33	.67
5	.33	.67
2	.14	.86
2	.14	.86
1	.57	.43
4	.33	.67
3	.33	.67
4	.33	.67
6	.8	.2
2	.14	.86
1	.57	.43
6	.8	.2
	7.31	10.69

Expectation Maximization

- Total count for “Red” : 7.31
- Red:
  - Total count for 1: 1.71
  - Total count for 2: 0.56
  - Total count for 3: 0.66
  - Total count for 4: 1.32
  - Total count for 5: 0.66
  - Total count for 6: 2.4
- **Updated probability of Red dice:**
  - **P(1 | Red) = 1.71/7.31 = 0.234**
  - **P(2 | Red) = 0.56/7.31 = 0.077**
  - **P(3 | Red) = 0.66/7.31 = 0.090**
  - **P(4 | Red) = 1.32/7.31 = 0.181**
  - **P(5 | Red) = 0.66/7.31 = 0.090**
  - **P(6 | Red) = 2.40/7.31 = 0.328**

Called	P(red|X)	P(blue|X)
6	.8	.2
4	.33	.67
5	.33	.67
1	.57	.43
2	.14	.86
3	.33	.67
4	.33	.67
5	.33	.67
2	.14	.86
2	.14	.86
1	.57	.43
4	.33	.67
3	.33	.67
4	.33	.67
6	.8	.2
2	.14	.86
1	.57	.43
6	.8	.2
	7.31	10.69

Expectation Maximization

- Total count for "Blue" : 10.69
- Blue:
  - Total count for 1: 1.29

Called	P(red|X)	P(blue|X)
6	.8	.2
4	.33	.67
5	.33	.67
1	.57	.43
2	.14	.86
3	.33	.67
4	.33	.67
5	.33	.67
2	.14	.86
2	.14	.86
1	.57	.43
4	.33	.67
3	.33	.67
4	.33	.67
6	.8	.2
2	.14	.86
1	.57	.43
6	.8	.2
	7.31	10.69

Expectation Maximization

- Total count for “Blue” : 10.69
- Blue:
  - Total count for 1: 1.29
  - Total count for 2: 3.44

Called	P(red|X)	P(blue|X)
6	.8	.2
4	.33	.67
5	.33	.67
1	.57	.43
2	.14	.86
3	.33	.67
4	.33	.67
5	.33	.67
2	.14	.86
2	.14	.86
1	.57	.43
4	.33	.67
3	.33	.67
4	.33	.67
6	.8	.2
2	.14	.86
1	.57	.43
6	.8	.2
	7.31	10.69

Expectation Maximization

- Total count for “Blue” : 10.69
- Blue:
  - Total count for 1: 1.29
  - Total count for 2: 3.44
  - Total count for 3: 1.34

Called	P(red|X)	P(blue|X)
6	.8	.2
4	.33	.67
5	.33	.67
1	.57	.43
2	.14	.86
3	.33	.67
4	.33	.67
5	.33	.67
2	.14	.86
2	.14	.86
1	.57	.43
4	.33	.67
3	.33	.67
4	.33	.67
6	.8	.2
2	.14	.86
1	.57	.43
6	.8	.2
	7.31	10.69

Expectation Maximization

- Total count for “Blue” : 10.69
- Blue:
  - Total count for 1: 1.29
  - Total count for 2: 3.44
  - Total count for 3: 1.34
  - Total count for 4: 2.68

Called	P(red|X)	P(blue|X)
6	.8	.2
4	.33	.67
5	.33	.67
1	.57	.43
2	.14	.86
3	.33	.67
4	.33	.67
5	.33	.67
2	.14	.86
2	.14	.86
1	.57	.43
4	.33	.67
3	.33	.67
4	.33	.67
6	.8	.2
2	.14	.86
1	.57	.43
6	.8	.2
	7.31	10.69

Expectation Maximization

- Total count for “Blue” : 10.69
- Blue:
  - Total count for 1: 1.29
  - Total count for 2: 3.44
  - Total count for 3: 1.34
  - Total count for 4: 2.68
  - Total count for 5: 1.34

Called	P(red|X)	P(blue|X)
6	.8	.2
4	.33	.67
5	.33	.67
1	.57	.43
2	.14	.86
3	.33	.67
4	.33	.67
5	.33	.67
2	.14	.86
2	.14	.86
1	.57	.43
4	.33	.67
3	.33	.67
4	.33	.67
6	.8	.2
2	.14	.86
1	.57	.43
6	.8	.2
	7.31	10.69

