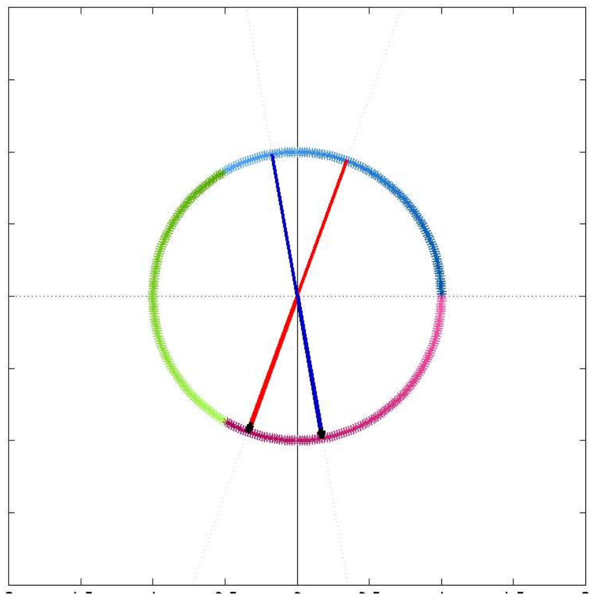
Machine Learning for Signal Processing

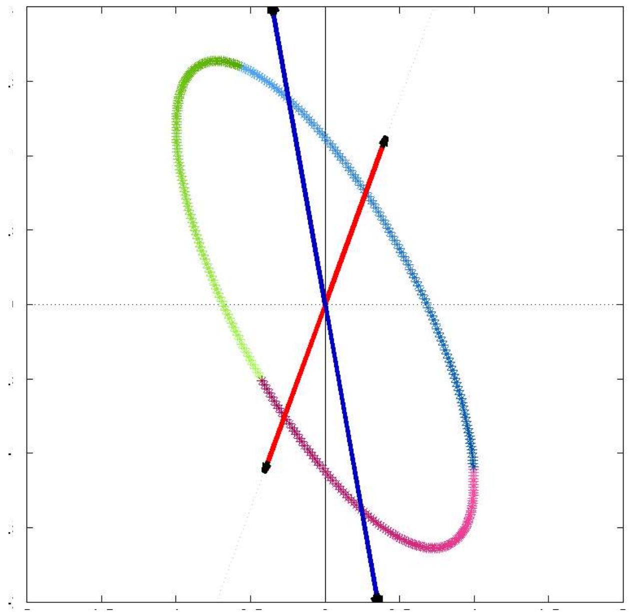
Data driven representations: 1. Eigenrepresentations

Instructor: Bhiksha Raj

Linear Algebra Reminders: 1

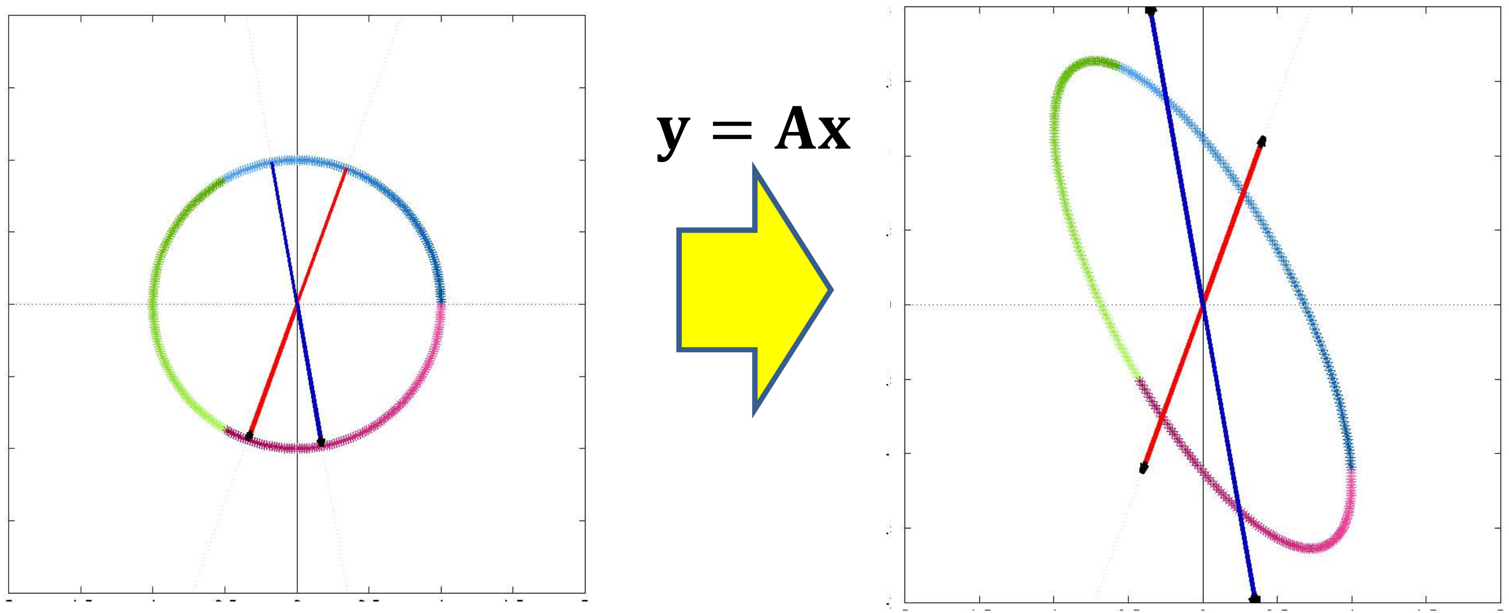


$\mathbf{y} = \mathbf{A}\mathbf{x}$



- A matrix transforms a sphereoid to an ellipsoid
- The Eigenvectors of the matrix are the vectors who do not change direction during this transformation

Linear Algebra Reminders: 1.5



- Any square matrix $\mathbf{A}$ can be "Eigen decomposed" as

$$\mathbf{A} = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1}$$

  - $\mathbf{V}$ is the set of Eigen vectors. $\boldsymbol{\Lambda}$ is a diagonal matrix of scaling terms

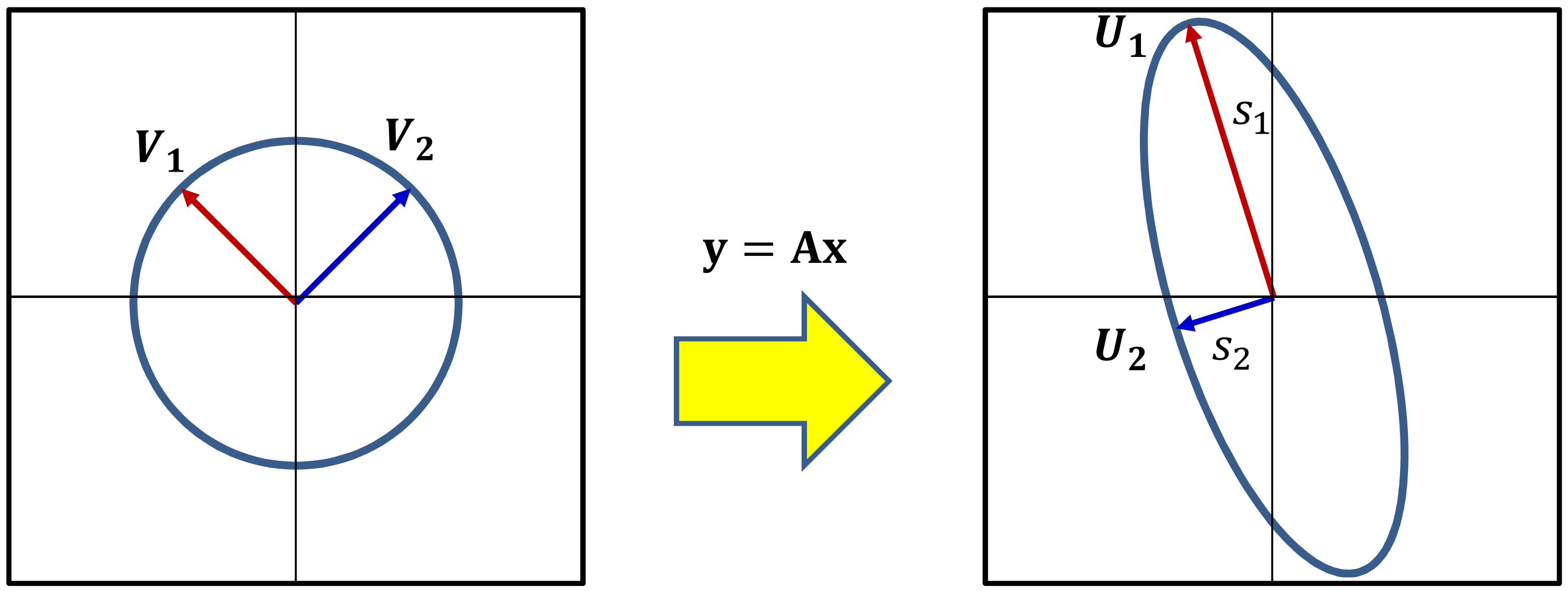
- If $\mathbf{A}$ is symmetric, we will get

$$\mathbf{A} = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{T}$$

  - The vectors in $\mathbf{V}$ are orthogonal to one another. $\mathbf{V}$ is an *orthogonal matrix*
  - $\mathbf{V}\mathbf{V}^{T} = \mathbf{V}^{T}\mathbf{V} = \mathbf{I}$

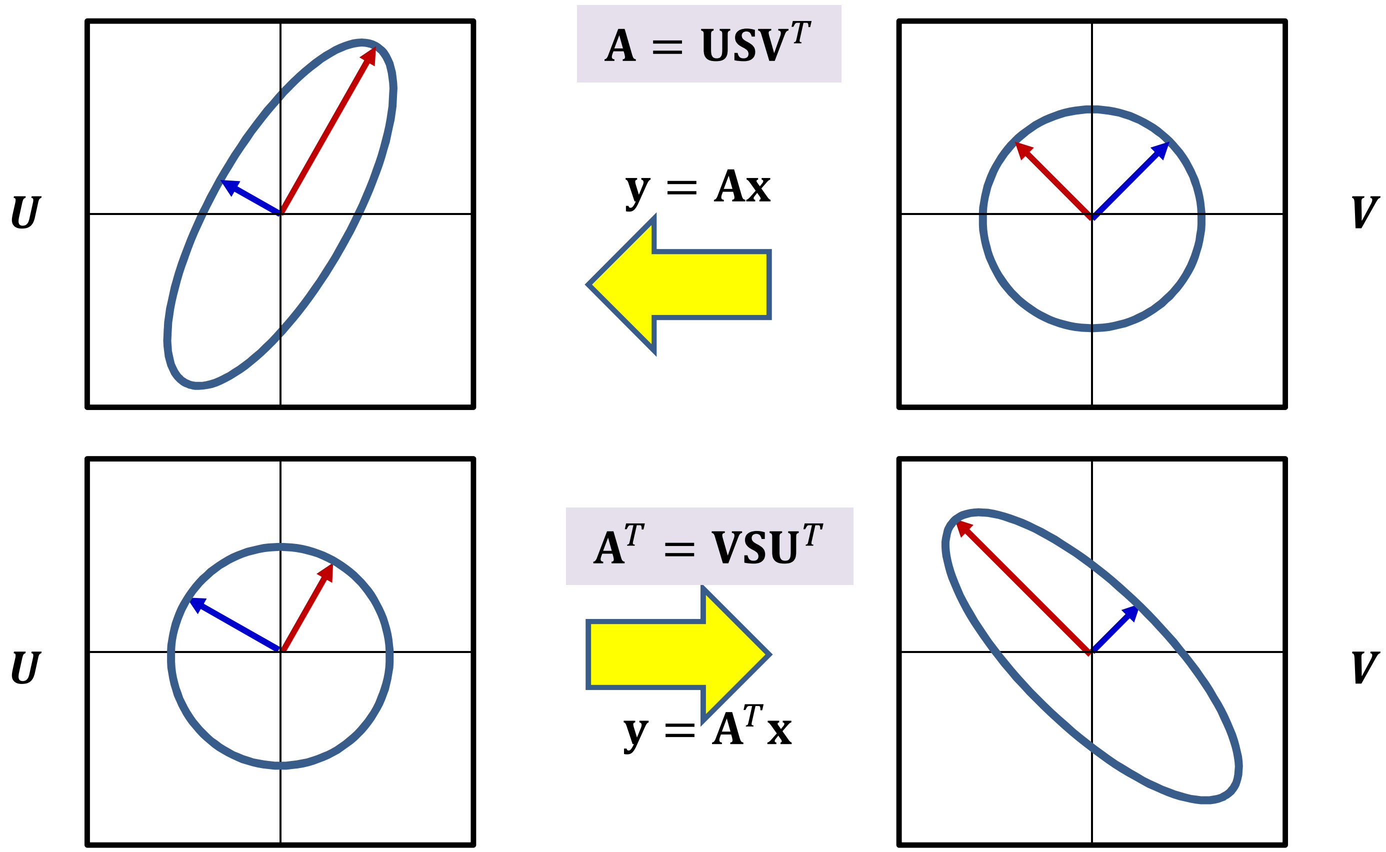
Linear Algebra Reminders: 2

$$\mathbf{A} = \mathbf{U}\mathbf{S}\mathbf{V}^T$$



- A matrix transforms the orthogonal set of right singular vectors to the orthogonal set of left singular vectors
  - These are the major axes of the ellipsoid obtained from the sphereoid
  - The scaling factors are the singular values

Linear Algebra Reminders: 2

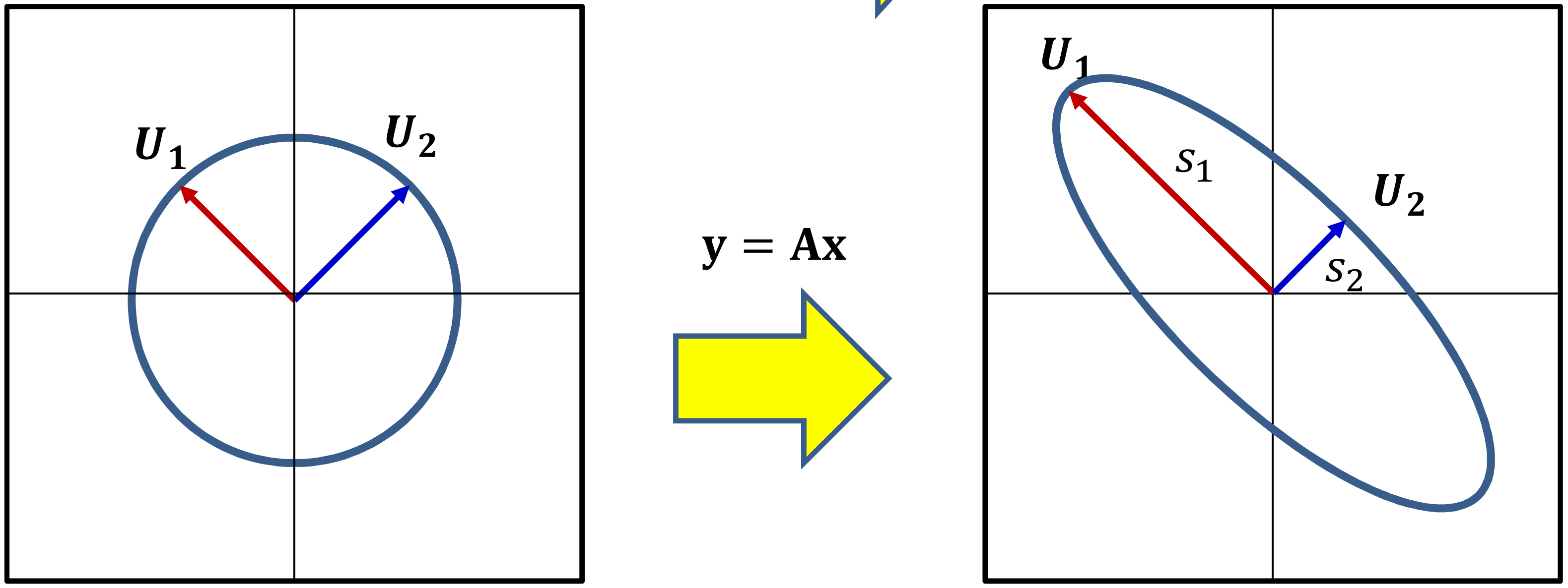


$\mathbf{A} = \mathbf{USV}^T$
$\mathbf{y} = \mathbf{Ax}$
U
V
$\mathbf{A}^T = \mathbf{VSU}^T$
$\mathbf{y} = \mathbf{A}^T\mathbf{x}$
U
V

- A matrix transforms the orthogonal set of right singular vectors to the orthogonal set of left singular vectors
  - These are the major axes of the ellipsoid obtained from the sphereoid
  - The scaling factors are the singluar values
- The *transpose* of a matrix transforms the left singular vectors to the right singular vectors

Linear Algebra Reminders: 3

$\mathbf{A} = \mathbf{A}^T$ $\mathbf{USV}^T = \mathbf{VSU}^T$ $\Rightarrow$ $\mathbf{U} = \mathbf{V}$ $\boldsymbol{A} = \mathbf{USU}^T$



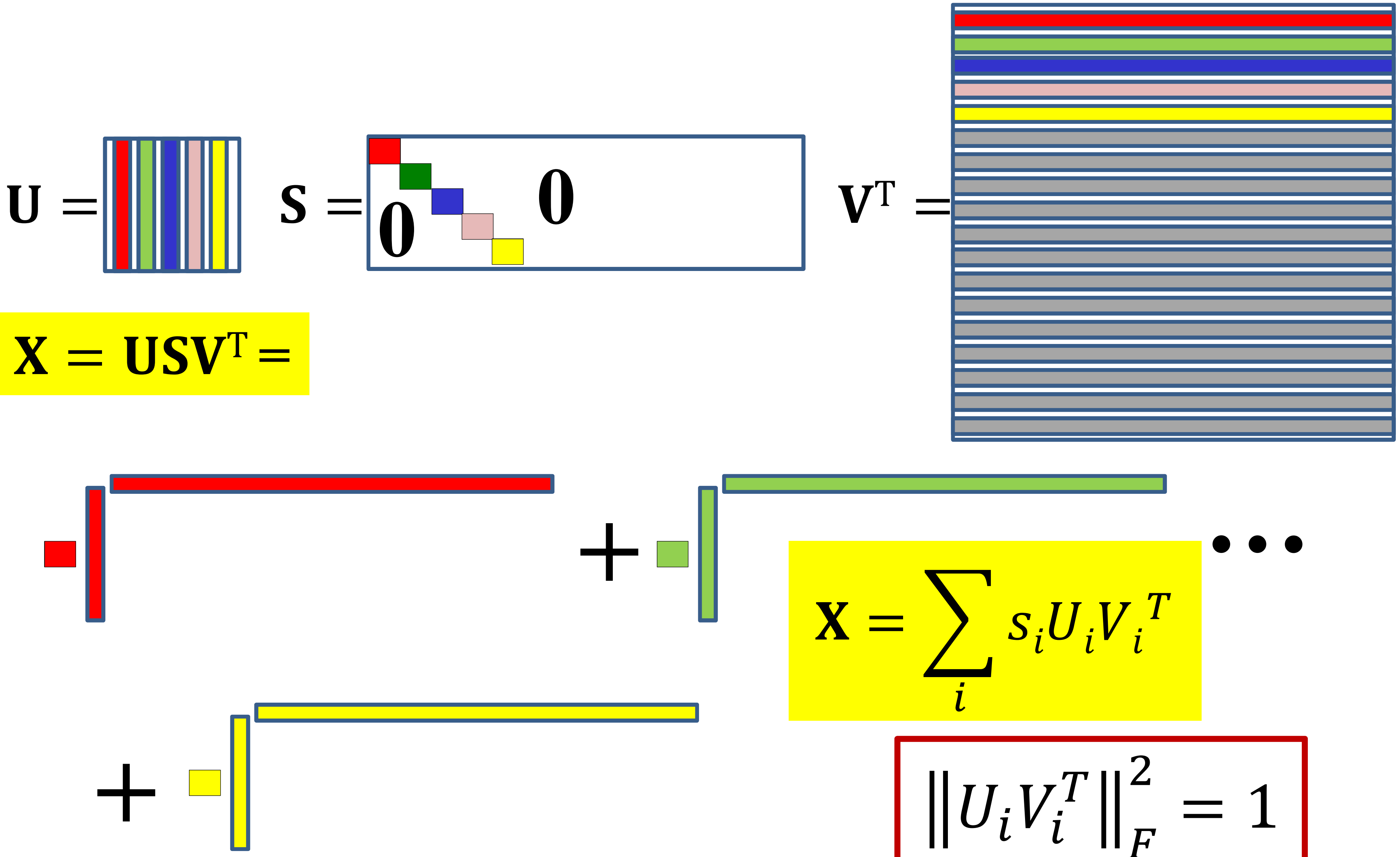
- For a symmetric matrix left and right singular vectors are identical
  - Orthogonal vectors which do not change direction from the transform
  - These are the major axes of the ellipsoid obtained from a sphereoid
- These are also the *eigenvectors* of the matrix
  - Since they do not change direction
  - SVD gives you Eigen decomposition, with $\boldsymbol{\Lambda} = \mathbf{S}^2$

Linear Algebra Reminders: 4 –> SVD

- SVD decomposes a matrix into a the sum of a sequence of “unit-energy” matrices weighted by the corresponding singular values

- Retaining only the “high-singular-value” components retains most of the energy in the matrix

SVD on data-container matrices

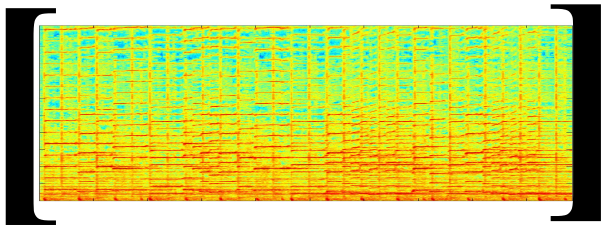


$$\mathbf{X} = \mathbf{USV}^{\mathrm{T}} =$$

$$\mathbf{X} = \sum_i s_i U_i V_i^T$$

$$\left\| U_i V_i^T \right\|_F^2 = 1$$

SVD decomposes the data



$$\mathbf{X} = [X_1\ X_2\ \ \cdots X_N]$$

$$\mathbf{X} = \mathbf{USV}^\mathrm{T}$$

$$\mathbf{X} = s_1 U_1 V_1^T + s_2 U_2 V_2^T + s_3 U_3 V_3^T + s_4 U_4 V_4^T + \ldots$$

- Each left singular vector and the corresponding right singular vector contribute one “basic” component to the data
- The “magnitude” of its contribution is the corresponding singular value

Expanding the SVD

$$\mathbf{X} = s_1 U_1 V_1^T + s_2 U_2 V_2^T + s_3 U_3 V_3^T + s_4 U_4 V_4^T + \ldots$$

- Each left singular vector and the corresponding right singular vector contribute on “basic” component to the data
- The “magnitude” of its contribution is the corresponding singular value
- Low singular-value components contribute little, if anything
  - Carry little information
  - Are often just “noise” in the data

Expanding the SVD

$$\mathbf{X} = s_1 U_1 V_1^T + s_2 U_2 V_2^T + s_3 U_3 V_3^T + s_4 U_4 V_4^T + \ldots$$

$$\mathbf{X} \approx s_1 U_1 V_1^T + s_2 U_2 V_2^T$$

- Low singular-value components contribute little, if anything
  - Carry little information
  - Are often just "noise" in the data
- Data can be recomposed using only the "major" components with minimal change of value
  - Minimum squared error between original data and recomposed data
  - Sometimes eliminating the low-singular-value components will, in fact "clean" the data

Linear Algebra recall

- What is $\mathbf{x}^T\mathbf{y}$
  - When $\mathbf{y}$ is unit length

Linear Algebra recall

- What is $\mathbf{x}^T\mathbf{y}$
  - When $\mathbf{y}$ is unit length

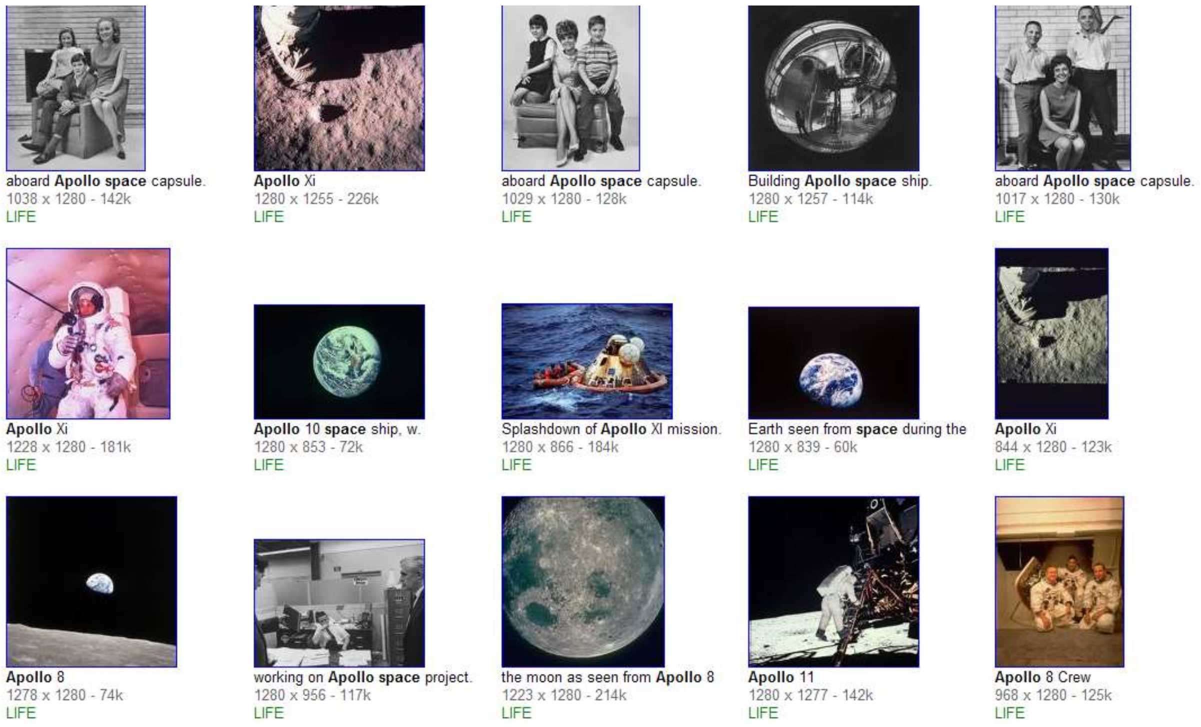
- What is the projection of $\mathbf{x}$ onto $\mathbf{y}$
  - When $\mathbf{y}$ is unit length

Linear Algebra recall

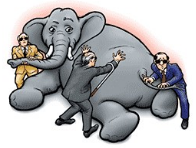
- What is $\mathbf{x}^T\mathbf{y}$
  - When $\mathbf{y}$ is unit length
- What is the projection of $\mathbf{x}$ onto $\mathbf{y}$
  - When $\mathbf{y}$ is unit length
- What is the projection of $\mathbf{x}$ onto $\mathbf{Y} = [\mathbf{y}_1\mathbf{y}_1 \ldots \mathbf{y}_K]$
  - When$\mathbf{Y}$ is an orthogonal matrix

- **On with the topic for today…**

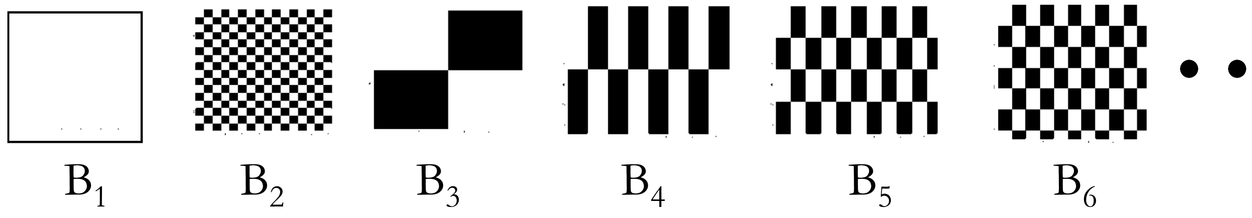
Recall: Representing images



- The most common element in the image: background
  - Or rather large regions of relatively featureless shading
  - Uniform sequences of numbers



Adding more bases



- Checkerboards with different variations

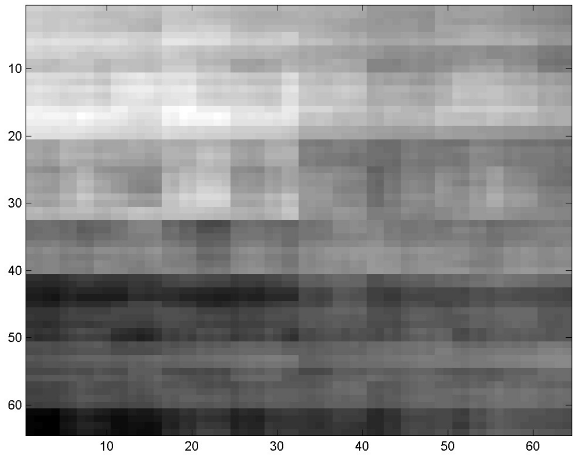
$$\mathrm{Im}age \approx w_1B_1 + w_2B_2 + w_3B_3 + \ldots$$

$$W = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ . \\ . \end{bmatrix} \qquad B = [B_1 \;\; B_2 \;\; B_3]$$

$$BW \approx \mathrm{Im}age$$

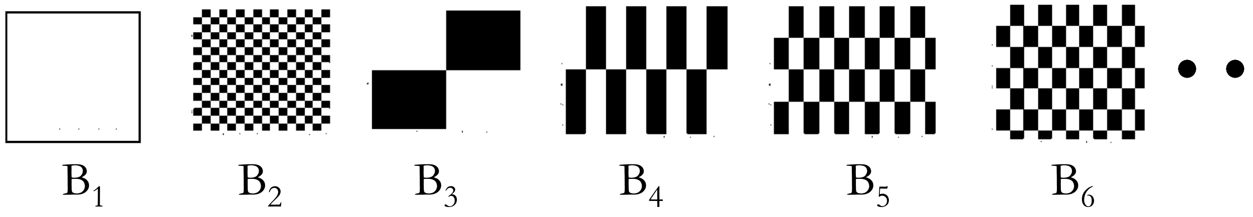
$$W = pinv(B)\,\mathrm{Im}age$$

$$PROJECTION = BW$$



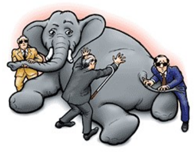
Getting closer at 625 bases!

"Bases"



$$image \approx w_1 B_1 + w_2 B_2 + w_3 B_3 + \ldots$$

- "Bases" are the "standard" units such that all instances can be expressed a weighted combinations of these units
- Ideal requirements: Bases must be orthogonal
- Checkerboards are one choice of bases
  - Orthogonal
  - But not "smooth"
- Other choices of bases: Complex exponentials, Wavelets, etc..

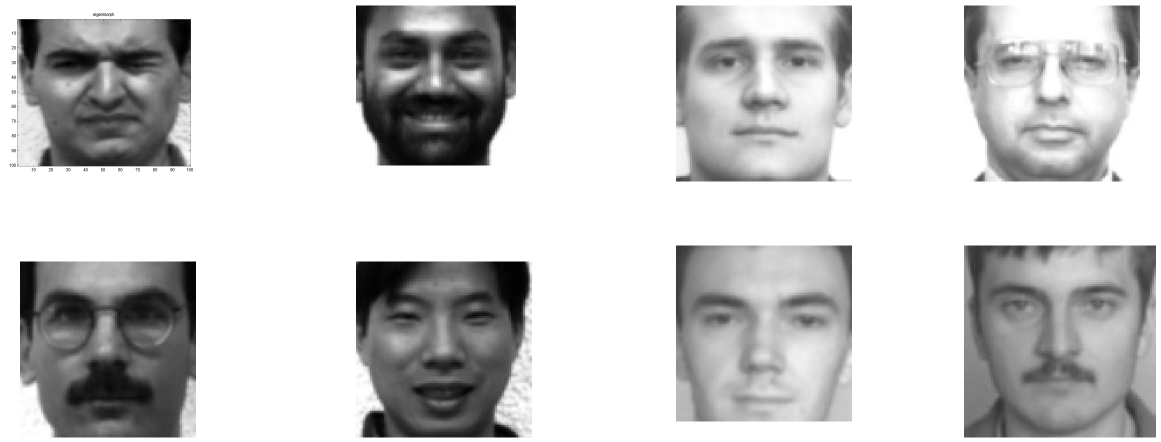


Data specific bases?

- **Issue**: **The bases we have considered so far are *data agnostic***
  - Checkerboards, Complex exponentials, Wavelets..
  - We use the same bases regardless of the data we analyze
    - Image of face vs. Image of a forest
    - Segment of speech vs. Seismic rumble

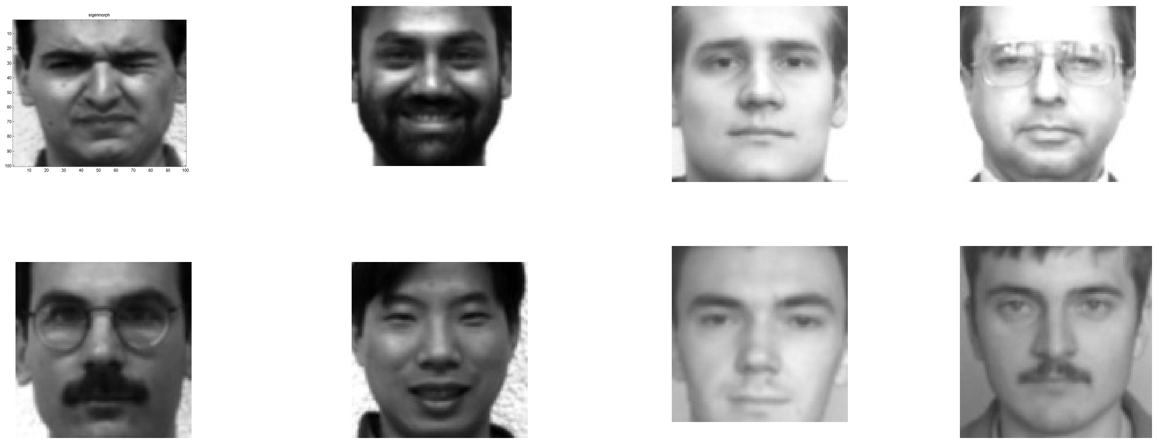
- How about data specific bases
  - Bases that consider the underlying data
    - E.g. is there something better than checkerboards to describe faces
    - Something better than complex exponentials to describe music?

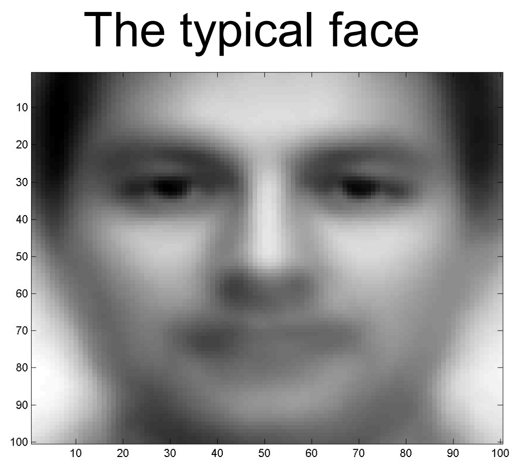
Data-specific description of faces



- A collection of images
  - All normalized to 100x100 pixels
- What is common among all of them?
  - Do we have a common descriptor?

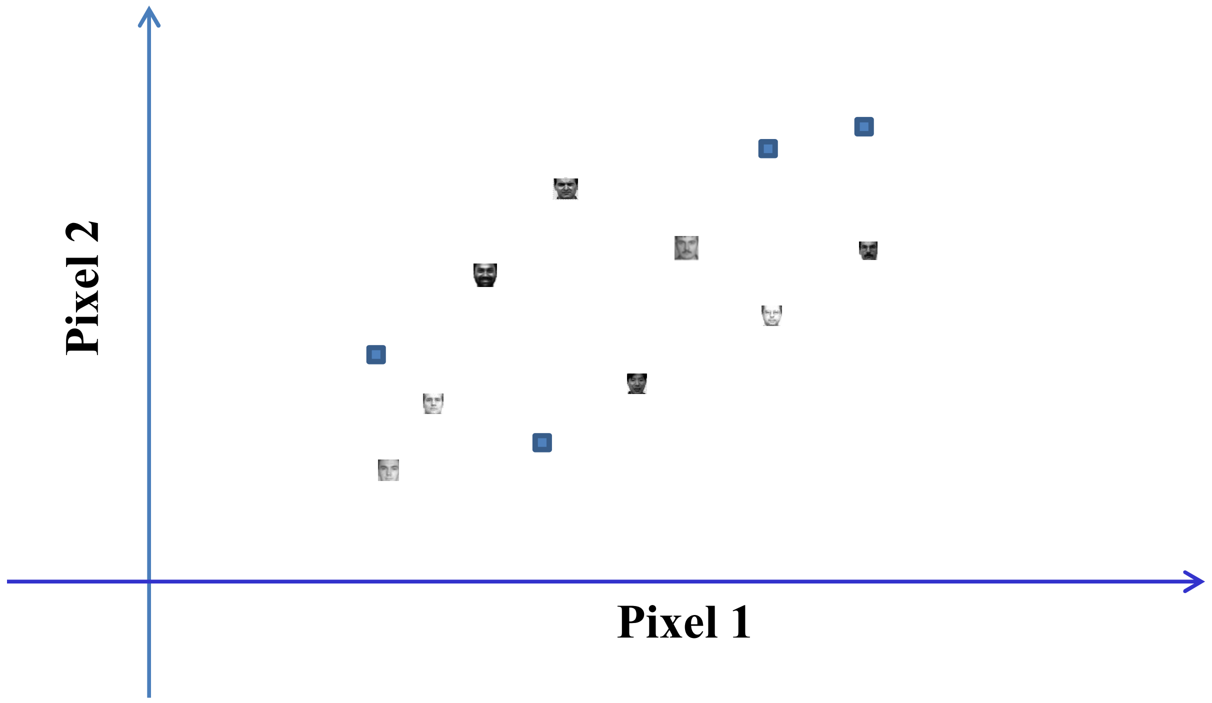
A typical face





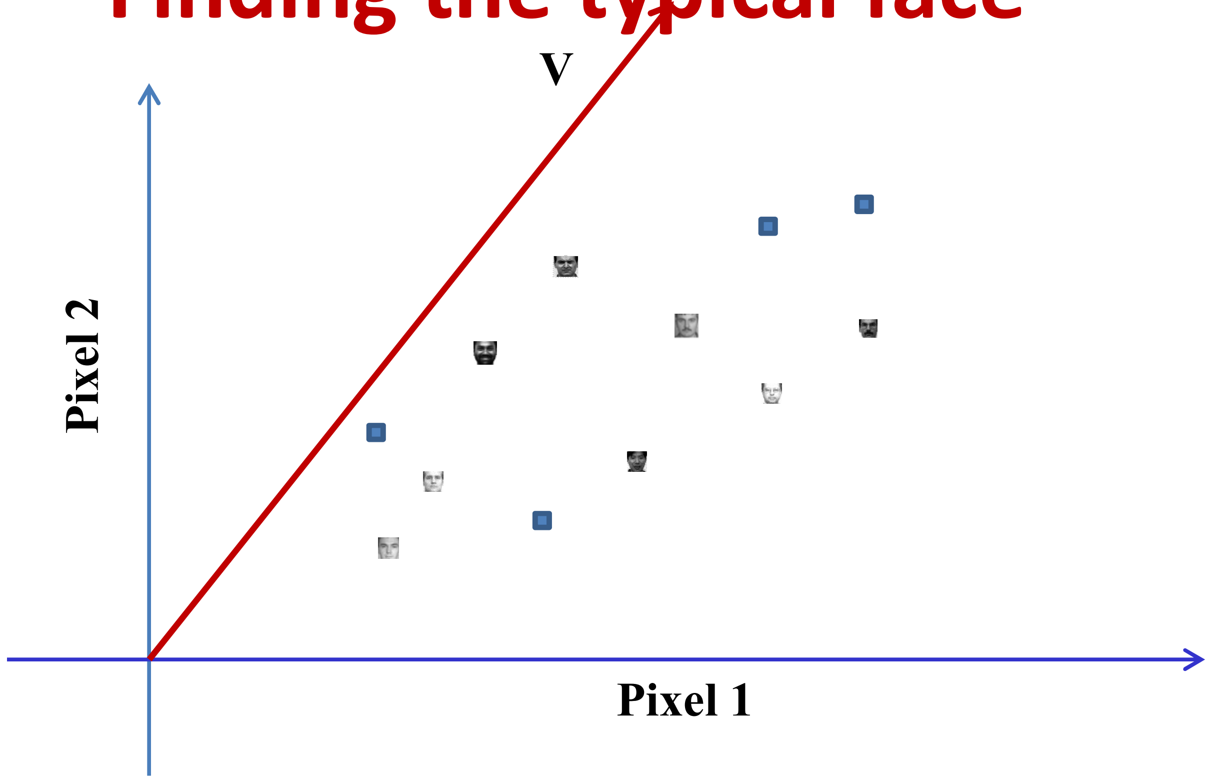
- **Assumption: There is a "typical" face that captures most of what is common to all faces**
  - Every face can be represented by a scaled version of a typical face
  - We will denote this face as V
- Approximate **every** face f as $\mathrm{f} = \mathrm{w_f}\ \mathrm{V}$
- Estimate V to minimize the squared error
  - How? What is V?

Abstracting the problem: Finding the typical face



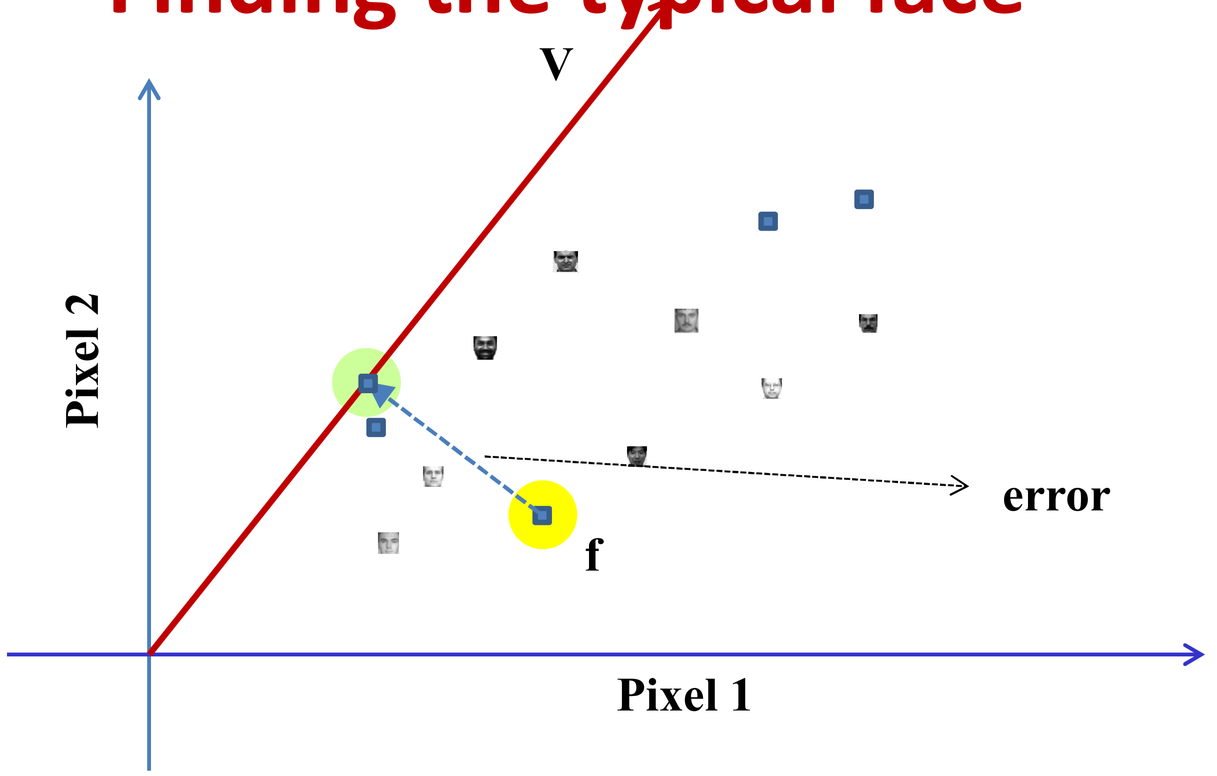
- Each "point" represents a face in "pixel space"

Abstracting the problem: Finding the typical face



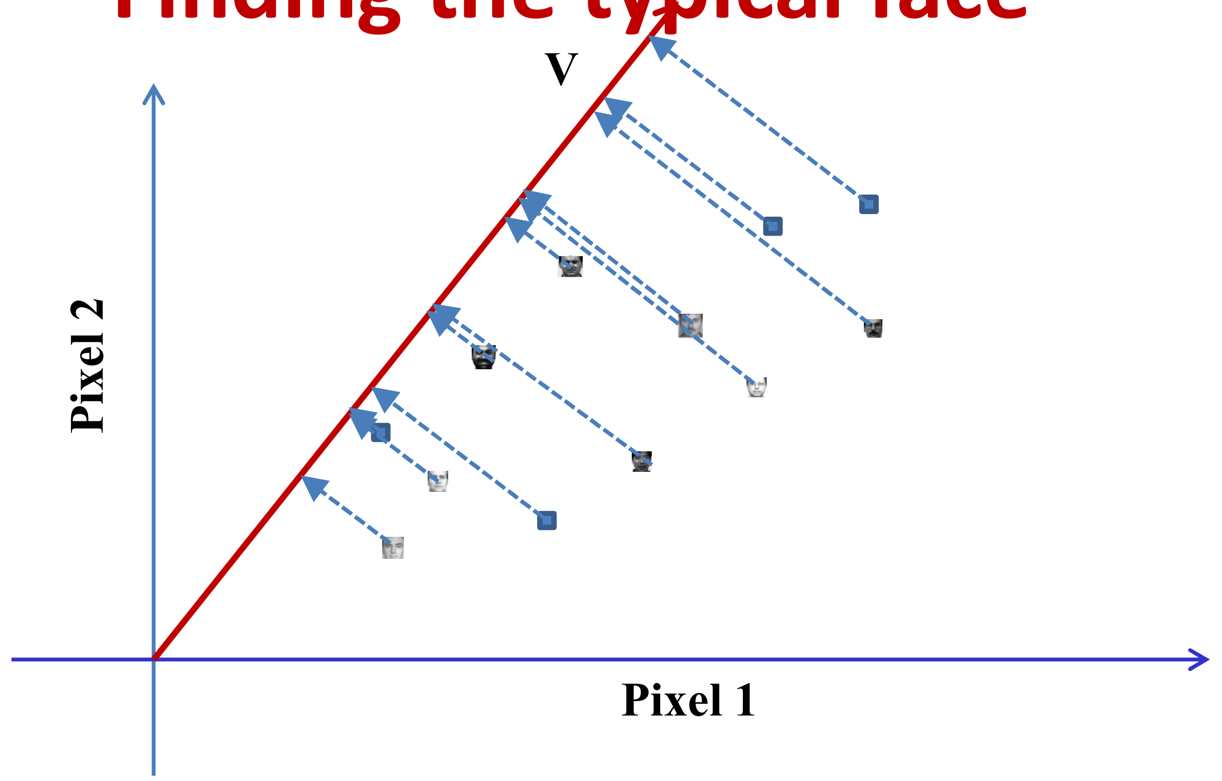
- Each “point” represents a face in “pixel space”
- Any “typical face” V is a vector in this space