Expectation Maximization

- Total count for “Blue” : 10.69
- Blue:
  - Total count for 1: 1.29
  - Total count for 2: 3.44
  - Total count for 3: 1.34
  - Total count for 4: 2.68
  - Total count for 5: 1.34
  - Total count for 6: 0.6

Called	P(red|X)	P(blue|X)
6	.8	.2
4	.33	.67
5	.33	.67
1	.57	.43
2	.14	.86
3	.33	.67
4	.33	.67
5	.33	.67
2	.14	.86
2	.14	.86
1	.57	.43
4	.33	.67
3	.33	.67
4	.33	.67
6	.8	.2
2	.14	.86
1	.57	.43
6	.8	.2
	7.31	10.69

Expectation Maximization

- Total count for "Blue" : 10.69
- Blue:
  - Total count for 1: 1.29
  - Total count for 2: 3.44
  - Total count for 3: 1.34
  - Total count for 4: 2.68
  - Total count for 5: 1.34
  - Total count for 6: 0.6
- **Updated probability of Blue dice:**
  - **P(1 | Blue) = 1.29/11.69 = 0.122**
  - **P(2 | Blue) = 0.56/11.69 = 0.322**
  - **P(3 | Blue) = 0.66/11.69 = 0.125**
  - **P(4 | Blue) = 1.32/11.69 = 0.250**
  - **P(5 | Blue) = 0.66/11.69 = 0.125**
  - **P(6 | Blue) = 2.40/11.69 = 0.056**

Called	P(red|X)	P(blue|X)
6	.8	.2
4	.33	.67
5	.33	.67
1	.57	.43
2	.14	.86
3	.33	.67
4	.33	.67
5	.33	.67
2	.14	.86
2	.14	.86
1	.57	.43
4	.33	.67
3	.33	.67
4	.33	.67
6	.8	.2
2	.14	.86
1	.57	.43
6	.8	.2
	7.31	10.69

Expectation Maximization

- Total count for “Red” : 7.31
- Total count for “Blue” : 10.69
- Total instances = 18
  - Note 7.31+10.69 = 18
- We also revise our estimate for the probability that the caller calls out Red or Blue
  - i.e the fraction of times that he calls Red and the fraction of times he calls Blue
- P(Z=Red) = 7.31/18 = 0.41
- P(Z=Blue) = 10.69/18 = 0.59

Called	P(red|X)	P(blue|X)
6	.8	.2
4	.33	.67
5	.33	.67
1	.57	.43
2	.14	.86
3	.33	.67
4	.33	.67
5	.33	.67
2	.14	.86
2	.14	.86
1	.57	.43
4	.33	.67
3	.33	.67
4	.33	.67
6	.8	.2
2	.14	.86
1	.57	.43
6	.8	.2
	7.31	10.69

The updated values

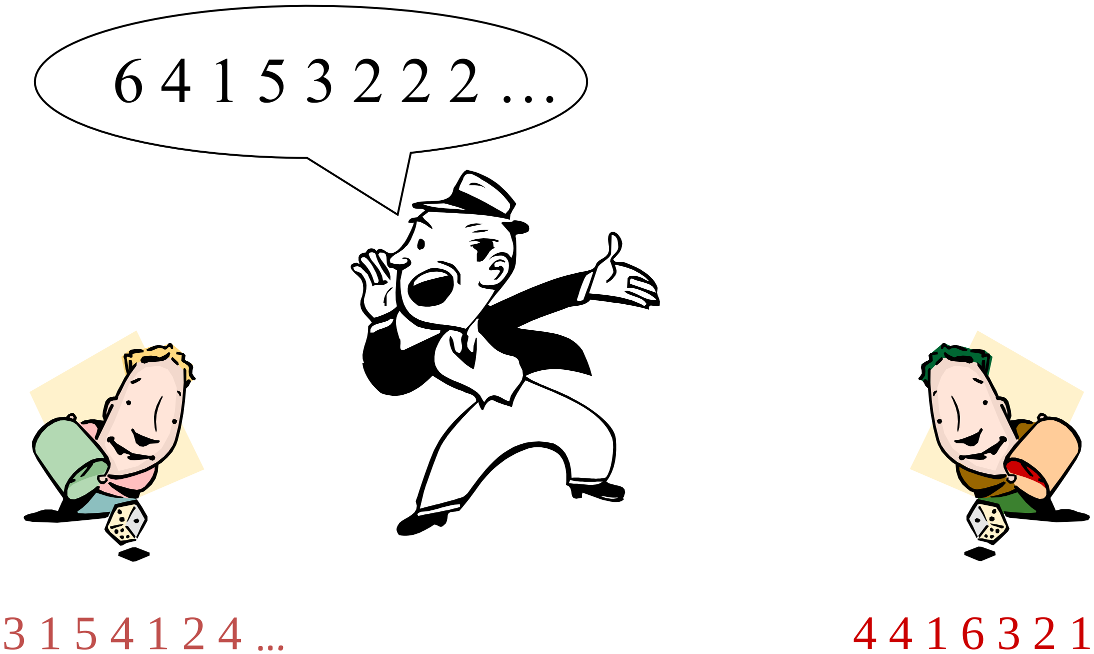
- Probability of Red dice:
  - P(1 | Red) = 1.71/7.31 = 0.234
  - P(2 | Red) = 0.56/7.31 = 0.077
  - P(3 | Red) = 0.66/7.31 = 0.090
  - P(4 | Red) = 1.32/7.31 = 0.181
  - P(5 | Red) = 0.66/7.31 = 0.090
  - P(6 | Red) = 2.40/7.31 = 0.328
- Probability of Blue dice:
  - P(1 | Blue) = 1.29/11.69 = 0.122
  - P(2 | Blue) = 0.56/11.69 = 0.322
  - P(3 | Blue) = 0.66/11.69 = 0.125
  - P(4 | Blue) = 1.32/11.69 = 0.250
  - P(5 | Blue) = 0.66/11.69 = 0.125
  - P(6 | Blue) = 2.40/11.69 = 0.056