Abstracting the problem: Finding the typical face



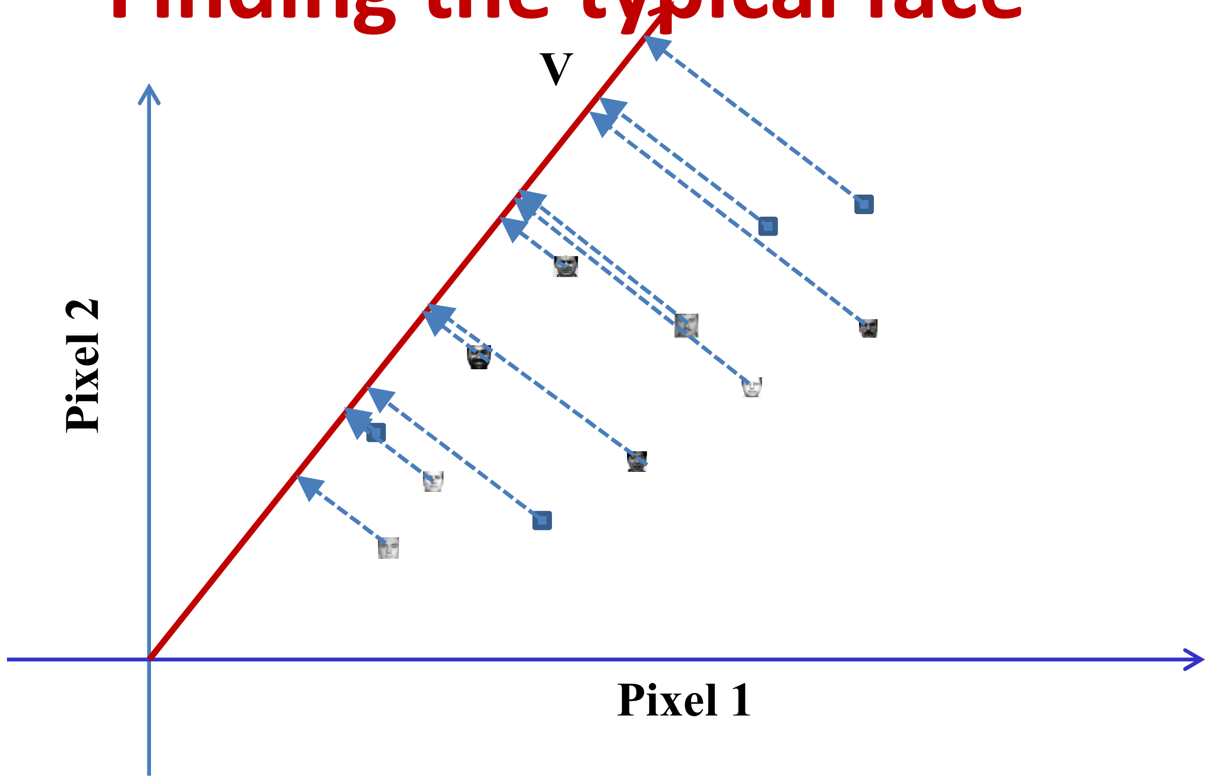
- Each "point" represents a face in "pixel space"
- The "typical face" V is a vector in this space
- The ***approximation*** $w_{f,}\ V$ for any face f is the *projection* of f onto V
- The distance between f and its projection $w_f V$ is the *projection error* for f

Abstracting the problem: Finding the typical face



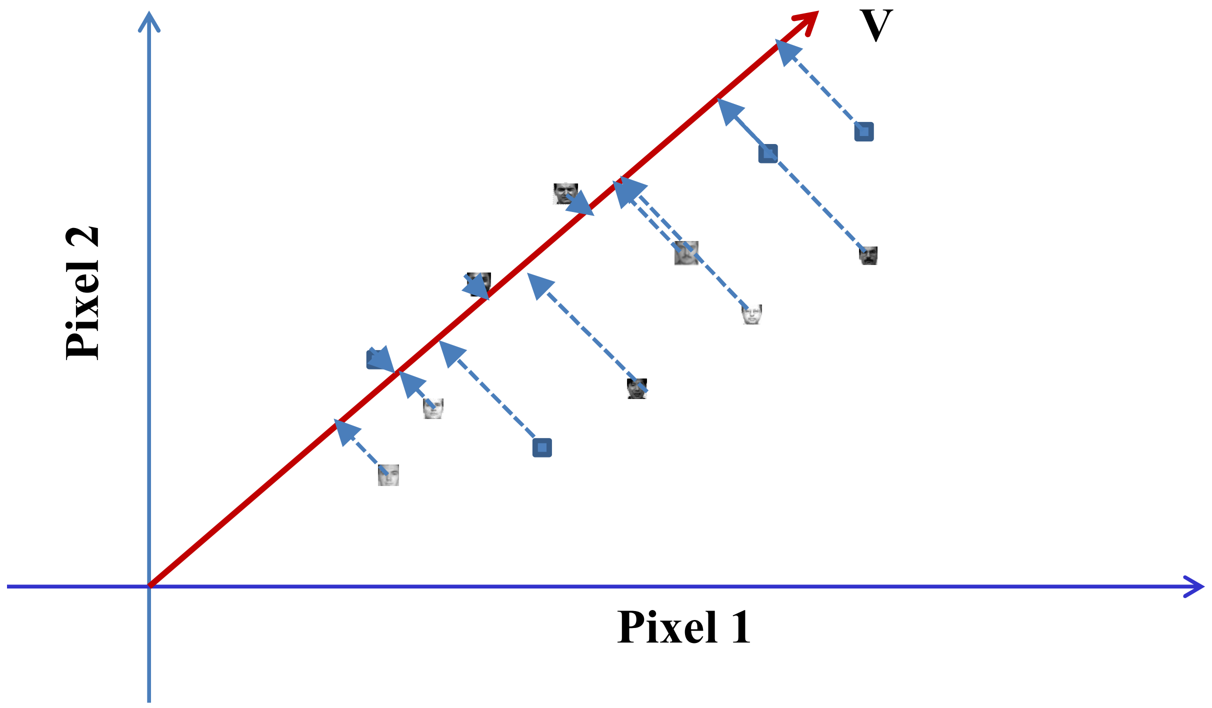
- *Every* face in our data will suffer error when approximated by its projection on V
- The total squared length of all error lines is the *total squared projection error*

Abstracting the problem: Finding the typical face



- The problem of finding the first typical face V_1:
  Find the V for which the total projection error is minimum!

Abstracting the problem: Finding the typical face



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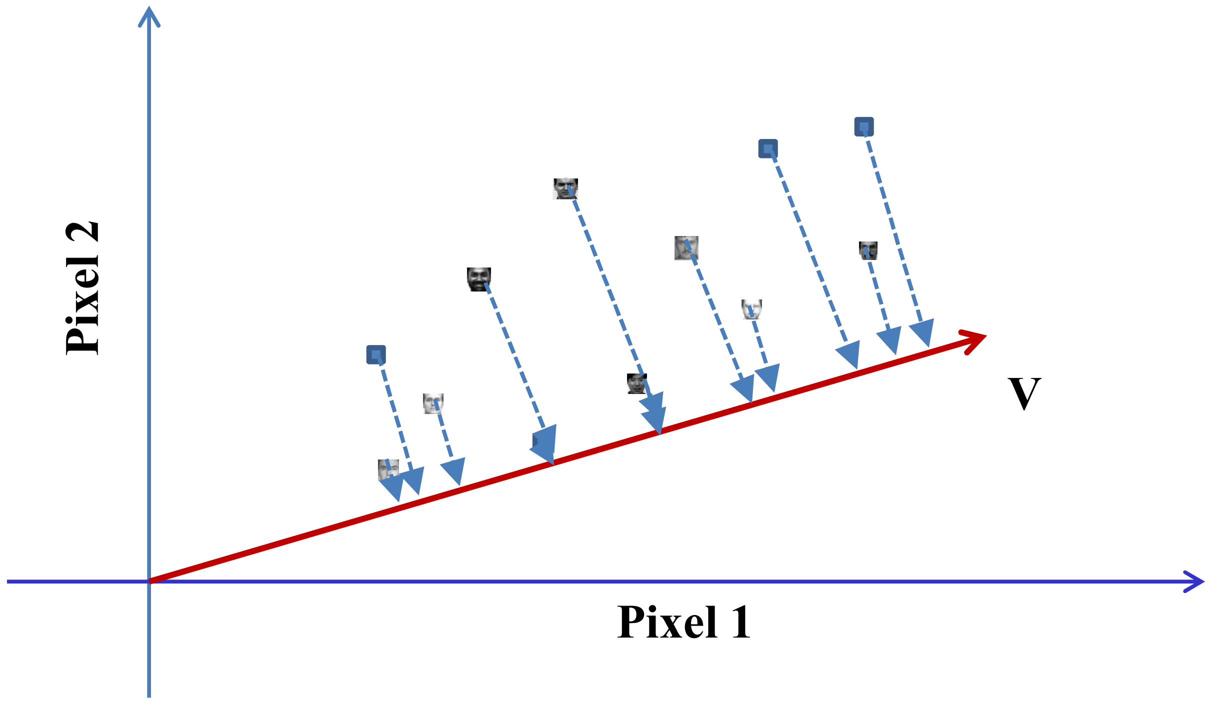

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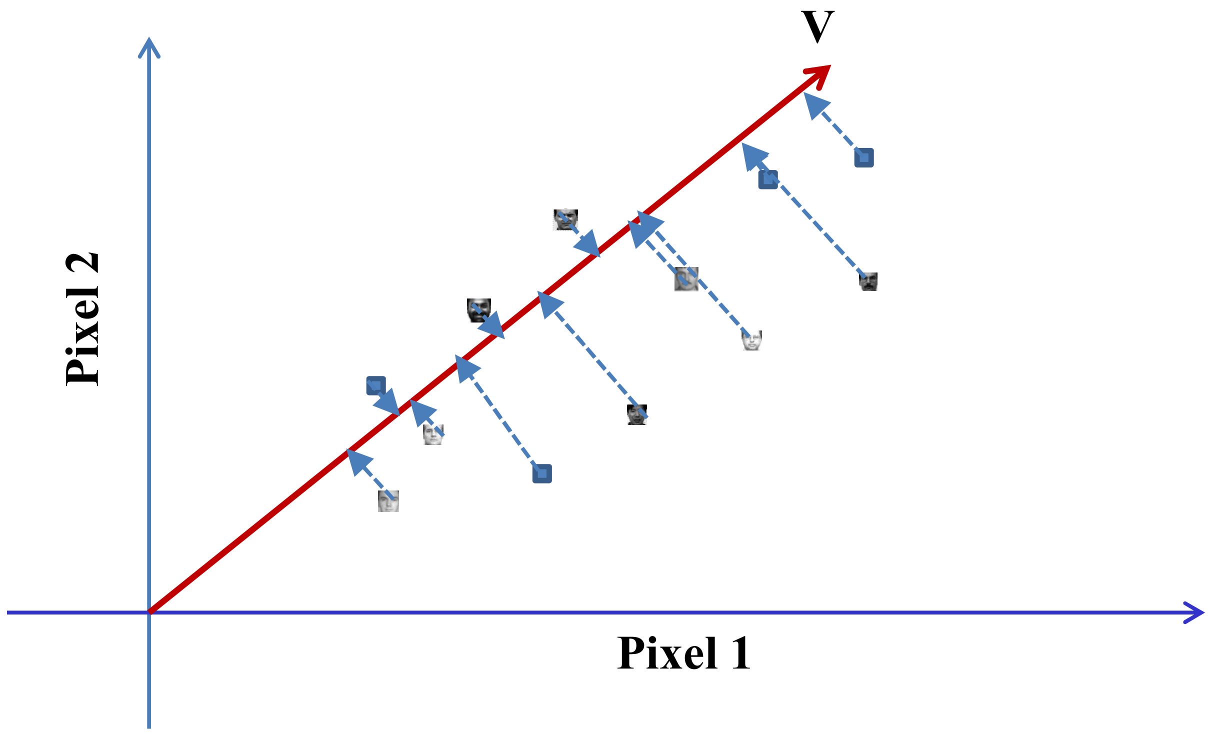
Abstracting the problem: Finding the typical face



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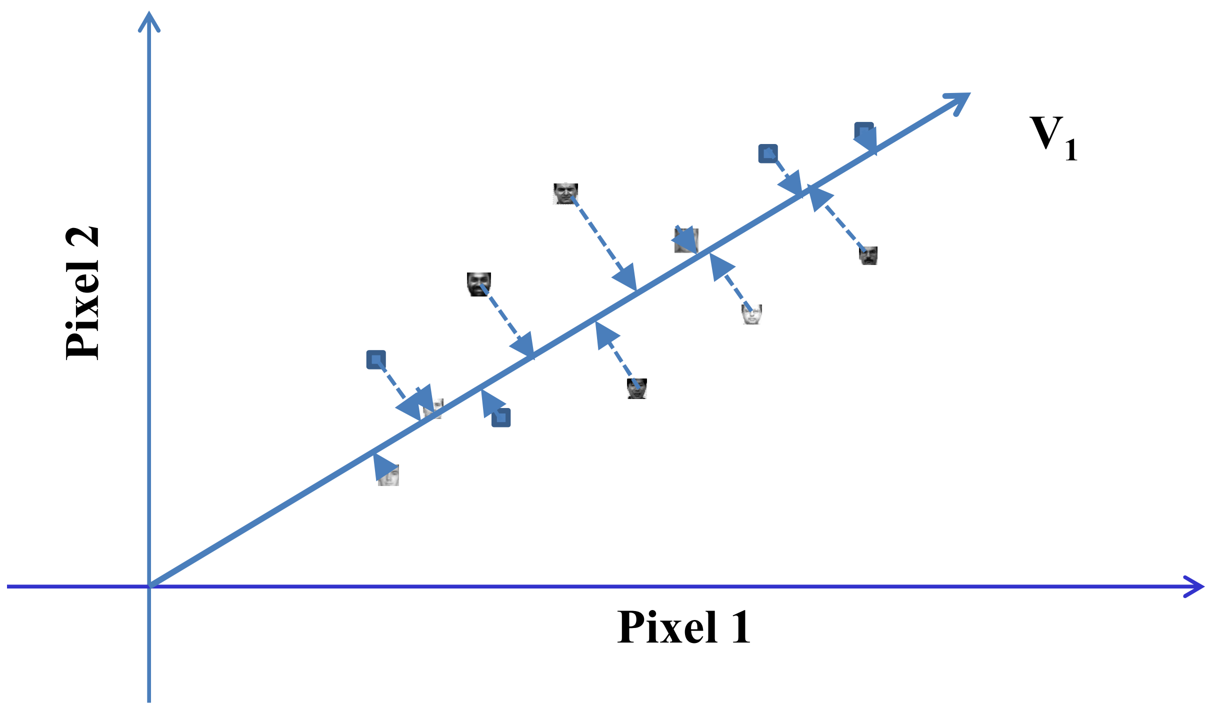
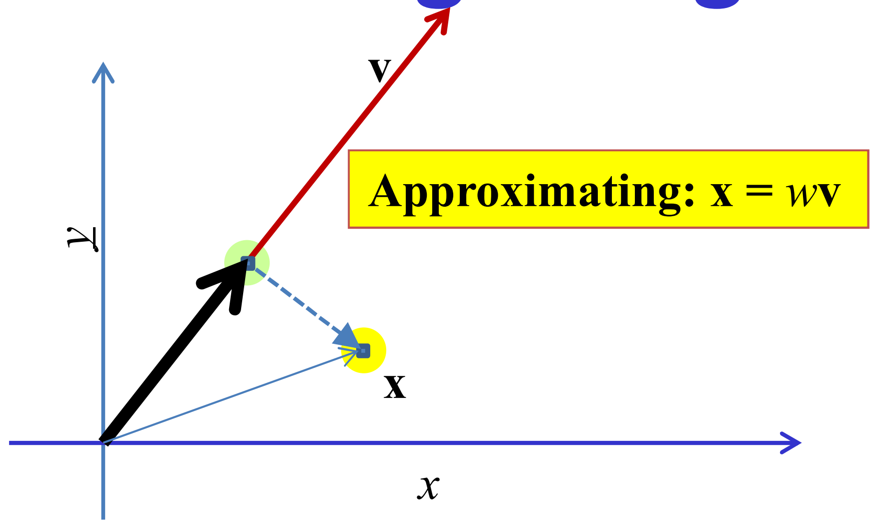
Abstracting the problem: Finding the typical face



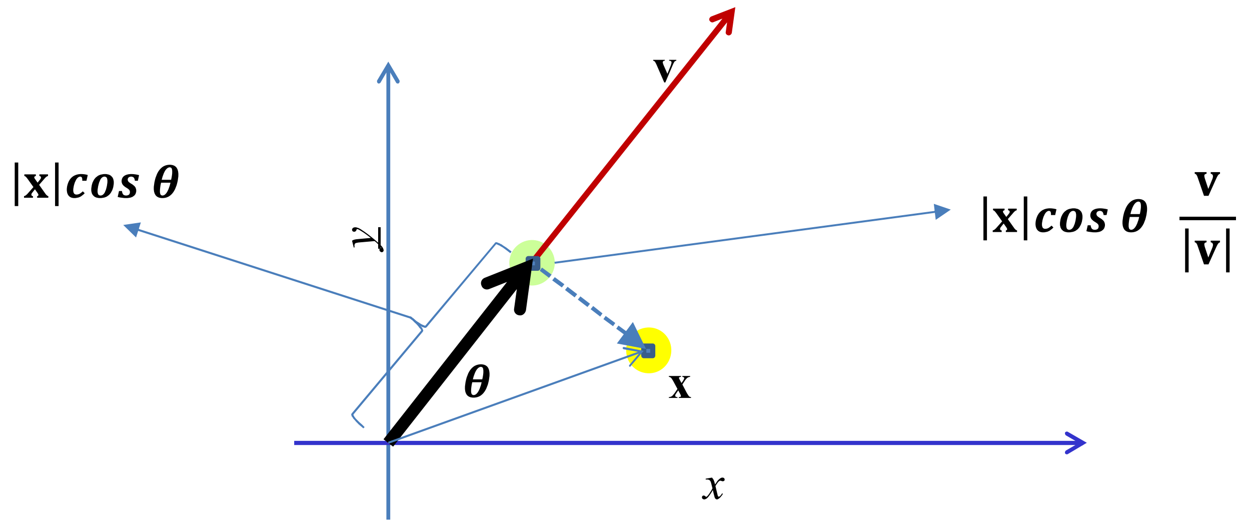
- The problem of finding the first typical face V_1:
  Find the V for which the total projection error is minimum!
- This "minimum squared error" V is our "best" first typical face
- **It is also the first *Eigen face***

Formalizing the Problem: Error from approximating a single vector



- Consider: approximating $\mathbf{x} = w\mathbf{v}$
  - E.g **x** is a face, and "**v**" is the "typical face"
- Finding an approximation $w\mathbf{v}$ which is closest to **x**
  - In a Euclidean sense
  - Basically projecting **x** onto **v**

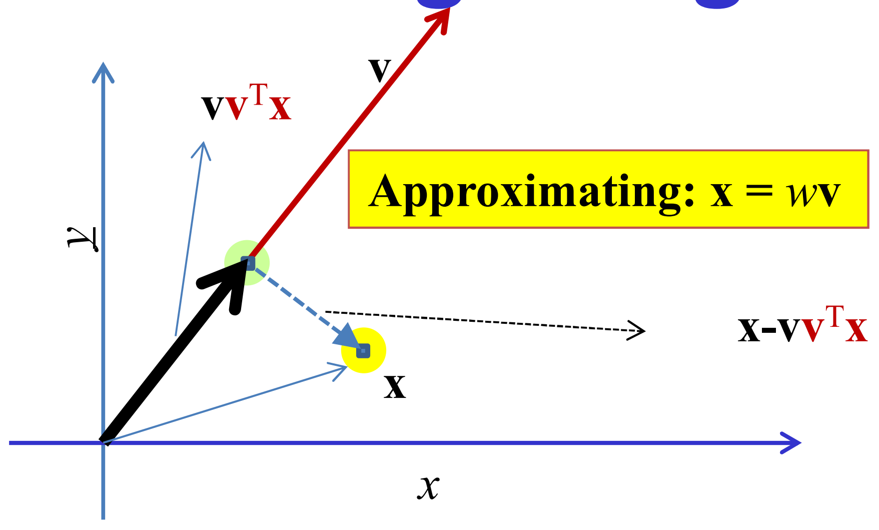
Projection of a vector on another



$\mathbf{v}$
$|\mathbf{x}|\boldsymbol{cos\ \theta}$
$|\mathbf{x}|\boldsymbol{cos\ \theta}\ \frac{\mathbf{v}}{|\mathbf{v}|}$
y
$\boldsymbol{\theta}$
$\mathbf{x}$
x

- The black arrow is the *projection* of $\mathbf{x}$ on $\mathbf{v}$
- $\frac{\mathbf{v}}{|\mathbf{v}|}$ is a *unit* vector in the direction of v
- $\mathbf{x}_{proj} = |\mathbf{x}|\boldsymbol{cos\ \theta}\frac{\mathbf{v}}{|\mathbf{v}|} = |\mathbf{x}||\mathbf{v}|\boldsymbol{cos\ \theta}\frac{\mathbf{v}}{|\mathbf{v}|^2}$
  $= \mathbf{x}^{\mathrm{T}}\mathbf{v}\frac{\mathbf{v}}{|\mathbf{v}|^2}$

Formalizing the Problem: Error from approximating a single vector



v
$\mathbf{v}\mathbf{v}^T\mathbf{x}$
Approximating: $\mathbf{x} = w\mathbf{v}$
y
$\mathbf{x}-\mathbf{v}\mathbf{v}^T\mathbf{x}$
$\mathbf{x}$
x

- Projection of a vector **x** on to a vector **v**

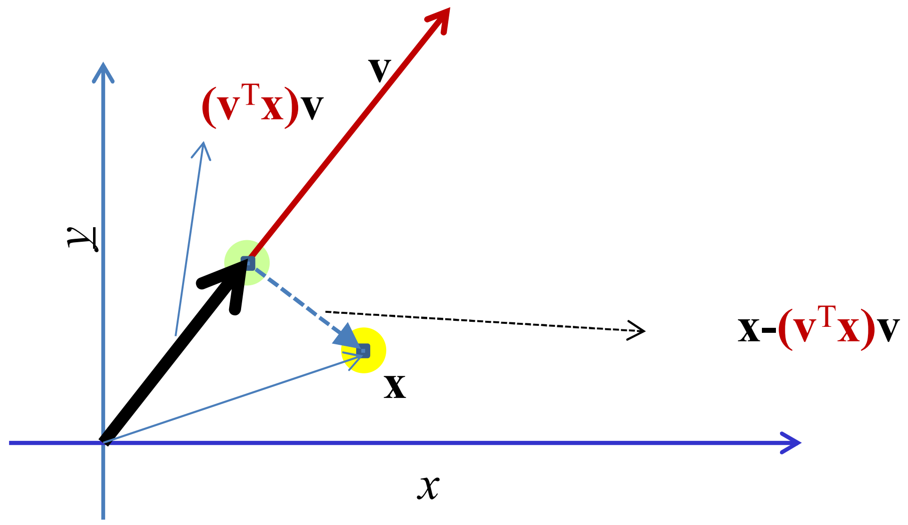
$$\hat{\mathbf{x}} = \frac{\mathbf{x}^T\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v}$$

- Assuming **v** is of unit length: $\hat{\mathbf{x}} = (\mathbf{x}^T\mathbf{v})\mathbf{v}$

$$error = \mathbf{x} - \hat{\mathbf{x}} = \mathbf{x} - (\mathbf{x}^T\mathbf{v})\mathbf{v}$$

$$\text{squared error} = \left\|\mathbf{x} - (\mathbf{x}^T\mathbf{v})\mathbf{v}\right\|^2$$

Error from approximating a single vector



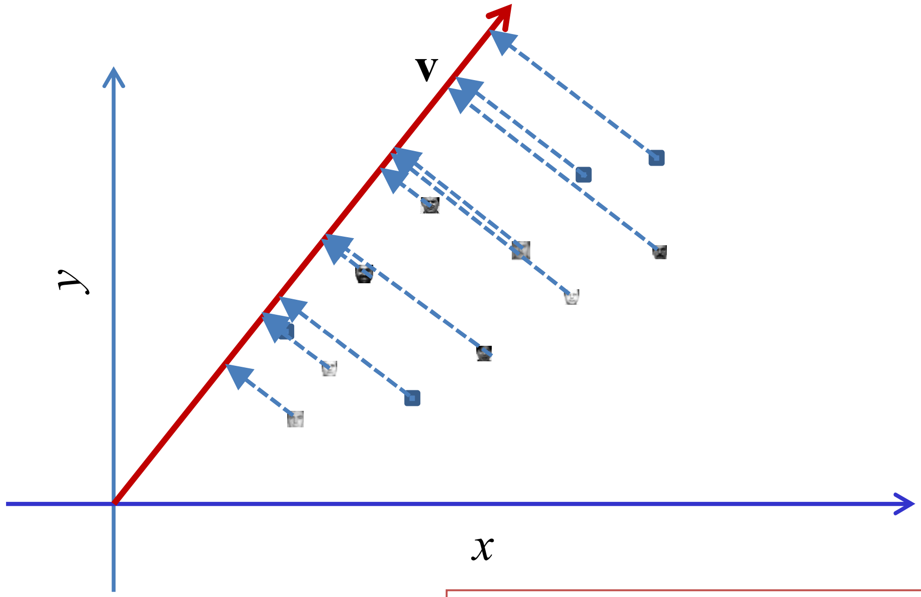
- Projection $\hat{\mathbf{x}} = (\mathbf{x}^T\mathbf{v})\mathbf{v}$
- Squared length of projection

$$\|\hat{\mathbf{x}}\|^2 = (\mathbf{x}^T\mathbf{v})^2 = (\mathbf{x}^T\mathbf{v})^T(\mathbf{x}^T\mathbf{v}) = \mathbf{v}^T\mathbf{x}\mathbf{x}^T\mathbf{v}$$