- P(Z=Red) = 7.31/18 = 0.41
- P(Z=Blue) = 10.69/18 = 0.59

Called	P(red|X)	P(blue|X)
6	.8	.2
4	.33	.67
5	.33	.67
1	.57	.43
2	.14	.86
3	.33	.67
4	.33	.67
5	.33	.67
2	.14	.86
2	.14	.86
1	.57	.43
4	.33	.67
3	.33	.67
4	.33	.67
6	.8	.2
2	.14	.86
1	.57	.43
6	.8	.2

THE UPDATED VALUES CAN BE USED TO REPEAT THE PROCESS. ESTIMATION IS AN ITERATIVE PROCESS

The Dice Shooter Example



1. Initialize $P(Z)$, $P(X \mid Z)$
2. Estimate $P(Z \mid X)$ for each Z, for each called out number
   - Associate X with each value of Z, with weight $P(Z \mid X)$
3. Re-estimate $P(X \mid Z)$ for every value of X and Z
4. Re-estimate $P(Z)$
5. If not converged, return to 2

In Squiggles

- Given a sequence of observations $O_1, O_2, ..$
  - N_X is the number of observations of number X
- Initialize P(Z), P(X|Z) for dice Z and numbers X
- Iterate:
  - For each number X:

$$P(Z \mid X) = \frac{P(X \mid Z)P(Z)}{\sum_{Z'} P(Z')P(X \mid Z')}$$

  - Update:

$$P(X \mid Z) = \frac{\sum_{O \text{ such that } O==X} P(Z \mid X)}{\sum_{O} P(Z \mid O)} = \frac{N_X P(Z \mid X)}{\sum_{X} N_X P(Z \mid X)}$$

$$P(Z) = \frac{\sum_{X} N_X P(Z \mid X)}{\sum_{Z'} \sum_{X} N_X P(Z' \mid X)}$$

Solutions may not be unique

- The EM algorithm will give us one of many solutions, all equally valid!
  - The probability of 6 being called out:

$$P(6) = \alpha P(6 \mid red) + \beta P(6 \mid blue) = \alpha P_r + \beta P_b$$

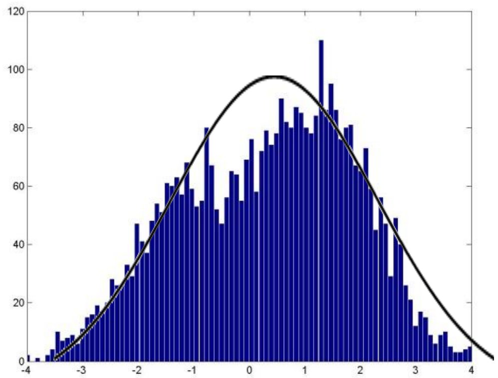
    - Assigns P_r as the probability of 6 for the red die
    - Assigns P_b as the probability of 6 for the blue die

  - The following too is a valid solution [FIX]

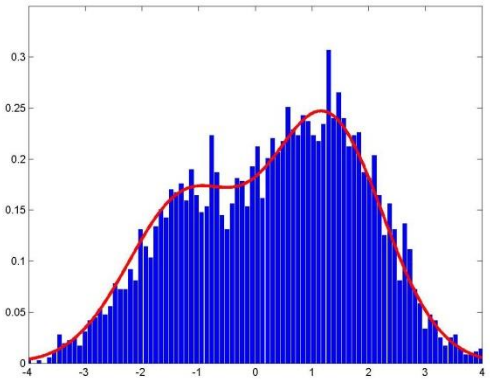
$$P(6) = 1.0\left(\alpha P_r + \beta P_b\right) + 0.0 anything$$

    - Assigns 1.0 as the a priori probability of the red die
    - Assigns 0.0 as the probability of the blue die
- The solution is NOT unique

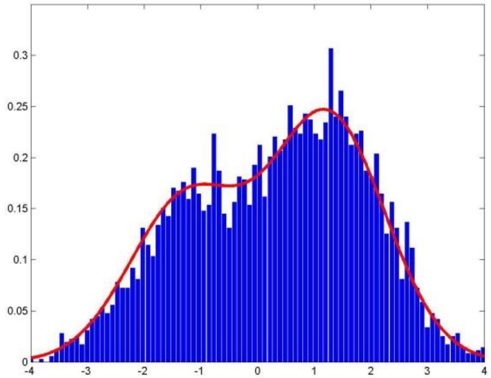
A more complex model: Gaussian mixtures



- A Gaussian mixture can represent data distributions far better than a simple Gaussian
- The two panels show the histogram of an unknown random variable
- The first panel shows how it is modeled by a simple Gaussian
- The second panel models the histogram by a mixture of two Gaussians
- Caveat: It is hard to know the optimal number of Gaussians in a mixture



A More Complex Model



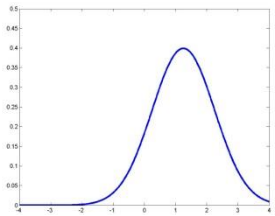
$$P(X) = \sum_k P(k) N(X; \mu_k, \Theta_k) = \sum_k \frac{P(k)}{\sqrt{(2\pi)^d |\Theta_k|}} \exp\left(-0.5(X-\mu_k)^T \Theta_k^{-1} (X-\mu_k)\right)$$

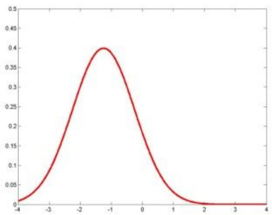
- Gaussian mixtures are often good models for the distribution of multivariate data
- Problem: Estimating the parameters, given a collection of data

Gaussian Mixtures: Generating model

$$P(X) = \sum_k P(k) N(X; \mu_k, \Theta_k)$$

6.1 1.4 5.3 1.9 4.2 2.2 4.9 0.5





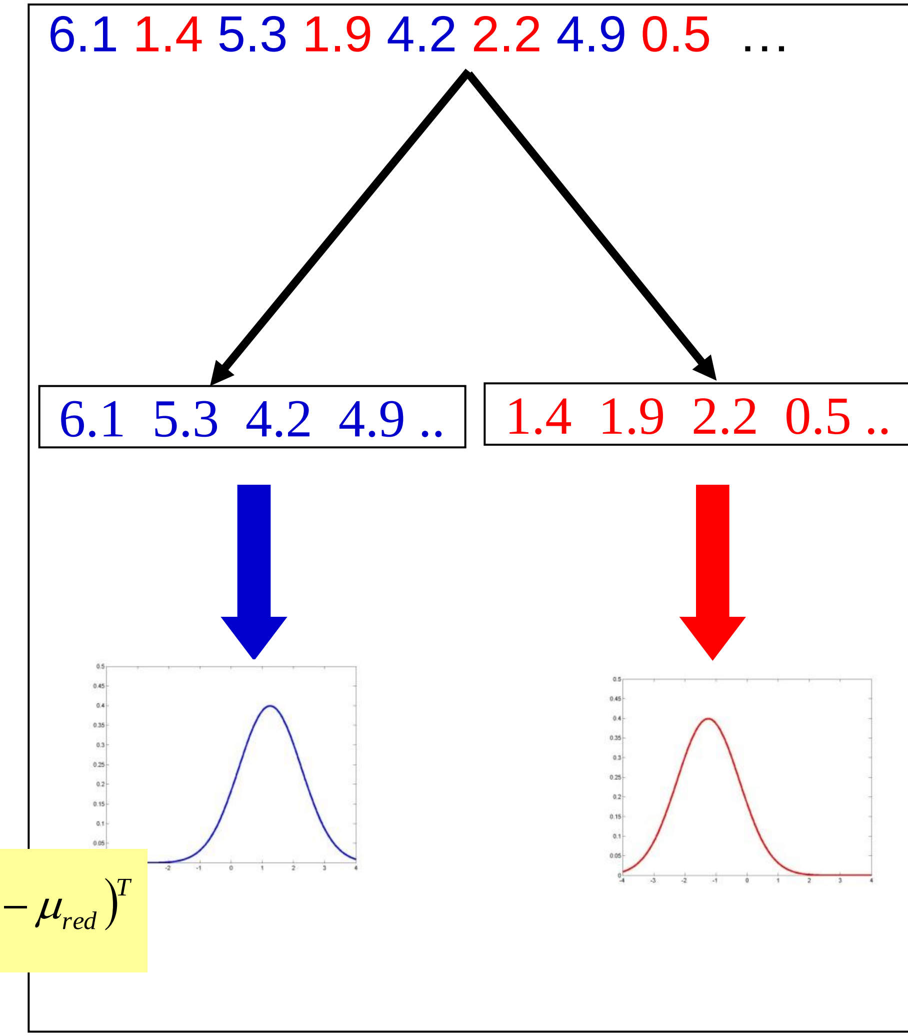
- The caller now has two Gaussians
  - At each draw he randomly selects a Gaussian, by the mixture weight distribution
  - He then draws an observation from that Gaussian
  - Much like the dice problem (only the outcomes are now real numbers and can be anything)