- Pythogoras theorem: Squared length of error $e(\mathbf{x}) = \|\mathbf{x}\|^2 - \|\hat{\mathbf{x}}\|^2$

$$e(\mathbf{x}) = \mathbf{x}^T\mathbf{x} - \mathbf{v}^T\mathbf{x}\mathbf{x}^T\mathbf{v}$$

Error for *many* vectors



- Error for one vector: $\boxed{e(\mathbf{x}) = \mathbf{x}^T\mathbf{x} - \mathbf{v}^T\mathbf{x}\mathbf{x}^T\mathbf{v}}$
- Error for many vectors

$$E = \sum_i e(\mathbf{x}_i) = \sum_i \mathbf{x}_i^T\mathbf{x}_i - \sum_i \mathbf{v}^T\mathbf{x}_i\mathbf{x}_i^T\mathbf{v} \qquad = \sum_i \mathbf{x}_i^T\mathbf{x}_i - \mathbf{v}^T\left(\sum_i \mathbf{x}_i\mathbf{x}_i^T\right)\mathbf{v}$$

- **Goal: Estimate v to minimize this error!**

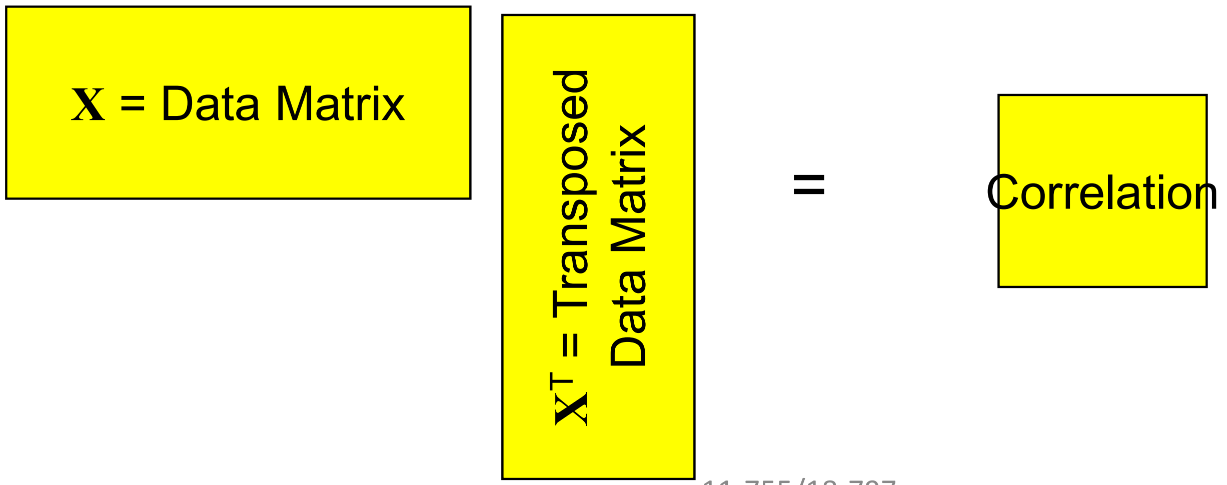
Definition: The correlation matrix

$$\left(\sum_i \mathbf{x}_i \mathbf{x}_i^T\right)$$

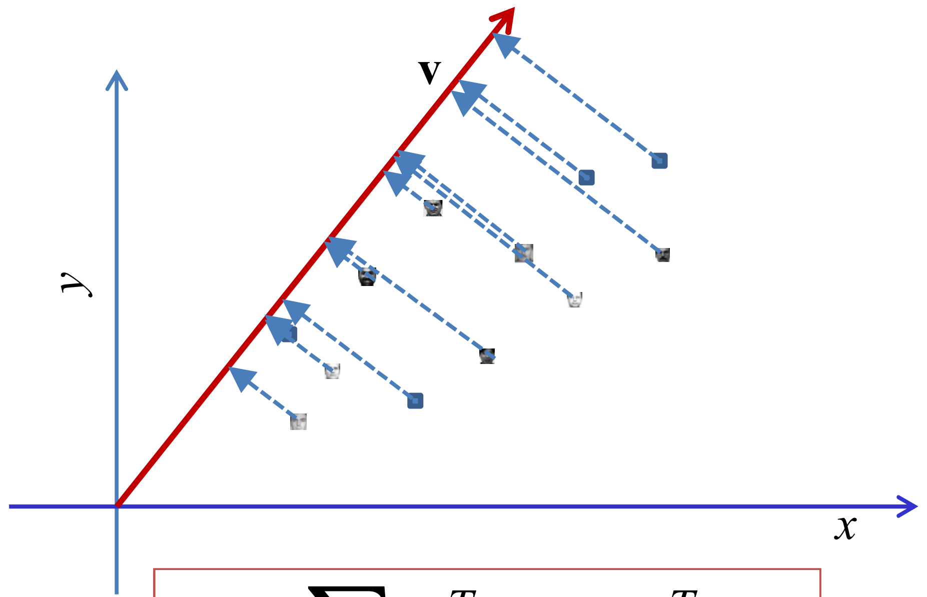
- The encircled term is the ***correlation matrix***

$$\mathbf{X} = \left[\mathbf{x}_1 \ \mathbf{x}_2 \ \ldots \ \mathbf{x}_N\right]$$

$$\sum_i \mathbf{x}_i \mathbf{x}_i^T = \mathbf{X}\mathbf{X}^T = \mathbf{R}$$



Error for *many* vectors



- Total error: $E = \sum_i \mathbf{x}_i^T \mathbf{x}_i - \mathbf{v}^T \mathbf{R} \mathbf{v}$
- Add constraint: $\mathbf{v}^T \mathbf{v} = 1$
- Constrained objective to minimize:

$$L = \sum_i \mathbf{x}_i^T \mathbf{x}_i - \mathbf{v}^T \mathbf{R} \mathbf{v} + \lambda\left(\mathbf{v}^T \mathbf{v} - 1\right)$$

Two Matrix Identities

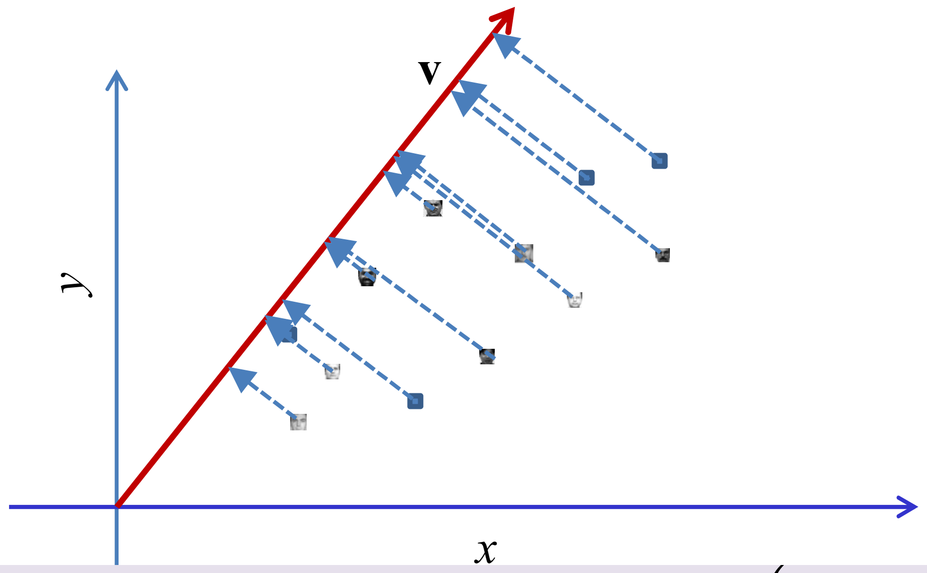
- Derivative w.r.t $\mathbf{v}$

$$L = \sum_i \mathbf{x}_i^T \mathbf{x}_i - \mathbf{v}^T \mathbf{R} \mathbf{v} + \lambda\left(\mathbf{v}^T \mathbf{v} - 1\right)$$

$$\nabla_{\mathbf{v}}\left(\mathbf{v}^T \mathbf{v}\right) = 2\mathbf{v}$$

$$\nabla_{\mathbf{v}} \mathbf{v}^T \mathbf{R} \mathbf{v} = 2\mathbf{R}\mathbf{v}$$

Minimizing error

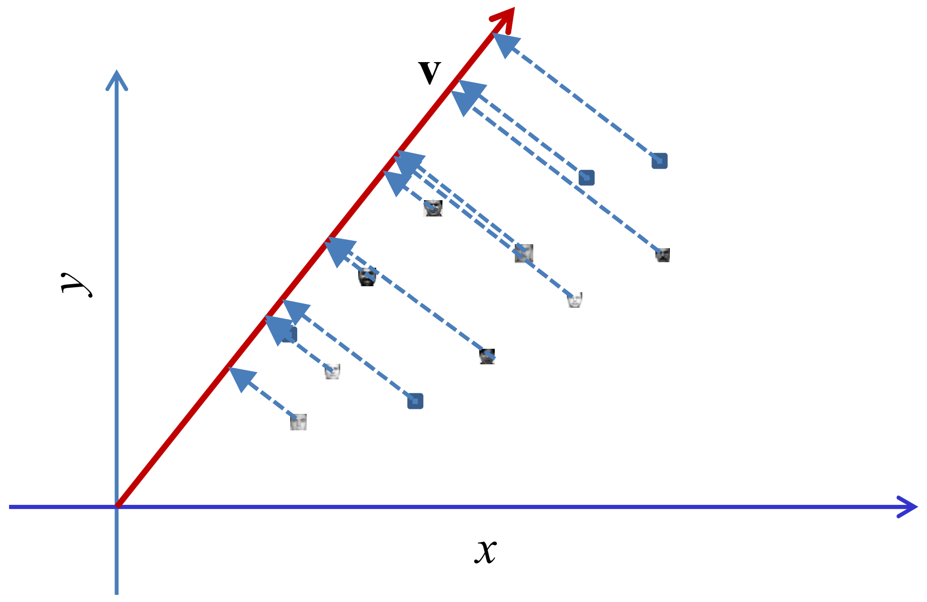


$$L = \sum_i \mathbf{x}_i^T \mathbf{x}_i - \mathbf{v}^T \mathbf{R} \mathbf{v} + \lambda\left(\mathbf{v}^T \mathbf{v} - 1\right)$$

- Differentiating w.r.t $\mathbf{v}$ and equating to 0

$$-2\mathbf{R}\mathbf{v} + 2\lambda\mathbf{v} = 0 \qquad \mathbf{R}\mathbf{v} = \lambda\mathbf{v}$$

The best "basis"



- The minimum-error basis is found by solving

$$\mathbf{Rv} = \lambda\mathbf{v}$$

- $\mathbf{v}$ is an Eigen vector of the correlation matrix $\mathbf{R}$
  - λ is the corresponding Eigen value

What about the total error?

$$E = \sum_i \mathbf{x}_i^T \mathbf{x}_i - \mathbf{v}^T \mathbf{R} \mathbf{v}$$

$$= \sum_i \mathbf{x}_i^T \mathbf{x}_i - \mathbf{v}^T \lambda \mathbf{v} = \sum_i \mathbf{x}_i^T \mathbf{x}_i - \lambda \mathbf{v}^T \mathbf{v}$$

$$\boxed{E = \sum_i \mathbf{x}_i^T \mathbf{x}_i - \lambda}$$

Minimizing the error

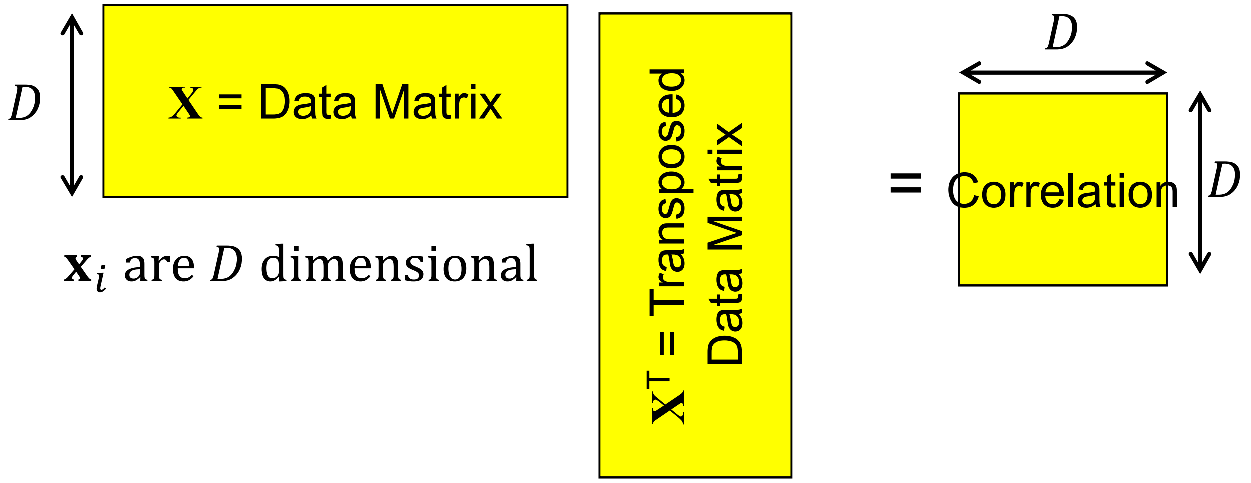
- The total error is $$E = \sum_i \mathbf{x}_i^T \mathbf{x}_i - \lambda$$
- We already know that the optimal basis is an Eigen vector
- The total error depends on the *negative* of the corresponding Eigen value
- To *minimize* error, we must *maximize* λ
- i.e. Select the Eigen vector with the largest Eigen value

A detour: The correlation matrix

A new definition:
The *correlation* matrix

$$\mathbf{X} = [\mathbf{x}_1 \ \mathbf{x}_2 \ \ldots \ \mathbf{x}_N]$$

$$\sum_i \mathbf{x}_i \mathbf{x}_i^T = \mathbf{X}\mathbf{X}^T = \mathbf{R}$$

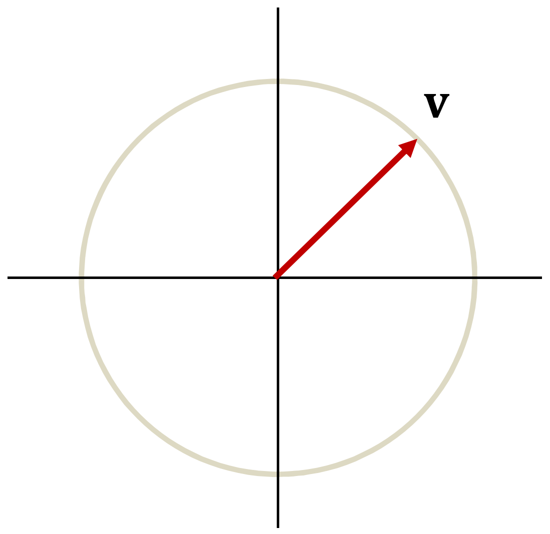


D
$\mathbf{X}$ = Data Matrix
$\mathbf{x}_i$ are D dimensional
$\mathbf{X}^T$ = Transposed Data Matrix
=
D
Correlation
D

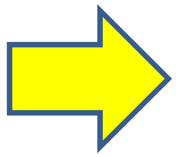
$$\mathbf{R} = \mathbf{R}^T$$

- For data-holder matrices: the product of a matrix and its transpose
  - Also equal to the sum of the outer products of the columns of the matrix
  - **The correlation matrix is symmetric**
  - **It quantifies the average dependence of individual *components* of the data on other components**

Interpreting the correlation matrix



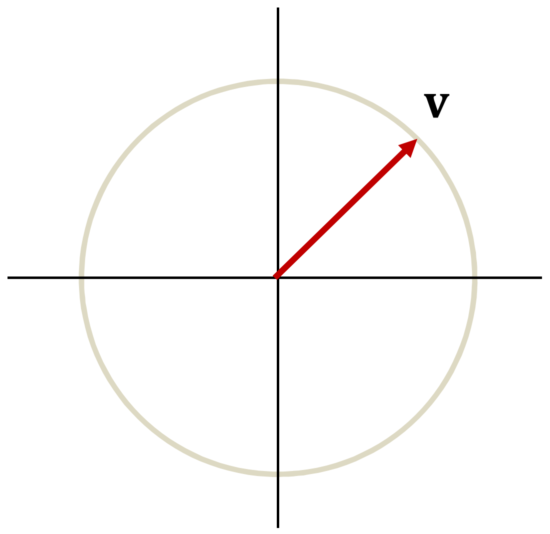
$\mathbf{y} = \mathbf{R}\mathbf{v}$



?

- Consider the effect of multiplying a unit vector by $\mathbf{R}$

Interpreting the correlation matrix



$\mathbf{y} = \mathbf{R}\mathbf{v}$

$$\mathbf{R}\mathbf{v} = \mathbf{X}\mathbf{X}^T\mathbf{v}$$

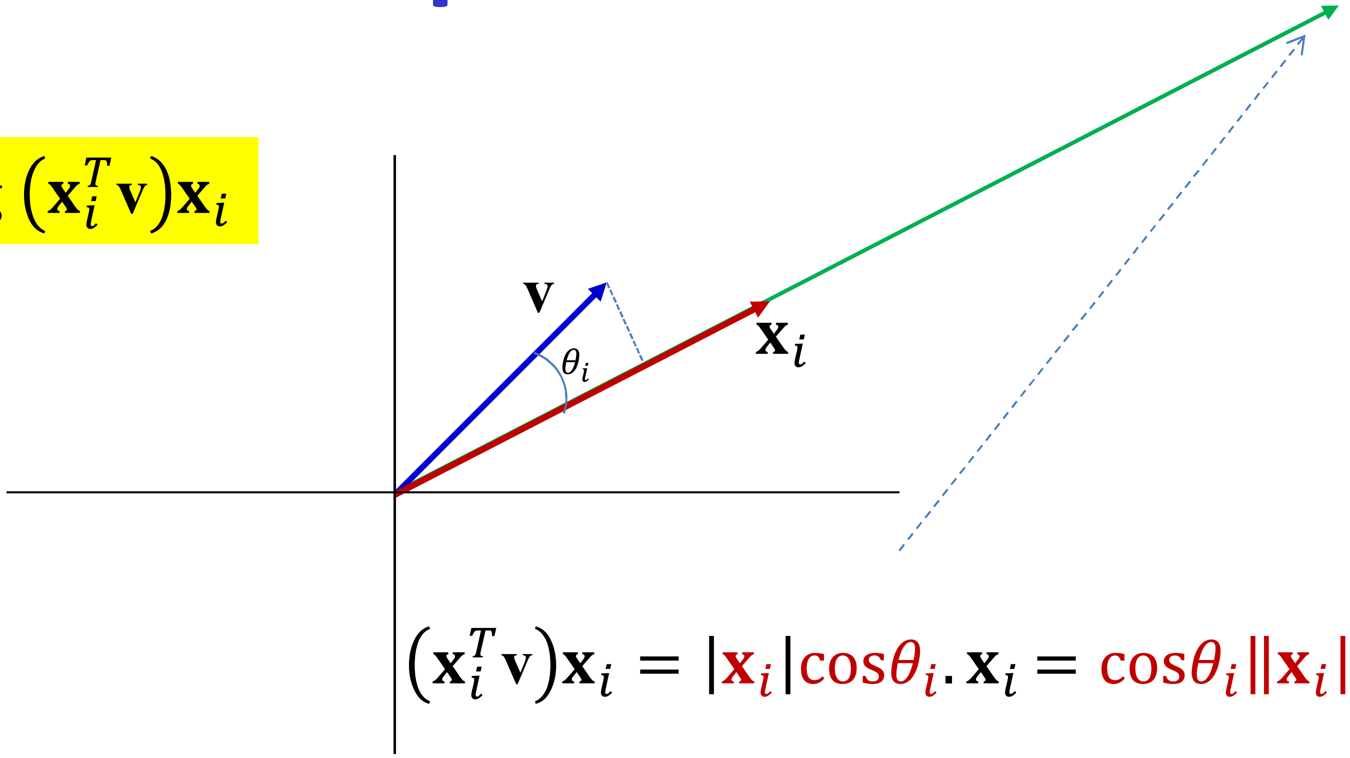
$$\mathbf{R}\mathbf{v} = \sum_i \mathbf{x}_i\,\mathbf{x}_i^T\,\mathbf{v}$$

$$\mathbf{R}\mathbf{v} = \sum_i (\mathbf{x}_i^T\mathbf{v})\mathbf{x}_i$$

- Consider the effect of multiplying a unit vector by $\mathbf{R}$

The inner product term

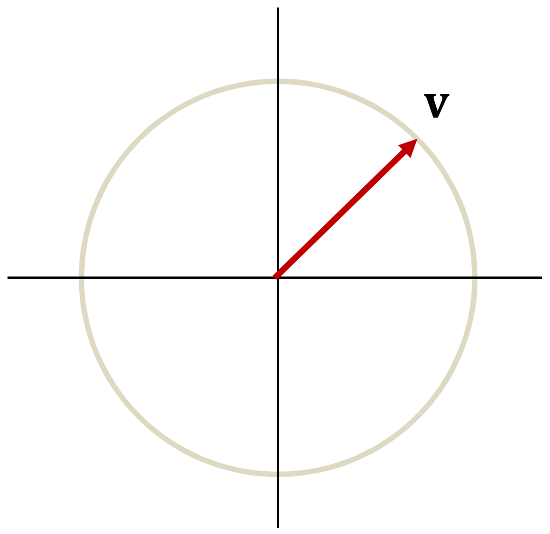
Understanding $(\mathbf{x}_i^T\mathbf{v})\mathbf{x}_i$



$$(\mathbf{x}_i^T\mathbf{v})\mathbf{x}_i = |\mathbf{x}_i|\cos\theta_i.\mathbf{x}_i = \cos\theta_i\|\mathbf{x}_i\|^2\bar{\mathbf{x}}_i$$

- Consider $(\mathbf{x}_i^T\mathbf{v})\mathbf{x}_i$
- This is the projection of unit vector $\mathbf{v}$ on $\mathbf{x}_i$, scaled by the squared length of $\mathbf{x}_i$

Interpreting the correlation matrix



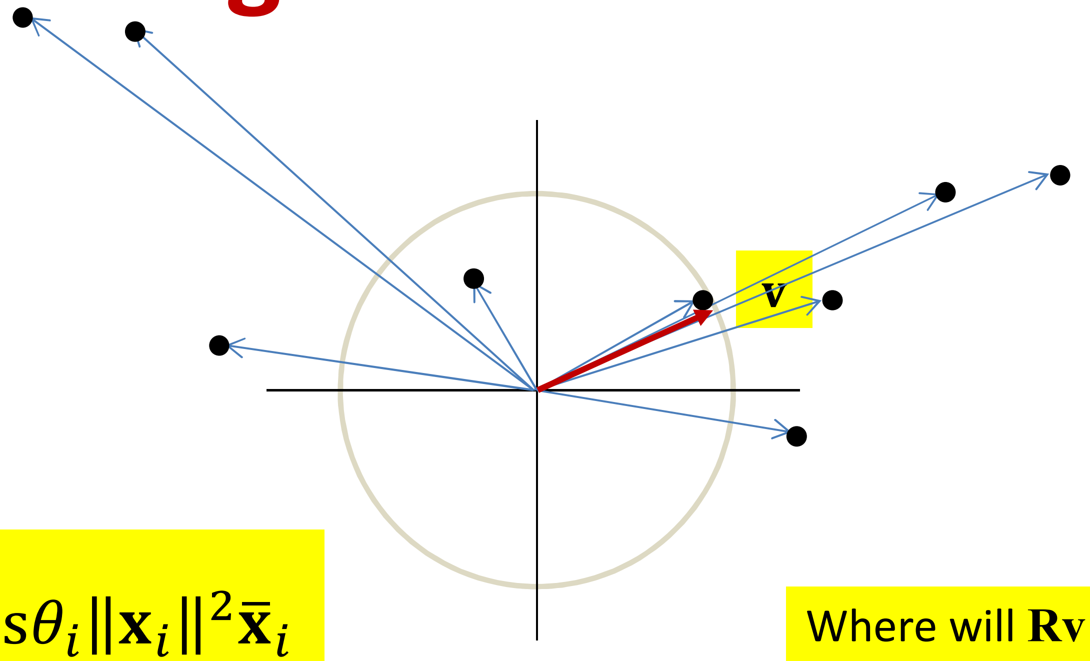
$\mathbf{y} = \mathbf{R}\mathbf{v}$

$$\mathbf{R}\mathbf{v} = \sum_i (\mathbf{x}_i^T \mathbf{v})\mathbf{x}_i$$

$$\boxed{\mathbf{R}\mathbf{v} = \sum_i \cos\theta_i \|\mathbf{x}_i\|^2 \bar{\mathbf{x}}_i}$$

- Consider the effect of multiplying a unit vector by $\mathbf{R}$

Interpreting the correlation matrix

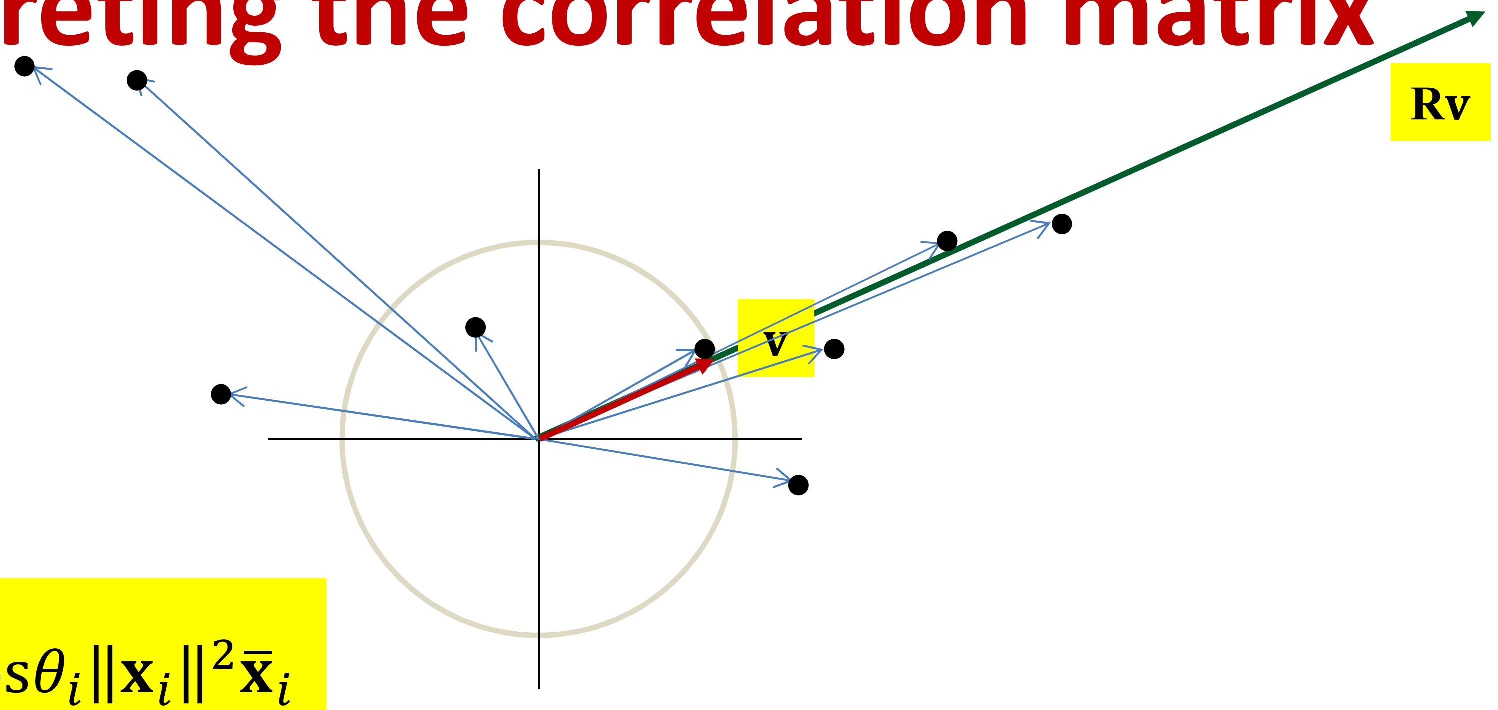


$$\mathbf{R}\mathbf{v} = \sum_{i} \cos\theta_i \|\mathbf{x}_i\|^2 \bar{\mathbf{x}}_i$$

Where will $\mathbf{R}\mathbf{v}$ be?

- Each unit vector is transformed to the sum of cosine-weighted squared-length versions of the individual vectors
  - Approximately the sum of the squared-length version of vectors that are close to it in angle

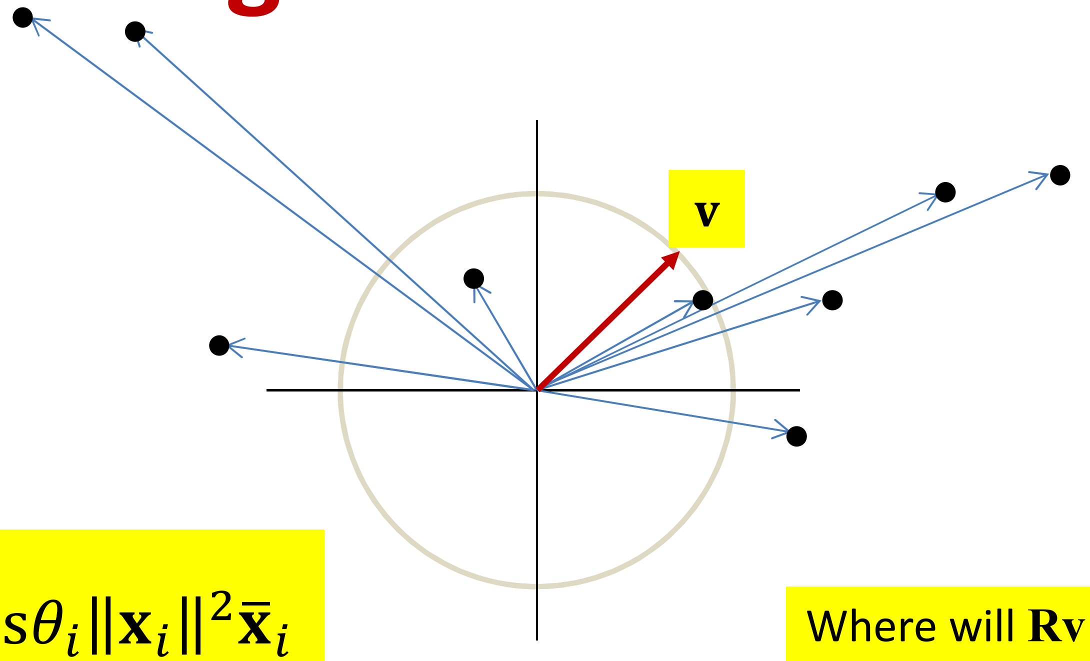
Interpreting the correlation matrix



$$\mathbf{Rv} = \sum_i \cos\theta_i \|\mathbf{x}_i\|^2 \bar{\mathbf{x}}_i$$

- Each unit vector is transformed to the sum of cosine-weighted squared-length versions of the individual vectors
  - Approximately the sum of the squared-length version of vectors that are close to it in angle

Interpreting the correlation matrix

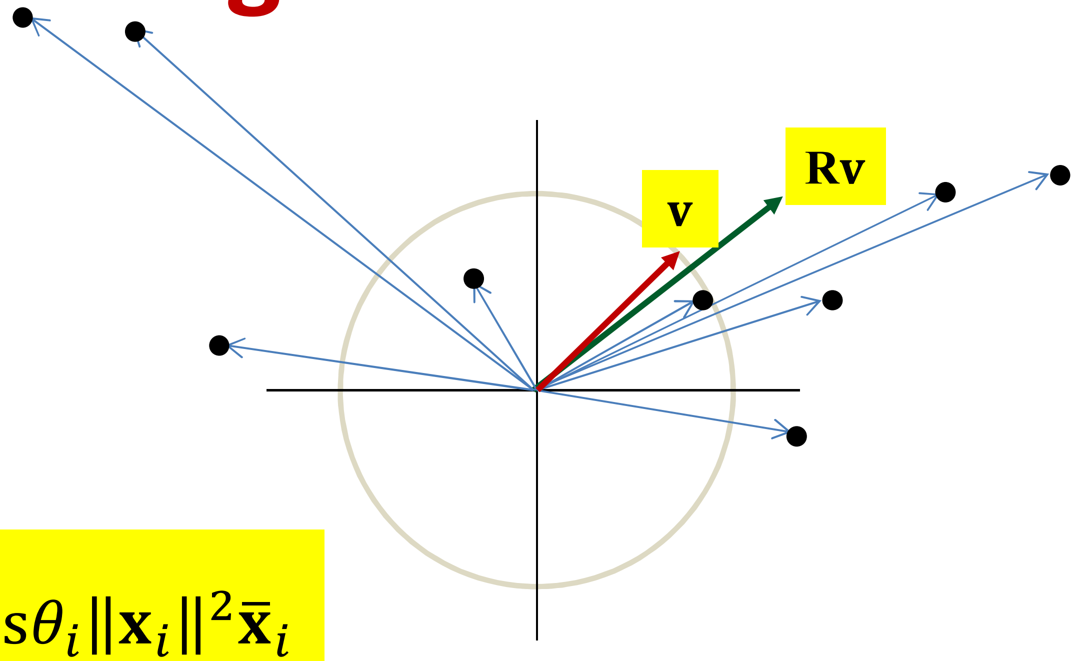


$$\mathbf{Rv} = \sum_i \cos\theta_i \|\mathbf{x}_i\|^2 \bar{\mathbf{x}}_i$$

Where will $\mathbf{Rv}$ be?

- Each unit vector is transformed to the sum of cosine-weighted squared-length versions of the individual vectors
  - Approximately the sum of the squared-length version of vectors that are close to it in angle

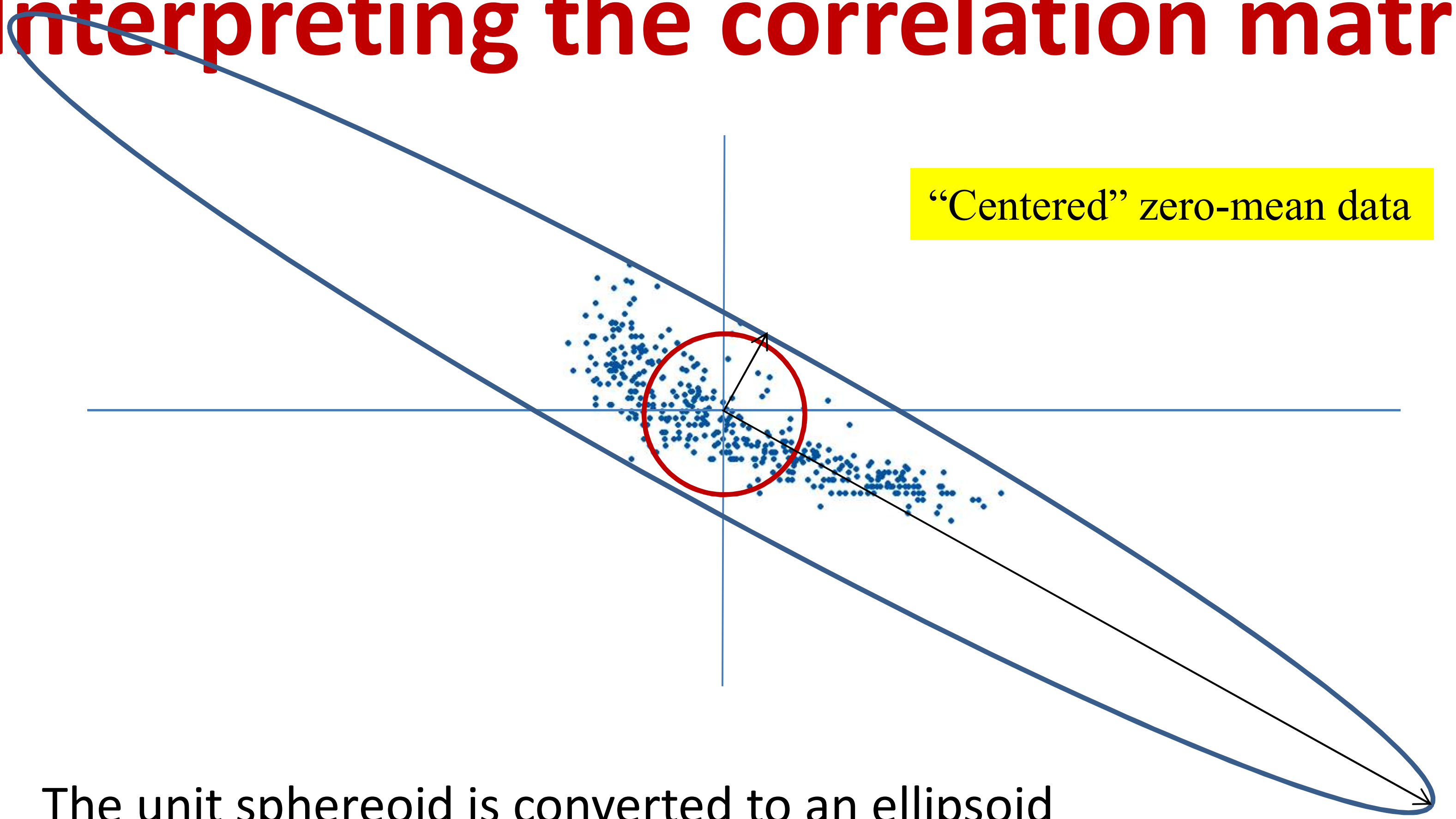
Interpreting the correlation matrix



$$\mathbf{Rv} = \sum_{i} \cos\theta_i \|\mathbf{x}_i\|^2 \bar{\mathbf{x}}_i$$

- Each unit vector is transformed to the sum of cosine-weighted squared-length versions of the individual vectors
  - Approximately the sum of the squared-length version of vectors that are close to it in angle

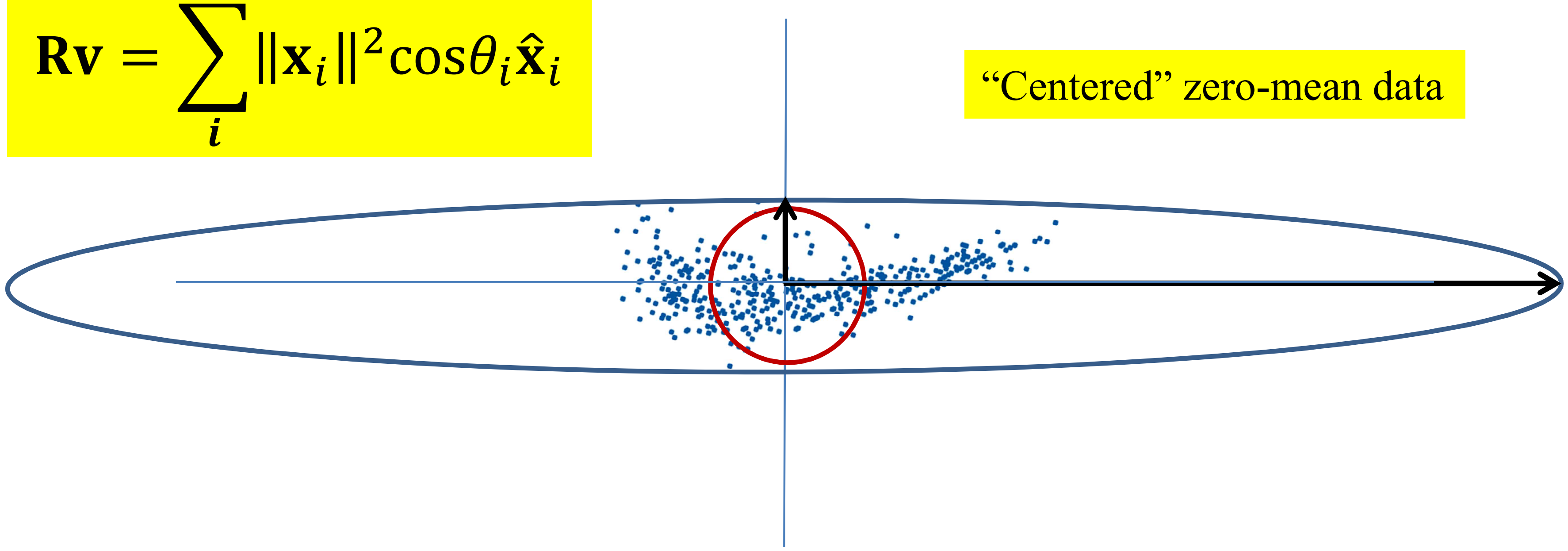
Interpreting the correlation matrix



- The unit sphereoid is converted to an ellipsoid
  - The major axes point to the directions of greatest energy
  - These are the *eigenvectors*
  - Their length is proportional to the *square* of the lengths of the data vectors
    - *Why?*

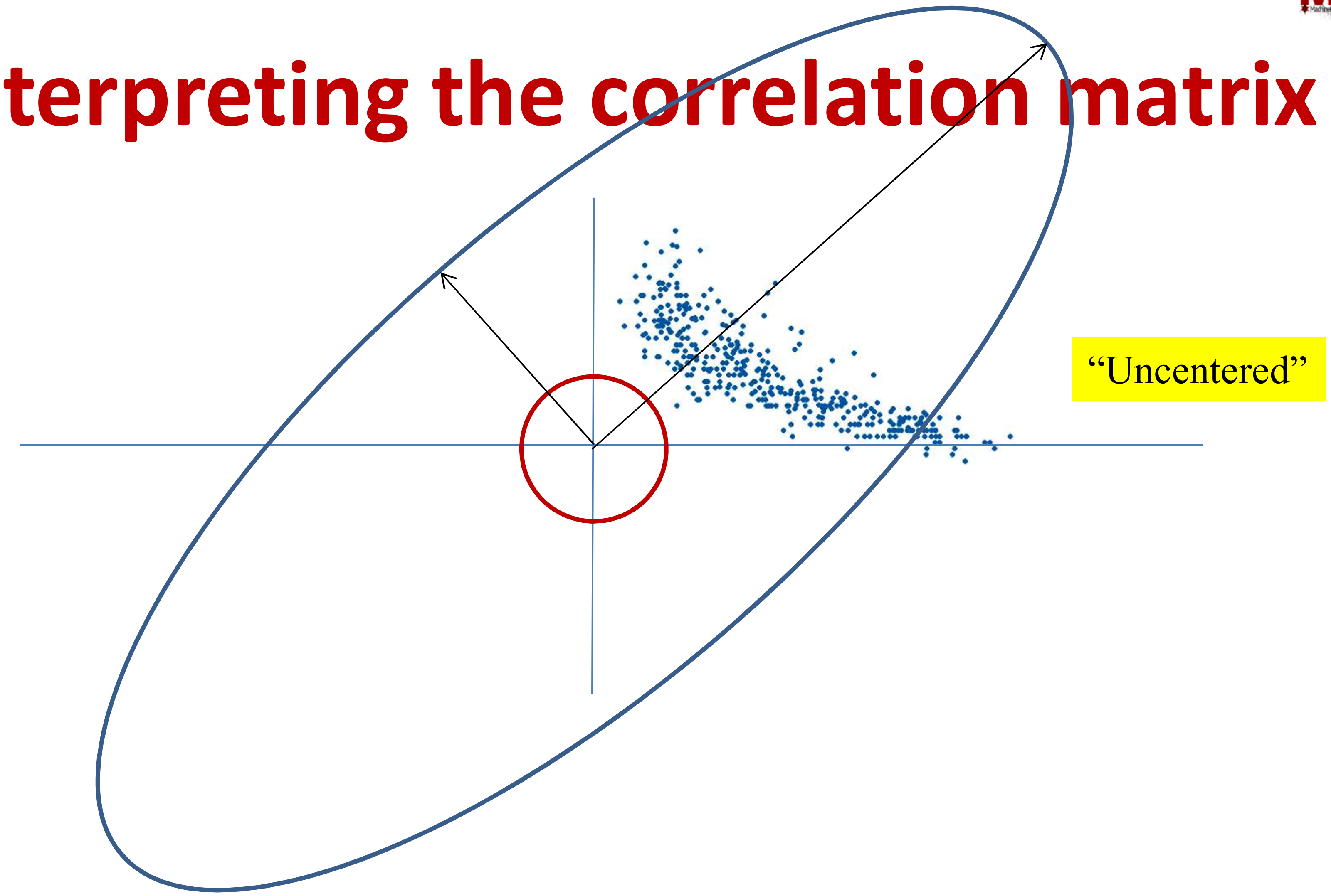
"Uncorrelated" data

$$\mathbf{Rv} = \sum_i \|\mathbf{x}_i\|^2 \cos\theta_i \hat{\mathbf{x}}_i$$



- When the scatter of the data is aligned to the axes, the transformed ellipse is also aligned to the axes
  - The data are "**uncorrelated**"

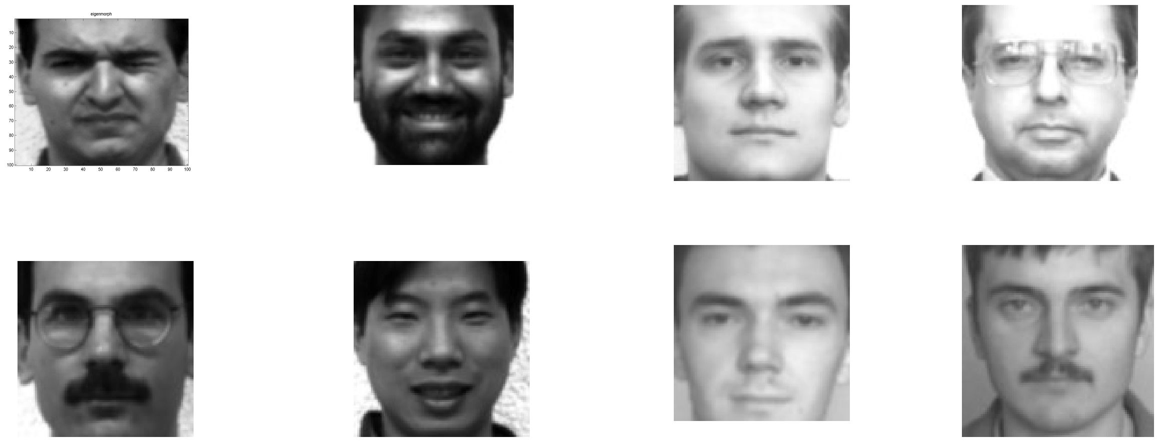
Interpreting the correlation matrix

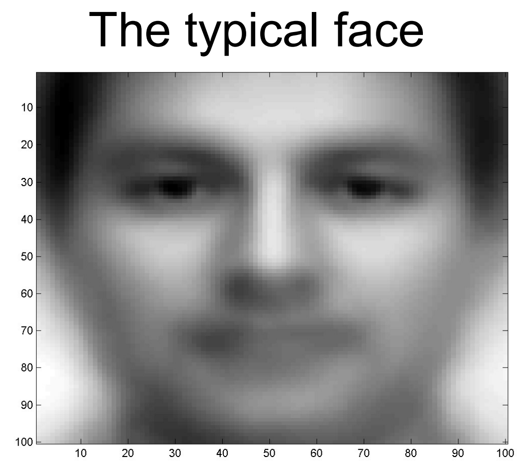


- For "uncentered" data..
  - Note although the vectors near the major axis are shorter, there are more of them, so the ellipse is wider in that direction

Returning to our problem..

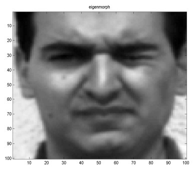
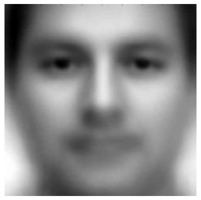
The typical face

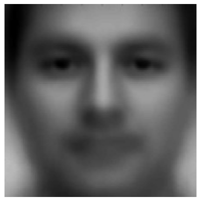


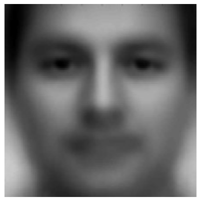
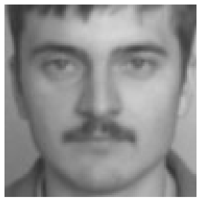
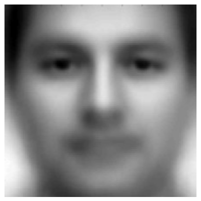


- Compute the correlation matrix for your data
  - Arrange them in matrix **X** and compute **R** = **XX**^{T}
- Compute the *principal* Eigen vector of R
  - The Eigen vector with the largest Eigen value
  - Explains most of the “energy” in the faces
- This is the typical face

The *approximation* with the first typical face

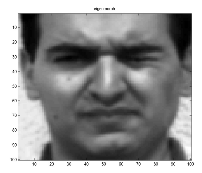



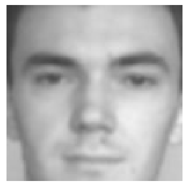
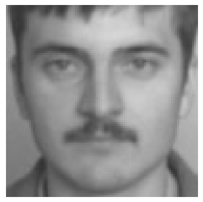


- The first typical face models some of the characteristics of the faces
  - Simply by scaling its grey level
- But the approximation has error
- Can we do better?