Estimating GMM with complete information

- Observation: A collection of numbers drawn from a mixture of 2 Gaussians
  - As indicated by the colors, we know which Gaussian generated what number

- Segregation: Separate the blue observations from the red

- From each set compute parameters for that Gaussian



$$\mu_{red} = \frac{1}{N_{red}} \sum_{i \in red} X_i$$

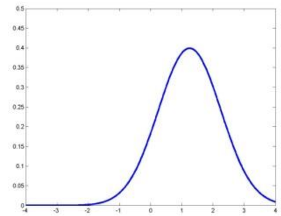
$$\Theta_{red} = \frac{1}{N_{red}} \sum_{i \in red} (X_i - \mu_{red})(X_i - \mu_{red})^T$$

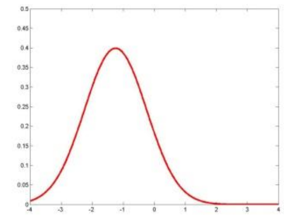
$$P(red) = \frac{N_{red}}{N}$$

Gaussian Mixtures: Generating model

$$P(X) = \sum_k P(k) N(X; \mu_k, \Theta_k)$$

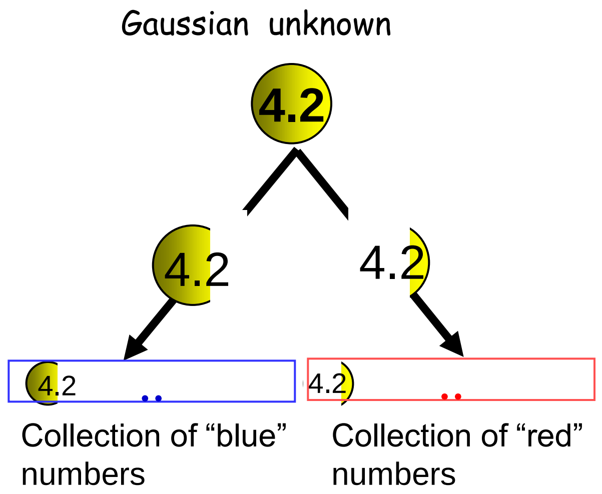
6.1 1.4 5.3 1.9 4.2 2.2 4.9 0.5





- Problem: In reality we will not know which Gaussian any observation was drawn from..
  - The color information is missing

Fragmenting the observation



- The identity of the Gaussian is not known!
- Solution: **Fragment the observation**
- Fragment size proportional to *a posteriori* probability

$$P(k \mid X) = \frac{P(X \mid k)P(k)}{\sum_{k'} P(k')P(X \mid k')} = \frac{P(k)N(X; \mu_k, \Theta_k)}{\sum_{k'} P(k')N(X; \mu_{k'}, \Theta_{k'})}$$

Expectation Maximization

- Initialize P(k), μ_k and Θ_k for both Gaussians
  - Important how we do this
  - Typical solution: Initialize means randomly, Θ_k as the global covariance of the data and P(k) uniformly
- Compute fragment sizes for each Gaussian, for each observation