The *second* typical face



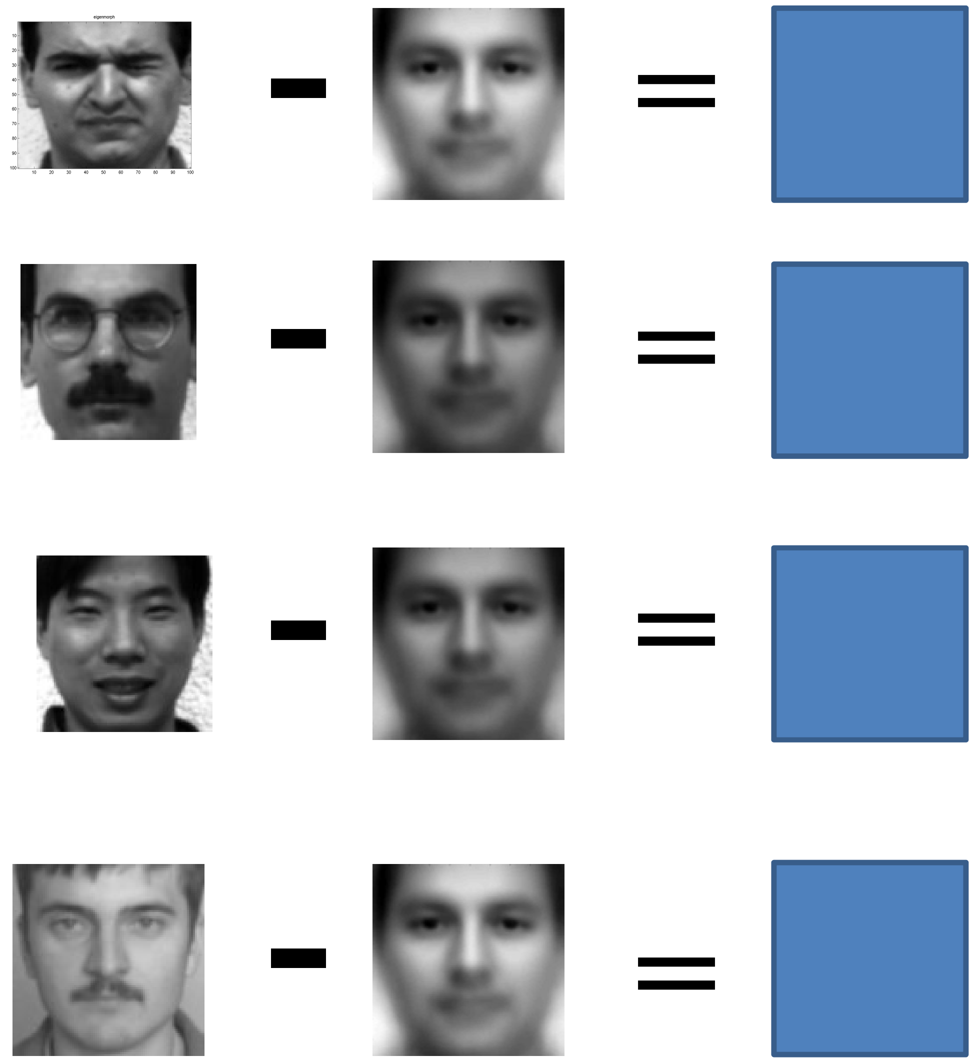
The *first* typical face

The *second* typical face?

?

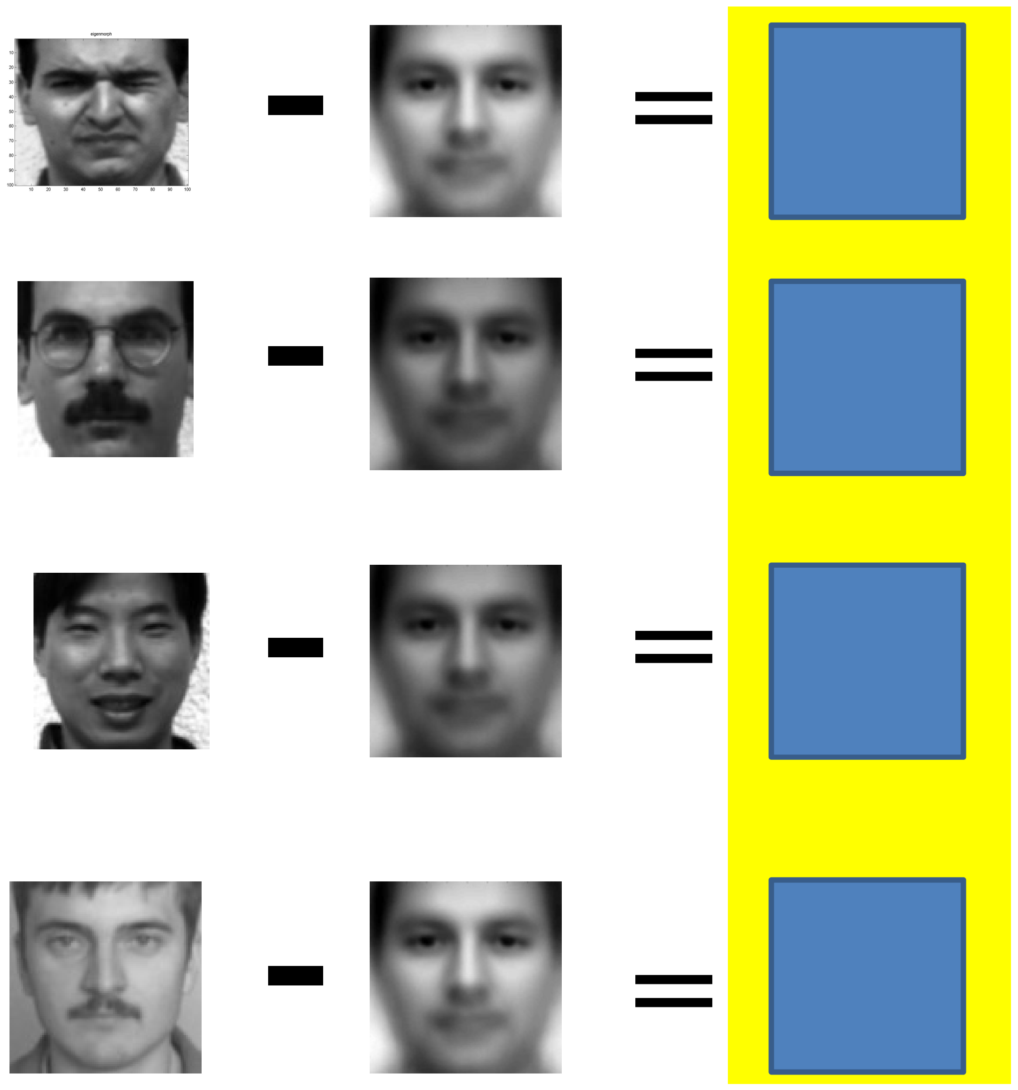
- Approximation with only one typical face V_1 has error
  - Approximating every face as $f = w_{f1} V_1$ is incomplete
- Lets add *second* face to explain this error
  - Add a *second* typical face V_1. Explain each face now as
  - $f = w_{f1} V_1 + w_{f2} V_2$
- How do we find this second face?

Solution: Iterate



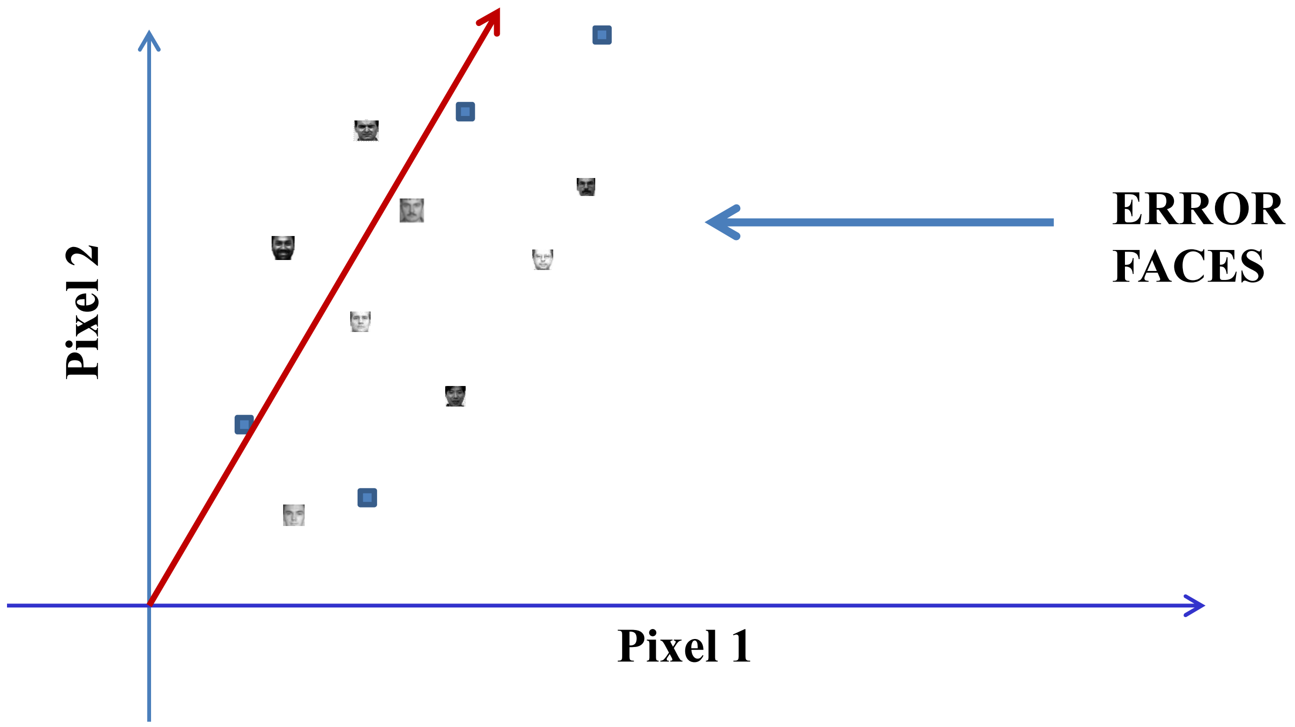
- Get the “error” faces by subtracting the first-level approximation from the original image

Solution: Iterate



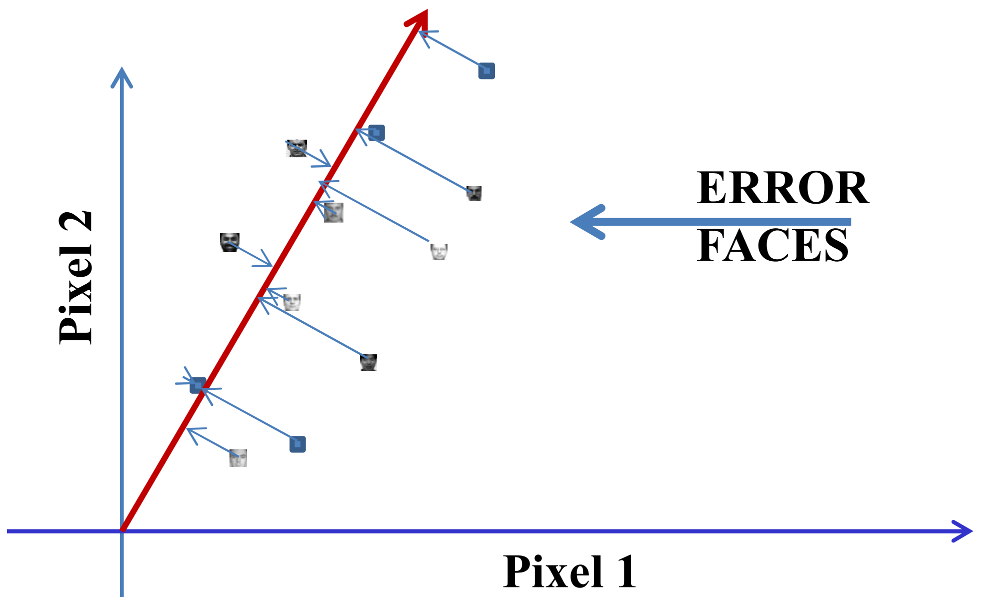
- Get the "error" faces by subtracting the first-level approximation from the original image
- Repeat the estimation on the "error" images

Abstracting the problem: Finding the *second* typical face



- Each "point" represents an *error* face in "pixel space"
- Find the vector V_2 such that the projection of these error faces on V_2 results in the least error

Minimizing error



The same math applies but now to the set of *error data points*

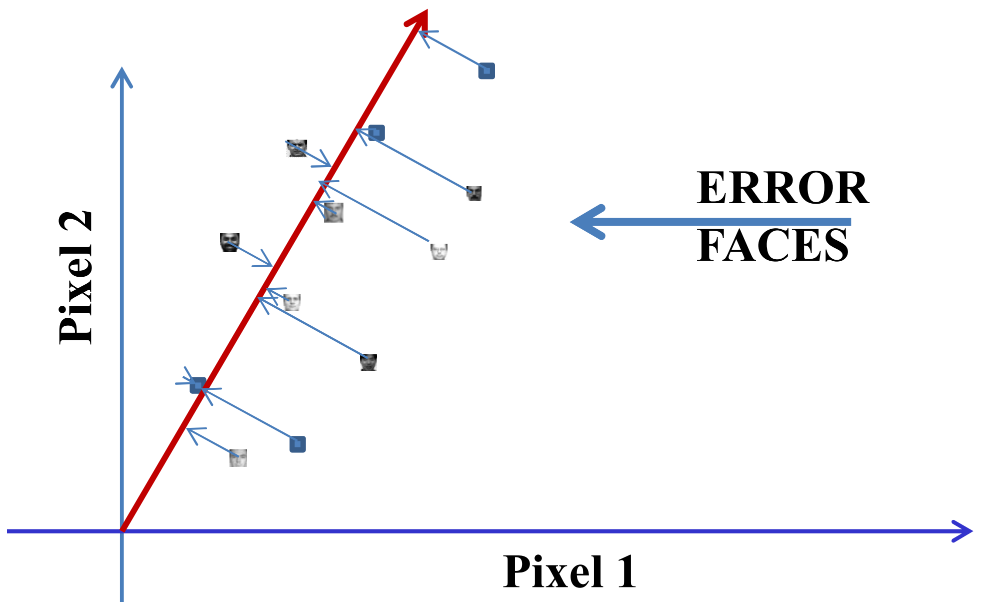
$$L = \sum_i \mathbf{e}_i^T \mathbf{e}_i - \mathbf{v}^T \sum_i \mathbf{e}_i \mathbf{e}_i^T \mathbf{v} + \lambda\left(\mathbf{v}^T \mathbf{v} - 1\right)$$

- Defining the autocorrelation of the error

$$\mathbf{R}_e = \sum \mathbf{e}\mathbf{e}^T$$

$$L = \sum_i \mathbf{e}_i^T \mathbf{e}_i - \mathbf{v}^T \mathbf{R}_e \mathbf{v} + \lambda\left(\mathbf{v}^T \mathbf{v} - 1\right)$$

Minimizing error



The same math applies but now to the set of *error data points*

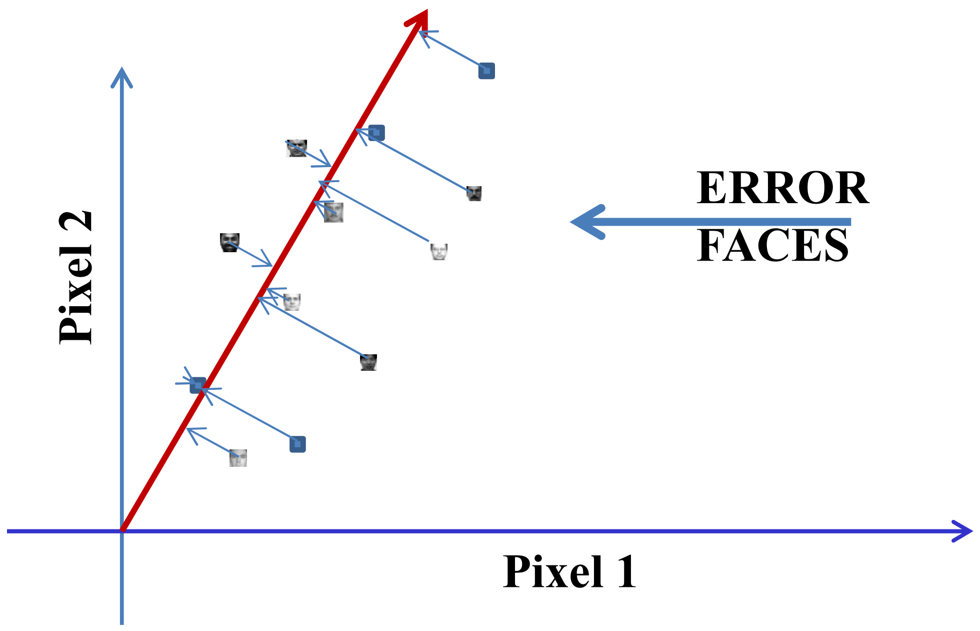
$$L = \sum_i \mathbf{e}_i^T \mathbf{e}_i - \mathbf{v}^T \mathbf{R}_e \mathbf{v} + \lambda\left(\mathbf{v}^T\mathbf{v} - 1\right)$$

- Differentiating w.r.t **v** and equating to 0

$$-2\mathbf{R}_e\mathbf{v} + 2\lambda\mathbf{v} = 0$$

$$\mathbf{R}_e\mathbf{v} = \lambda\mathbf{v}$$

Minimizing error



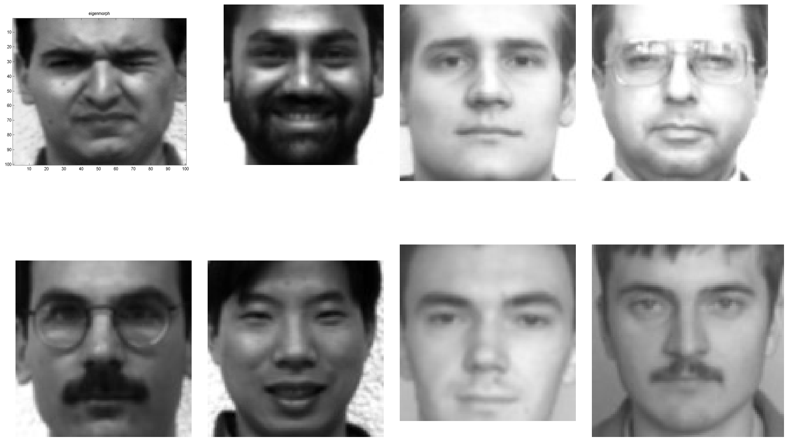
The same math applies but now to the set of *error data points*

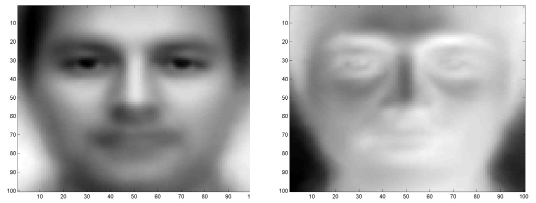
- The minimum-error basis is found by solving

$$\mathbf{R}_e \mathbf{v}_2 = \lambda \mathbf{v}_2$$

- $\mathbf{v}_2$ is an Eigen vector of the correlation matrix $\mathbf{R}_e$ corresponding to the largest eigen value λ of $\mathbf{R}_e$

Which gives us our second typical face

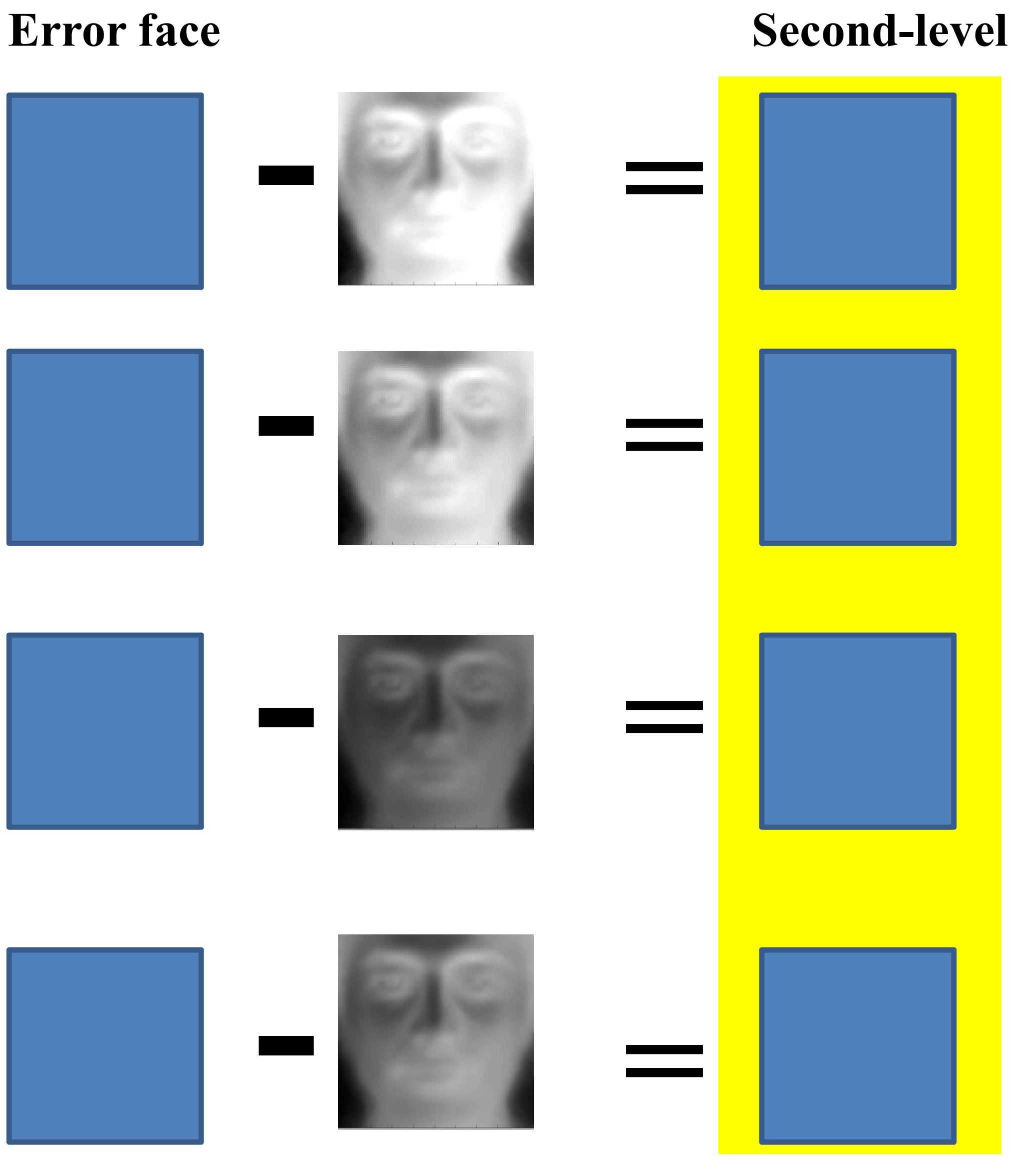




- But approximation with the two faces will *still* result in error
- So we need more typical faces to explain *this* error

- We can do this by subtracting the appropriately scaled version of the second “typical” face from the error images and repeating the process

Solution: Iterate



- Get the second-level "error" faces by subtracting the scaled second typical face from the first-level error
- Repeat the estimation on the second-level "error" images

An interesting property

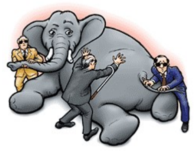
- Each “typical face” will be orthogonal to all other typical faces
  – Because each of them is learned to explain what the rest could not
  – None of these faces can explain one another!

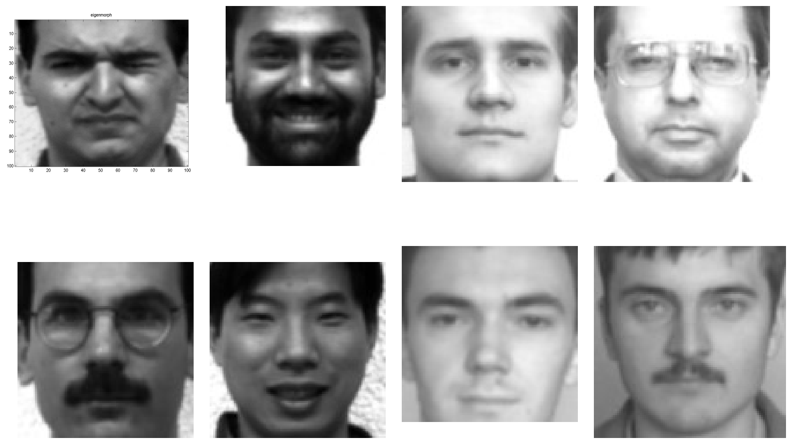
To add more faces

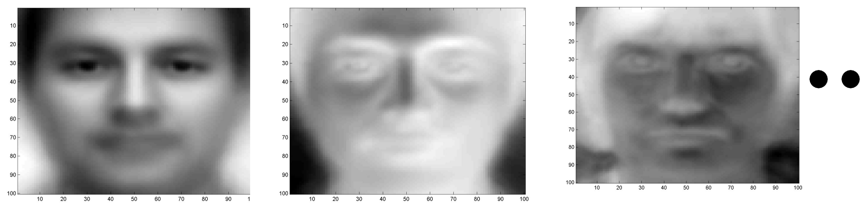
- We can continue the process, refining the error each time
  - An instance of a procedure is called "Gram-Schmidt" orthogonalization

- So what are we really doing?

A collection of least squares typical faces







- Assumption: There are a set of *K* “typical” faces that captures most of all faces
- Approximate **every** face f as $f = w_{f,1} V_1 + w_{f,2} V_2 + w_{f,3} V_3 +.. + w_{f,k} V_k$
  - V_2 is used to “correct” errors resulting from using only V_1. So on average

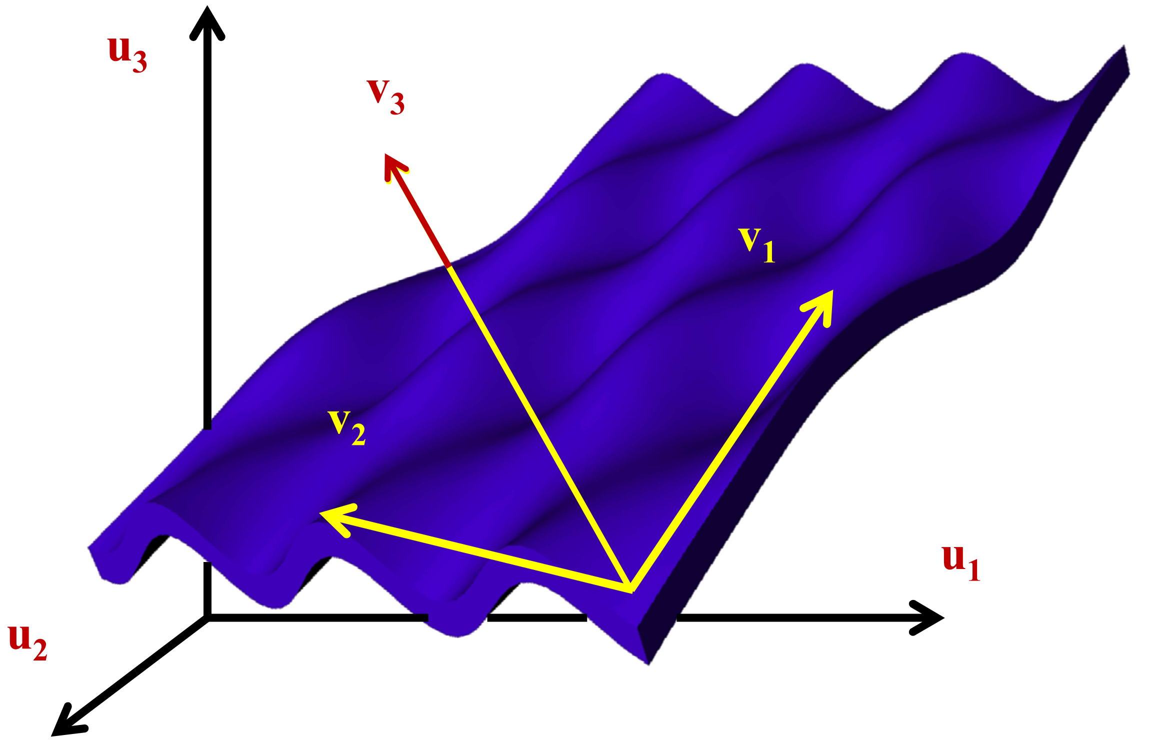
$$\left\|f-(w_{f,1}V_{f,1}+w_{f,2}V_{f,2})\right\|^2 < \left\|f-w_{f,1}V_{f,1}\right\|^2$$

  - V_3 corrects errors remaining after correction with V_2

$$\left\|f-(w_{f,1}V_{f,1}+w_{f,2}V_{f,2}+w_{f,3}V_{f,3})\right\|^2 < \left\|f-(w_{f,1}V_{f,1}+w_{f,2}V_{f,2})\right\|^2$$

  - And so on..
  - $\mathbf{V = [V_1\ V_2\ V_3]}$
- Estimate V to minimize the squared error
  - ***What is V?***

Recall: Basis based representation



- *The most important challenge* in ML: Find the best set of bases for a given data set

The Energy Compaction Property

- **Define "best"?**
- The description

$$X = w_1 B_1 + w_2 B_2 + w_3 B_3 + \ldots + w_N B_N$$

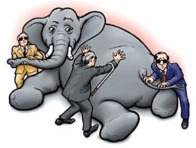
- The ideal:

$$\hat{X}_i \approx w_1 B_1 + w_2 B_2 + \ldots + w_i B_i \qquad Error_i = \left\| X - \hat{X}_i \right\|^2$$

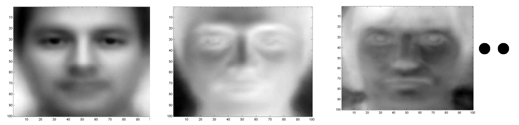
$$Error_i < Error_{i-1}$$

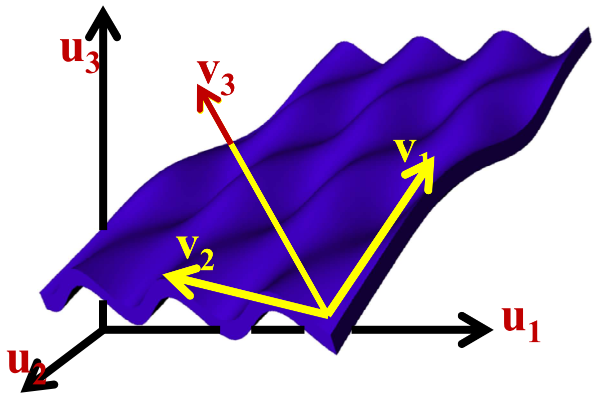
  - If the description is terminated at any point, we should still get most of the information about the data
    - No other set of bases should result in lower

$$Error_{i-1} - Error_i$$



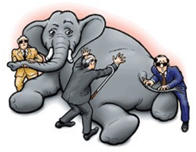
Finding the bases



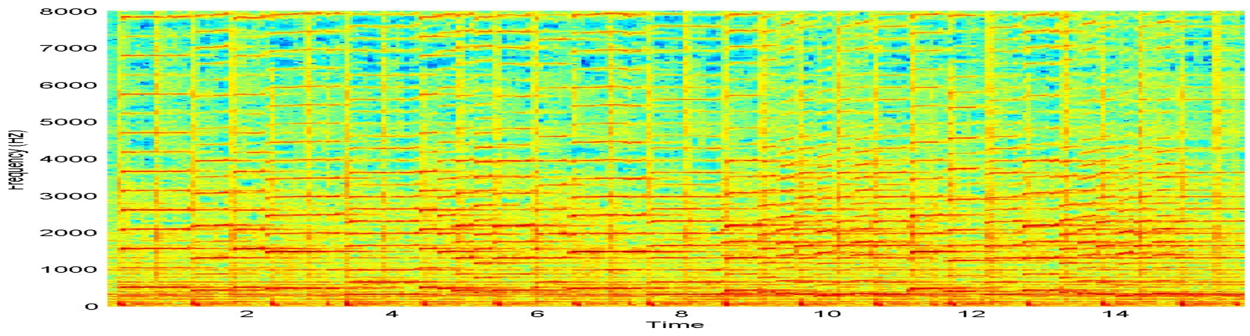


- Finding the optimal set of "typical faces" in this example is the problem of finding the optimal basis set for the data

A recollection

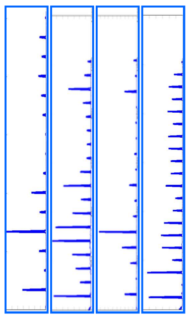


M =



S=Pinv(N) M

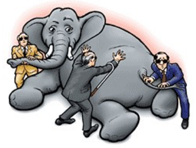
N =



$$U = NS \approx M$$

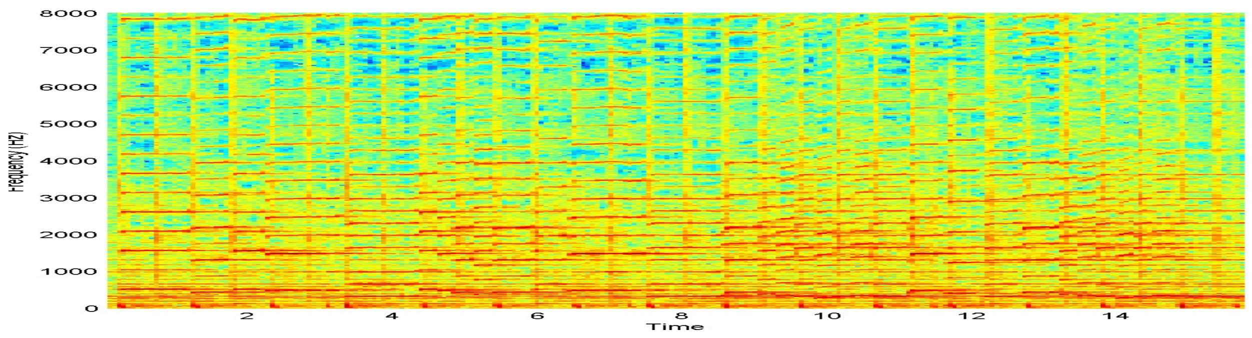
$$S = pinv(N)M$$

- Finding the best explanation of music M in terms of notes N
- Also finds the *score* S of M in terms of N

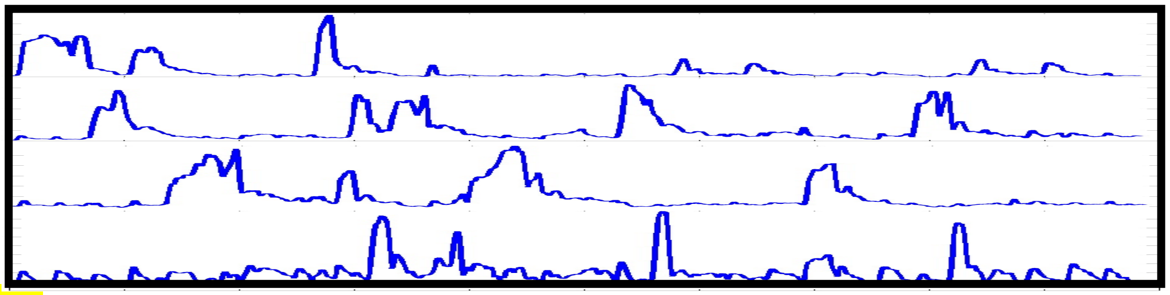


How about the other way?

M =



S =



N = M Pinv(S)

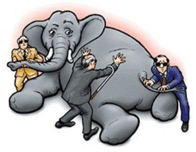
N = ?

U = ?

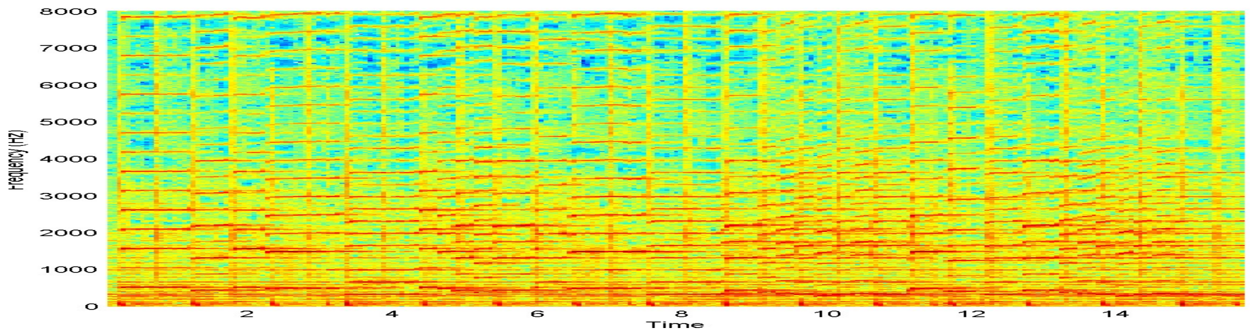
$$U = NS \approx M$$
$$N = M \ pinv(S)$$

- Finding the *notes* N given music M and score S
- Also finds best explanation of M in terms of S

Finding Everything



M =

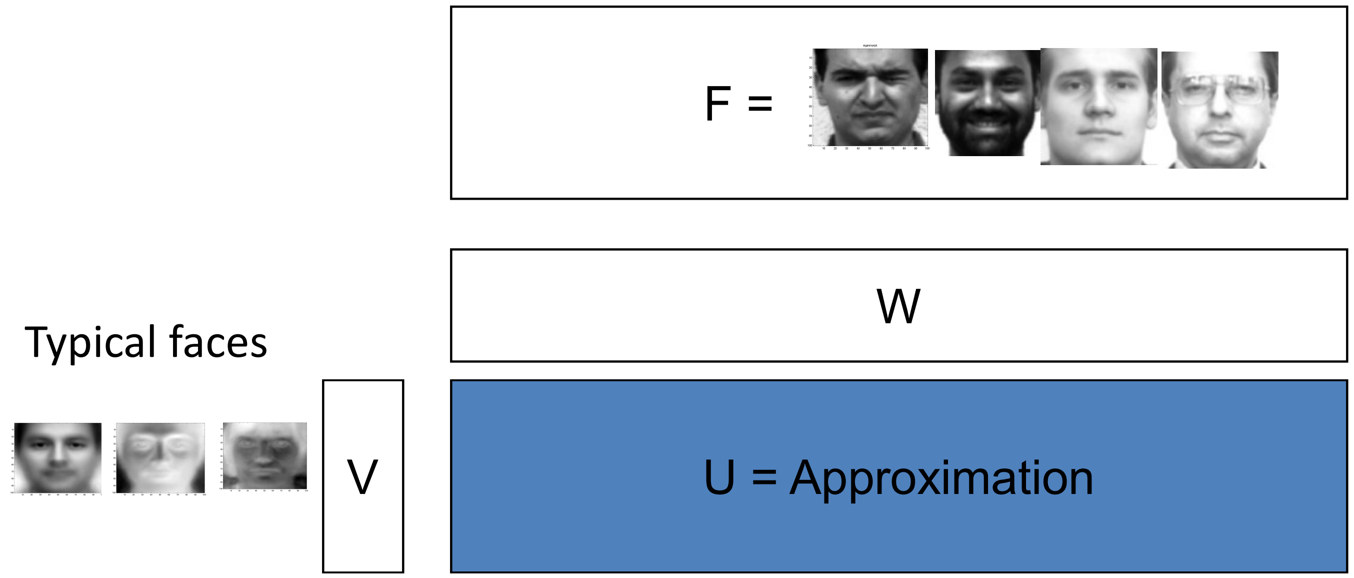


S = $\boxed{?}$

N = $\boxed{?}$ $\quad$ U = ? $\quad U = NS \approx M$

- Find the four notes and their score that generate the closest approximation to M

The Same Problem



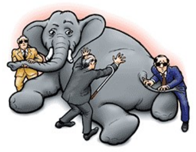
- Here U, V and W are all unknown and must be estimated
  - Such that the total squared error between F and U is minimized

- For each face f
  - $f = w_{f,1}V_1 + w_{f,2}V_2 + \cdots + w_{f,K}V_K$
- For the collection of faces $F \approx VW$
  - V is $D \times K$, W is $K \times N$
    - D is the number of pixels, N is the number of faces in the set

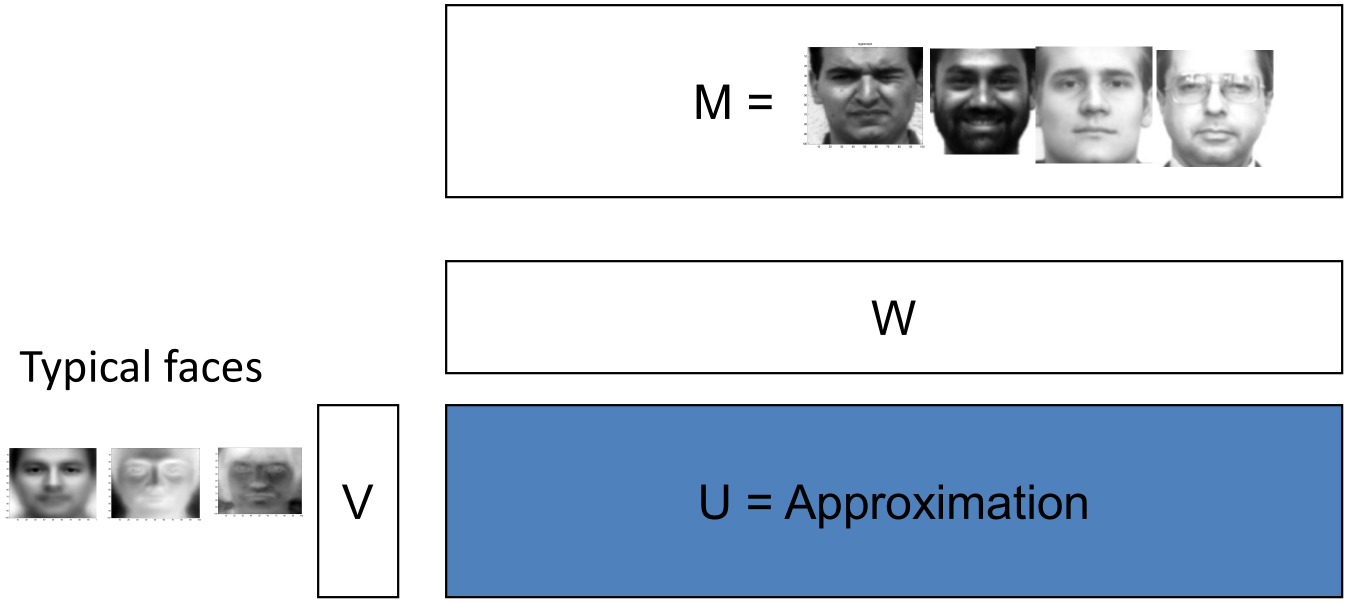
Finding the bases

- We just saw an incremental procedure for finding the bases
  - Finding one new basis at a time that explains residual error not explained by previous bases
  - An instance of a procedure is called "Gram-Schmidt" orthogonalization

- We can also do it all at once

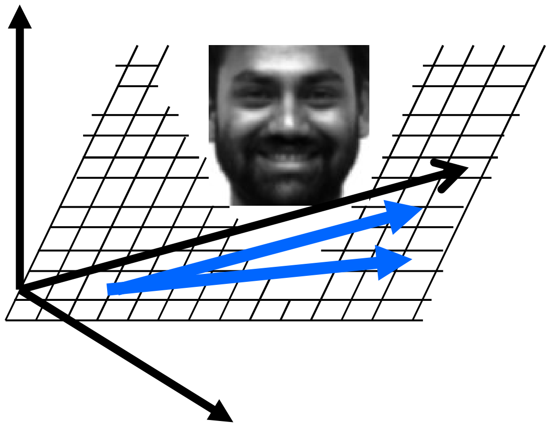


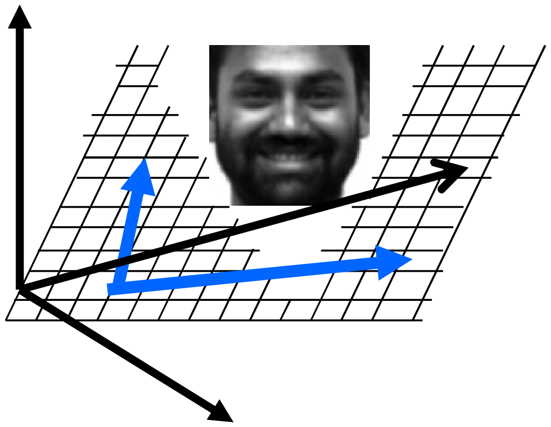
With many typical faces



- Approximate **every** face f as $f = w_{f,1} V_1 + w_{f,2} V_2 + \ldots + w_{f,k} V_k$
- Here W, V and U are ALL unknown and must be determined
  - Such that the squared error between U and M is minimum

With multiple bases



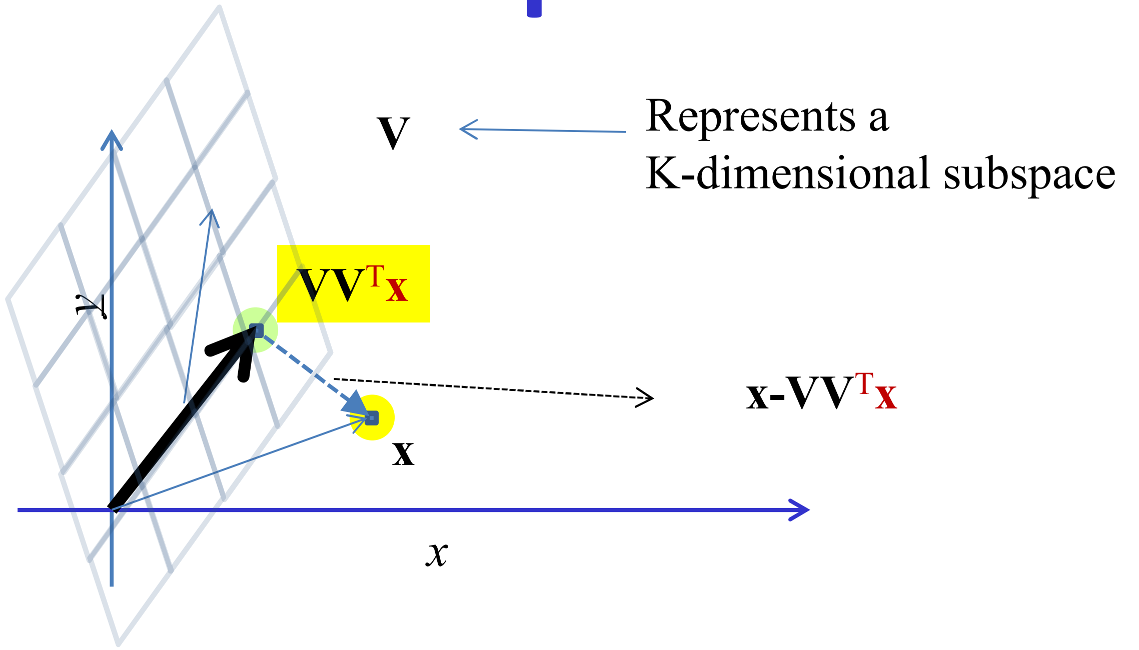


- **Assumption: all bases v_1 v_2 v_3.. are unit length**
- **Assumption: all bases are orthogonal to one another: $v_i^T v_j = 0$ if i != j**
  - We are trying to find the optimal K-dimensional subspace to project the data
  - Any set of basis vectors in this subspace will define the subspace
  - Constraining them to be orthogonal does not change this

- I.e. if $\mathbf{V} = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \ldots]$, $\quad \mathbf{V}^T\mathbf{V} = \mathbf{I}$
  - $\text{Pinv}(\mathbf{V}) = \mathbf{V}^T$