Number	P(red|X)	P(blue|X)
6.1	.81	.19
1.4	.33	.67
5.3	.75	.25
1.9	.41	.59
4.2	.64	.36
2.2	.43	.57
4.9	.66	.34
0.5	.05	.95

$$P(k \mid X) = \frac{P(k) N(X; \mu_k, \Theta_k)}{\sum_{k'} P(k') N(X; \mu_{k'}, \Theta_{k'})}$$

Expectation Maximization

- ***Each observation contributes only as much as its fragment size to each statistic***
- Mean(red) = (6.1*0.81 + 1.4*0.33 + 5.3*0.75 + 1.9*0.41 + 4.2*0.64 + 2.2*0.43 + 4.9*0.66 + 0.5*0.05) / (0.81 + 0.33 + 0.75 + 0.41 + 0.64 + 0.43 + 0.66 + 0.05)
  = 17.05 / 4.08 = 4.18

Number	P(red|X)	P(blue|X)
6.1	.81	.19
1.4	.33	.67
5.3	.75	.25
1.9	.41	.59
4.2	.64	.36
2.2	.43	.57
4.9	.66	.34
0.5	.05	.95
	4.08	3.92

- Var(red) = ((6.1-4.18)^2*0.81 + (1.4-4.18)^2*0.33 + (5.3-4.18)^2*0.75 + (1.9-4.18)^2*0.41 + (4.2-4.18)^2*0.64 + (2.2-4.18)^2*0.43 + (4.9-4.18)^2*0.66 + (0.5-4.18)^2*0.05) / (0.81 + 0.33 + 0.75 + 0.41 + 0.64 + 0.43 + 0.66 + 0.05)

$$P(red) = \frac{4.08}{8}$$

EM for Gaussian Mixtures

1. Initialize P(k), μ_k and Θ_k for all Gaussians
2. For each observation X compute *a posteriori* probabilities for all Gaussian

$$P(k \mid X) = \frac{P(k)N(X; \mu_k, \Theta_k)}{\sum_{k'} P(k')N(X; \mu_{k'}, \Theta_{k'})}$$

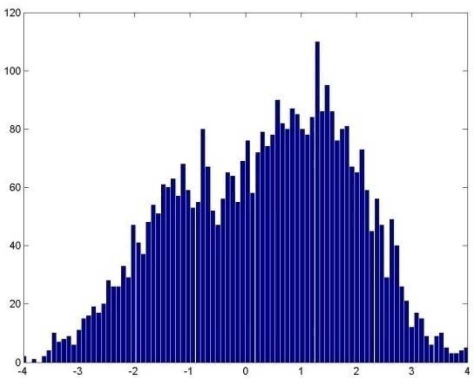
3. Update mixture weights, means and variances for all Gaussians

$$P(k) = \frac{\sum_X P(k|X)}{N} \qquad \mu_k = \frac{\sum_X P(k|X)\, X}{\sum_X P(k|X)} \qquad \Theta_k = \frac{\sum_X P(k|X)\,(X - \mu_k)^2}{\sum_X P(k|X)}$$

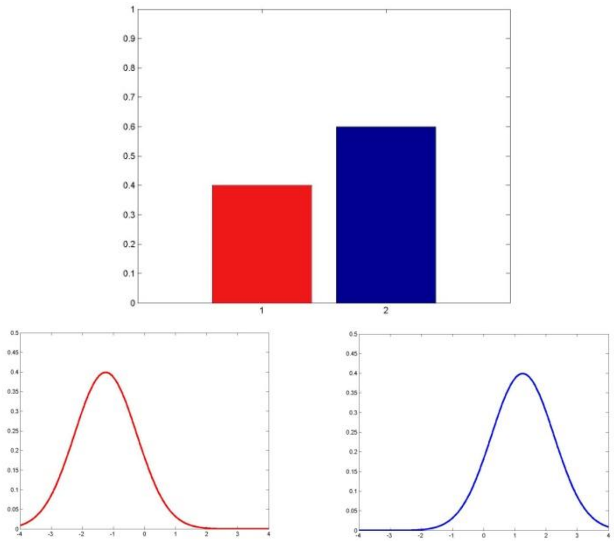
4. If not converged, return to 2

EM estimation of Gaussian Mixtures

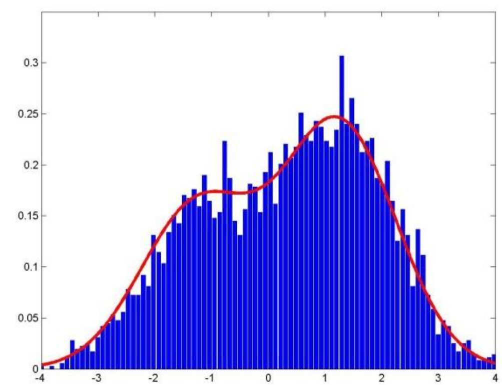
- An Example



Histogram of 4000 instances of a randomly generated data



Individual parameters of a two-Gaussian mixture estimated by EM



Two-Gaussian mixture estimated by EM

Expectation Maximization

- The same principle can be extended to mixtures of other distributions.

- E.g. Mixture of Laplacians: Laplacian parameters become

$$\mu_k = median(P(k \mid x)) \quad b_k = \frac{1}{\sum_x P(k \mid x)} \sum_x P(k \mid x) \mid x - \mu_k \mid$$

- In a mixture of Gaussians and Laplacians, Gaussians use the Gaussian update rules, Laplacians use the Laplacian rule

Expectation Maximization

- The EM algorithm is used whenever proper statistical analysis of a phenomenon requires the knowledge of a hidden or missing variable (or a set of hidden/missing variables)
  – The hidden variable is often called a “latent” variable

- Some examples:
  – Estimating mixtures of distributions
    - Only data are observed. The individual distributions and mixing proportions must both be learnt.
  – Estimating the distribution of data, when some attributes are missing
  – Estimating the dynamics of a system, based only on observations that may be a complex function of system state

Solve this problem:

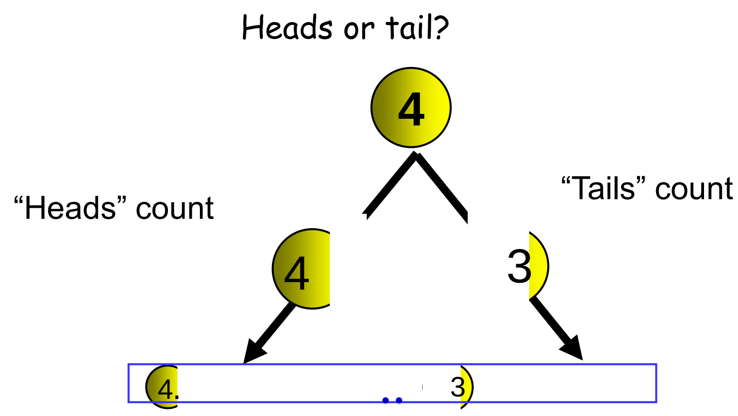
- Problem 1:
  - Caller rolls a dice and flips a coin

  - He calls out the number rolled if the coin shows head
  - Otherwise he calls the number+1
  - Determine p(heads) and p(number) for the dice from a collection of outputs

- Problem 2:

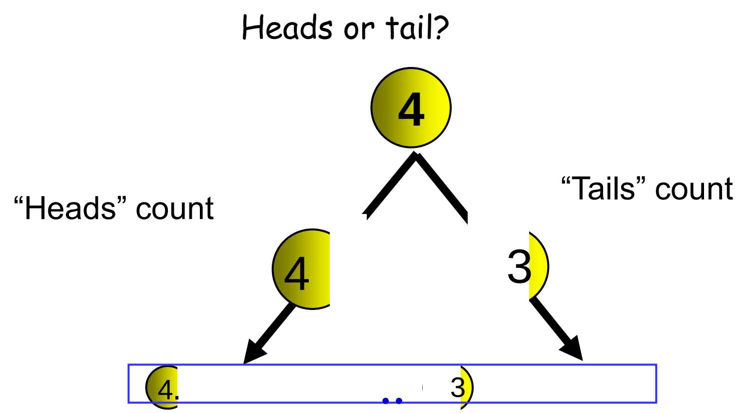
  - Caller rolls two dice
  - He calls out the sum
  - Determine P(dice) from a collection of ouputs

The dice and the coin



- Unknown: Whether it was head or tails

The dice and the coin

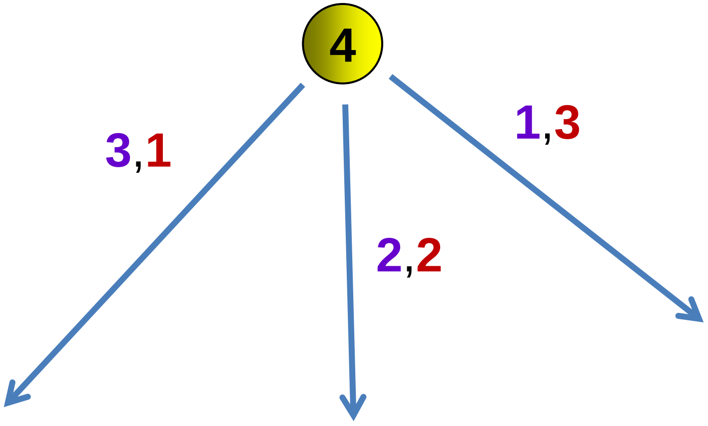


- Unknown: Whether it was head or tails

$$P(heads \mid N) = \frac{P(N)P(heads)}{P(N)P(heads) + P(N-1)P(tails)}$$

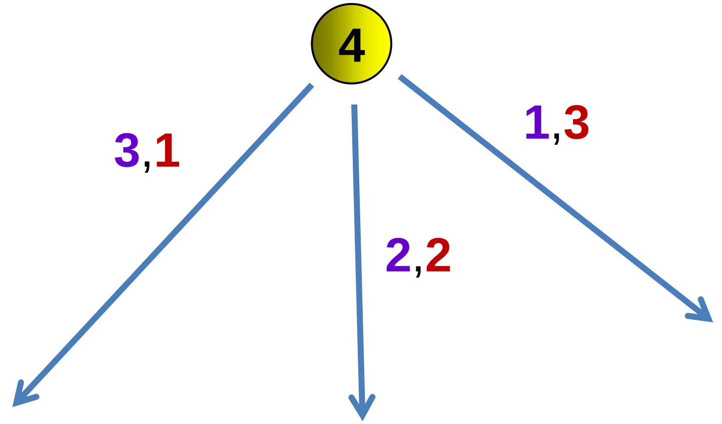
$$count(N) = \#N.P(heads \mid N) + \#(N-1).P(tails \mid N-1)$$

The two dice



- Unknown: How to partition the number
- $Count_{blue}(3) \mathrel{+}= P(3,1 \mid 4)$
- $Count_{blue}(2) \mathrel{+}= P(2,2 \mid 4)$
- $Count_{blue}(1) \mathrel{+}= P(1,3 \mid 4)$

The two dice



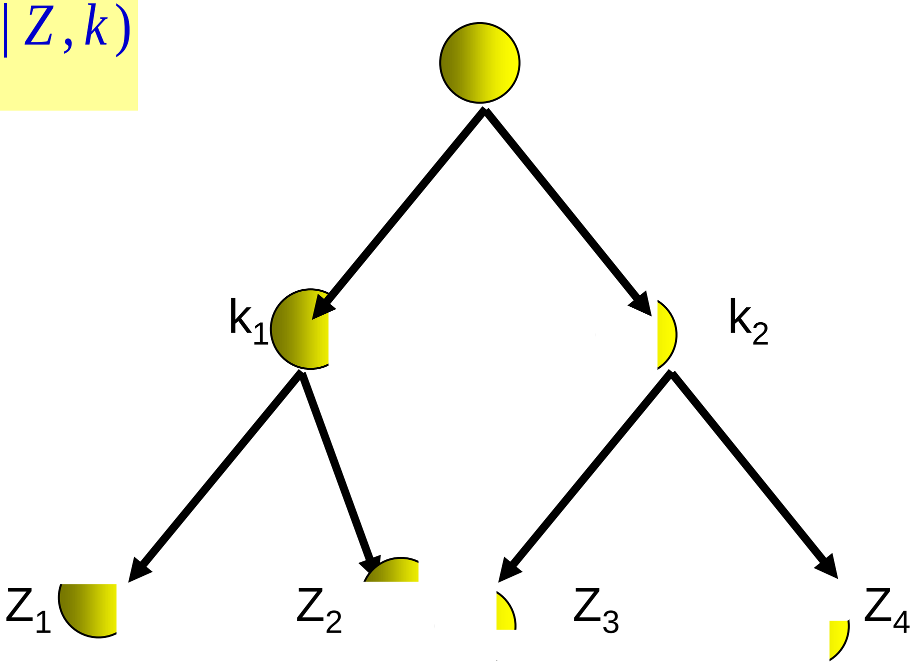
- Update rules

$$P(N, K-N \mid K) = \frac{P_1(N)P_2(K-N)}{\sum_{J=1}^{6} P_1(J)P_2(K-J)}$$

$$count_1(N) = \sum_{K=2}^{12} \#K.P(N, K-N \mid K)$$

Fragmentation can be hierarchical

$$P(X) = \sum_k P(k) \sum_Z P(Z \mid k) P(X \mid Z, k)$$



- E.g. mixture of mixtures
- Fragments are further fragmented..
  – Work this out

More later

- Will see a couple of other instances of the use of EM
- EM for signal representation: PCA and factor analysis
- EM for signal separation
- EM for parameter estimation

- EM for homework..