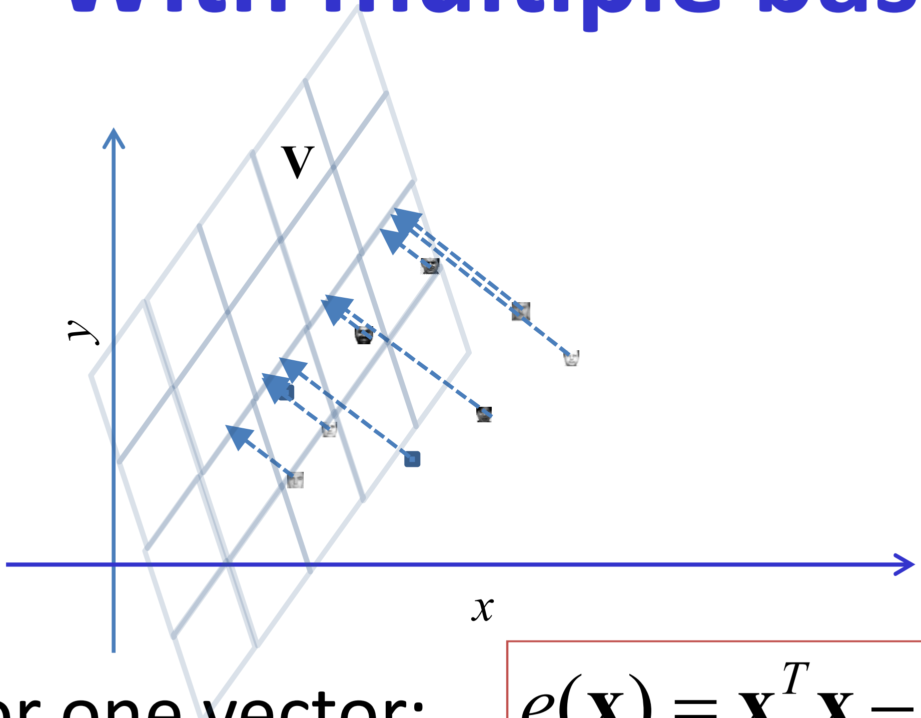
- Projection matrix for $\mathbf{V} = \mathbf{V}\text{Pinv}(\mathbf{V}) = \mathbf{V}\mathbf{V}^T$

With multiple bases



- Projection for a vector $\boxed{\hat{\mathbf{x}} = \mathbf{V}\mathbf{V}^T\mathbf{x}}$
- Error vector = $\boxed{\mathbf{x} - \hat{\mathbf{x}} = \mathbf{x} - \mathbf{V}\mathbf{V}^T\mathbf{x}}$
- Error length = $\boxed{e(\mathbf{x}) = \mathbf{x}^T\mathbf{x} - \mathbf{x}^T\mathbf{V}\mathbf{V}^T\mathbf{x}}$

With multiple bases



- Error for one vector: $\boxed{e(\mathbf{x}) = \mathbf{x}^T\mathbf{x} - \mathbf{x}^T\mathbf{V}\mathbf{V}^T\mathbf{x}}$
- Error for many vectors

$$E = \sum_i \mathbf{x}_i^T\mathbf{x}_i - \sum_i \mathbf{x}_i^T\mathbf{V}\mathbf{V}^T\mathbf{x}_i$$

- **Goal: Estimate $\mathbf{V}$ to minimize this error!**

Minimizing Error

- With constraint $\mathbf{V}^T\mathbf{V} = \mathbf{I}$, we get the modified objective

$$L = \sum_i \mathbf{x}_i^T \mathbf{x}_i - \sum_i \mathbf{x}_i^T \mathbf{V}\mathbf{V}^T \mathbf{x}_i + trace\left(\Lambda(\mathbf{V}^T\mathbf{V} - \mathbf{I})\right)$$

  - Λ is a symmetric Lagrangian matrix
  - Constraints are $\mathbf{v}_i^T\mathbf{v}_i = 1$ and $\mathbf{v}_i^T\mathbf{v}_j = 0$ for $i \neq j$

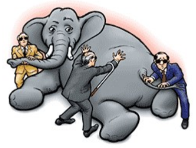
- Differentiating w.r.t $\mathbf{V}$ and equation to 0

$$-2\left(\sum_i \mathbf{x}_i^T \mathbf{x}_i\right)\mathbf{V} + 2\mathbf{V}\Lambda = 0 \quad \Rightarrow \quad \mathbf{R}\mathbf{V} = \mathbf{V}\Lambda$$

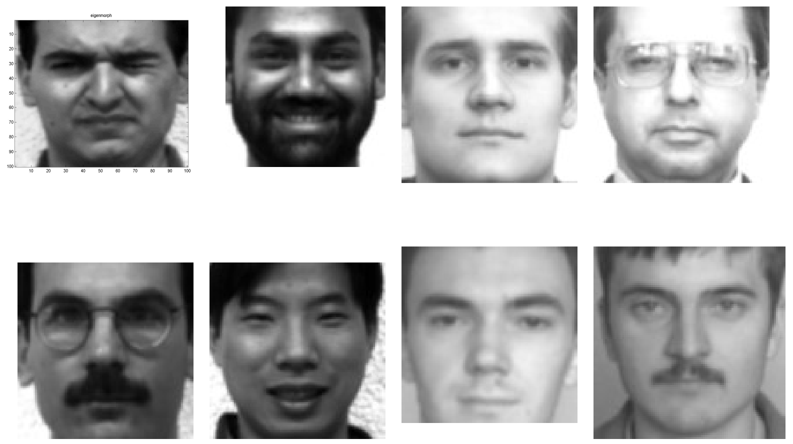
Finding the optimal K bases

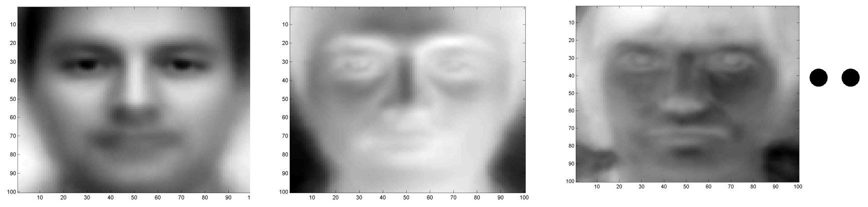
$$\mathbf{RV} = \Lambda\mathbf{V}$$

- Compute the Eigendecompsition of the correlation matrix
- Select *K* Eigen vectors
- But which *K*?
- Total error = $E = \sum_i \mathbf{x}_i^T \mathbf{x}_i - \sum_{j=1}^{K} \lambda_j$
- Select *K* eigen vectors corresponding to the *K* largest Eigen values



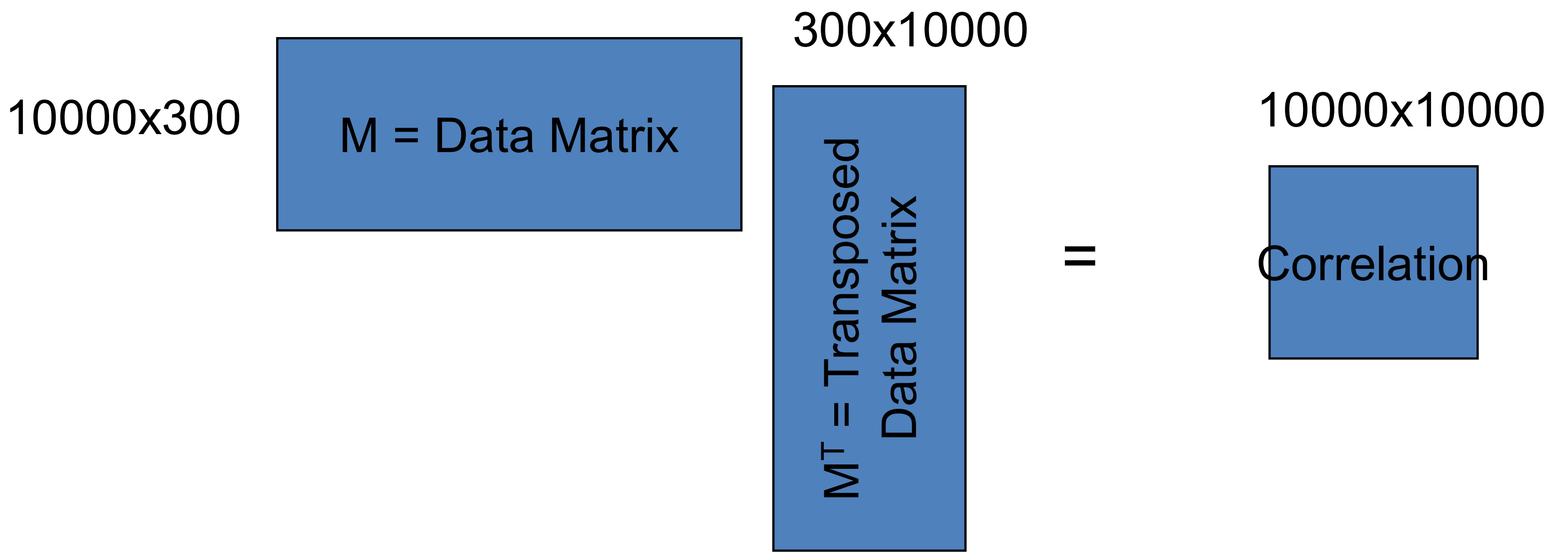
Eigen Faces!



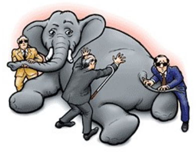


- Arrange your input data into a matrix $\mathbf{X}$
- Compute the correlation $\mathbf{R} = \mathbf{X}\mathbf{X}^{\mathrm{T}}$
- Solve the Eigen decomposition: $\mathbf{RV} = \Lambda\mathbf{V}$
- The Eigen vectors corresponding to the *K* largest eigen values are our optimal bases
- We will refer to these as *eigen faces.*

How many Eigen faces



- How to choose "K" (number of Eigen faces)
- Lay all faces side by side in vector form to form a matrix
  - In my example: 300 faces. So the matrix is 10000 x 300
- Multiply the matrix by its transpose
  - The correlation matrix is 10000x10000

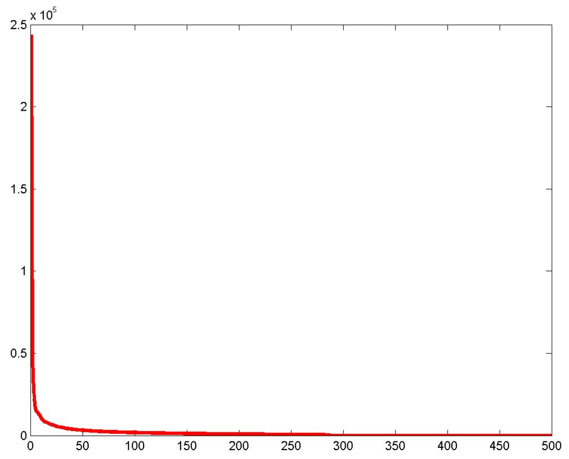


Eigen faces

[U,S] = eig(correlation)

$$S = \begin{bmatrix} \lambda_1 & . & 0 & . & 0 \\ 0 & \lambda_2 & 0 & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & . & 0 & . & \lambda_{10000} \end{bmatrix}$$

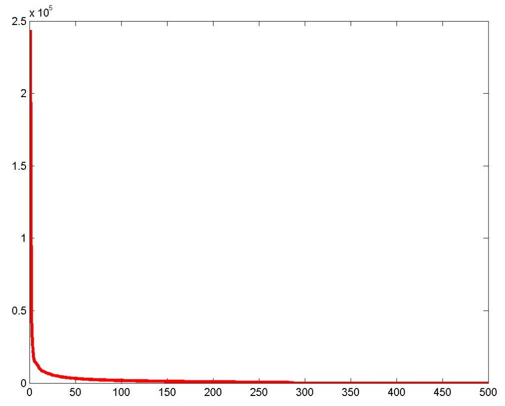
$$U = \begin{bmatrix} \rotatebox{90}{eigenface1} & \rotatebox{90}{eigenface2} & \bullet\bullet\bullet \end{bmatrix}$$

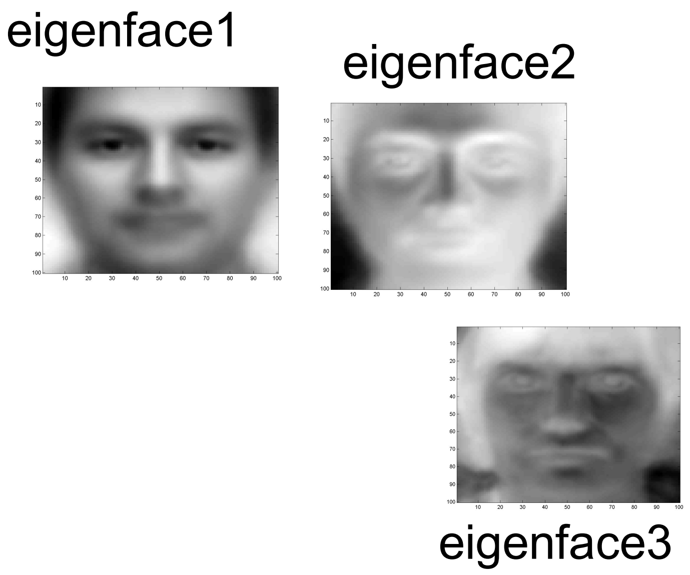


- Compute the eigen vectors
  - Only 300 of the 10000 eigen values are non-zero
    - Why?
- Retain eigen vectors with high eigen values (>0)
  - Could use a higher threshold

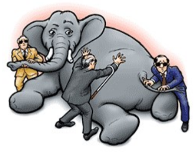
Eigen Faces

$$U = \begin{bmatrix} \text{eigenface1} & \text{eigenface2} & \cdots \end{bmatrix}$$

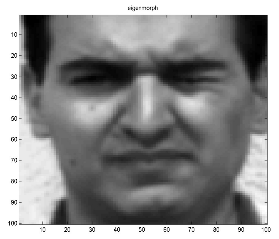


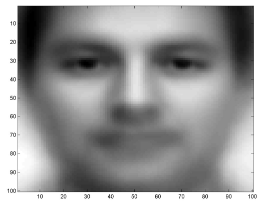
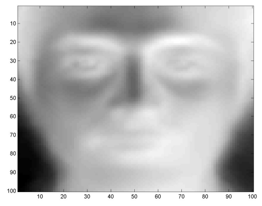
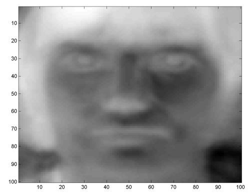


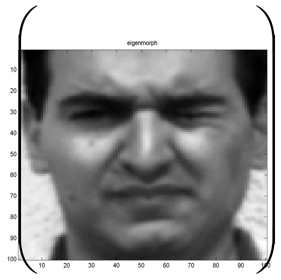
- The eigen vector with the highest eigen value is the first typical face
- The vector with the second highest eigen value is the second typical face.
- Etc.



Representing a face



$= w_1$  $+ w_2$  $+ w_3$  $\bullet\bullet\bullet$

Representation  $= [w_1\ w_2\ w_3\ \ldots.]^T$

- The weights with which the eigen faces must be combined to compose the face are used to represent the face!

Energy Compaction Example

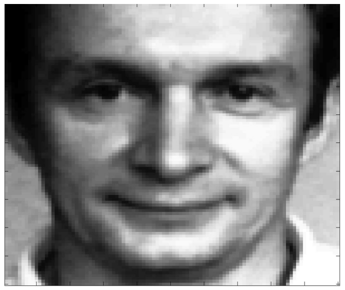
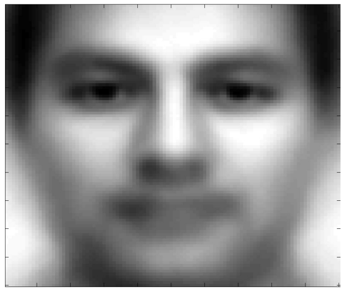
- One outcome of the "energy compaction principle": the approximations are recognizable

- Approximating a face with one basis:

$$f = w_1 \mathbf{v}_1$$

Energy Compaction Example

- One outcome of the "energy compaction principle": the approximations are recognizable

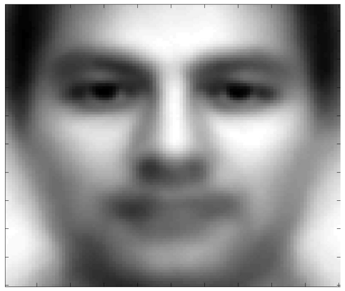
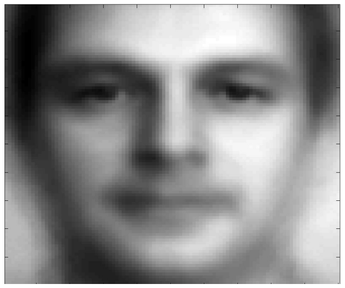



- Approximating a face with one Eigenface:

$$f = w_1 \mathbf{v}_1$$

Energy Compaction Example

- One outcome of the "energy compaction principle": the approximations are recognizable

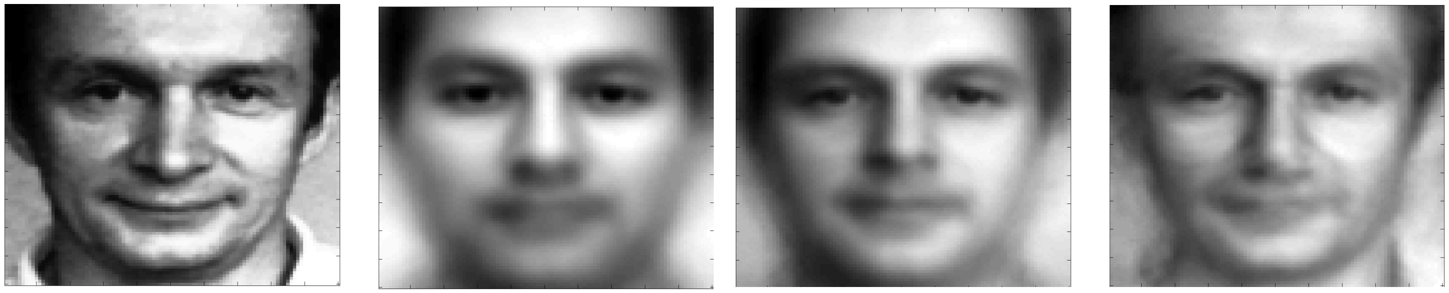



- Approximating a face with 10 eigenfaces:

$$f = w_1 \mathbf{v}_1 + w_2 \mathbf{v}_2 + \ldots w_{10} \mathbf{v}_{10}$$

Energy Compaction Example

- One outcome of the “energy compaction principle”: the approximations are recognizable

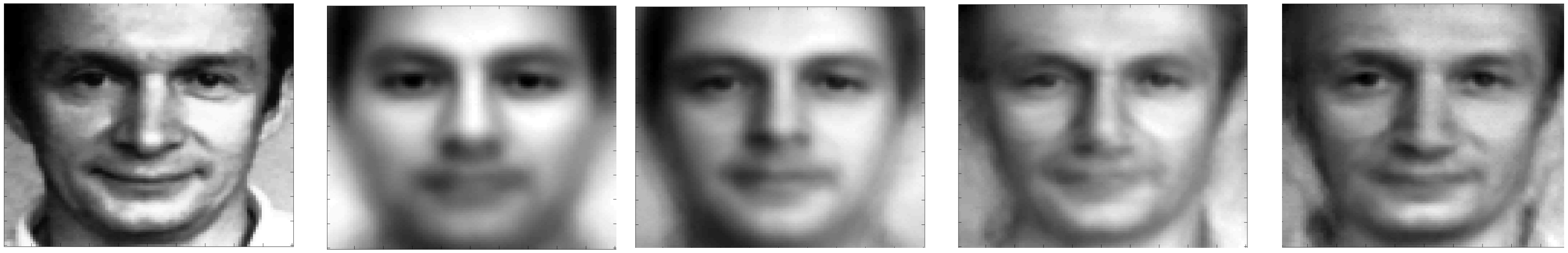


- Approximating a face with 30 eigenfaces:

$$f = w_1\mathbf{v}_1 + w_2\mathbf{v}_2 + \ldots + w_{10}\mathbf{v}_{10} + \ldots + w_{30}\mathbf{v}_{30}$$

Energy Compaction Example

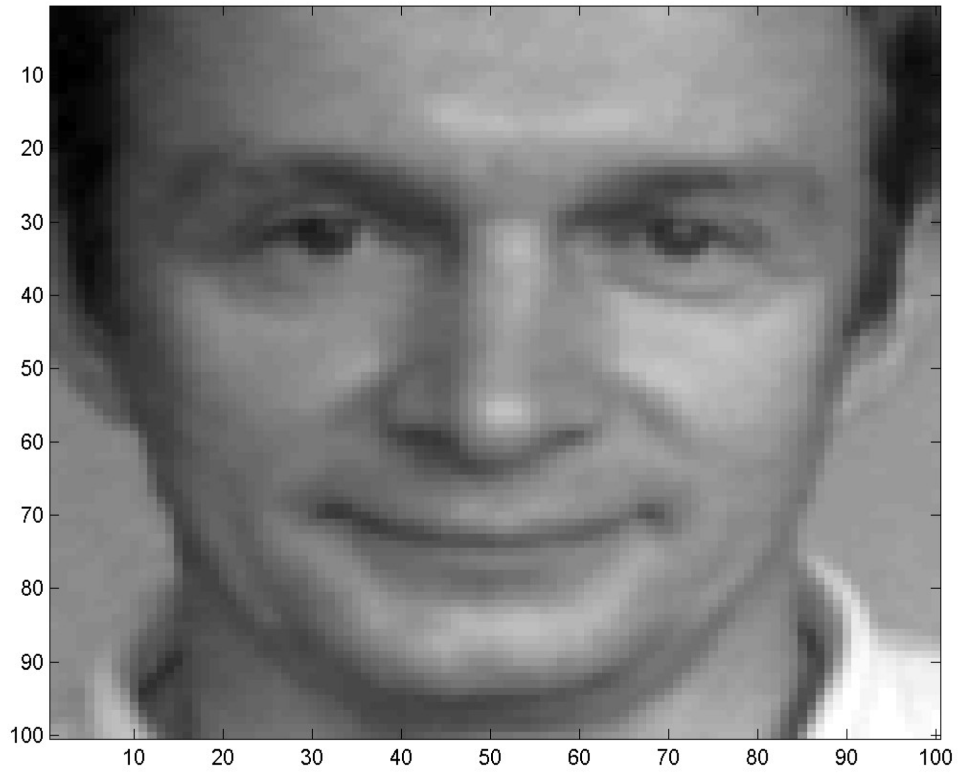
- One outcome of the “energy compaction principle”: the approximations are recognizable



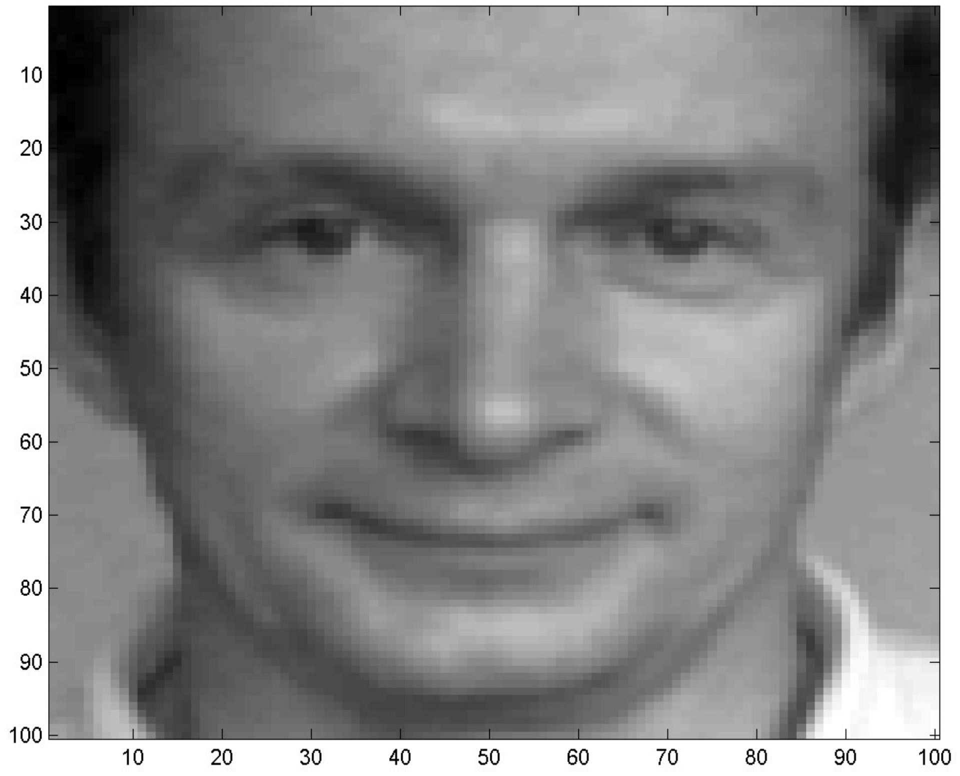
- Approximating a face with 60 eigenfaces:

$$f = w_1\mathbf{v}_1 + w_2\mathbf{v}_2 + \ldots + w_{10}\mathbf{v}_{10} + \ldots + w_{30}\mathbf{v}_{30} + \ldots + w_{60}\mathbf{v}_{60}$$

How did I do this?



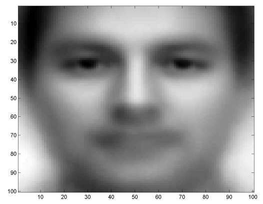
How did I do this?



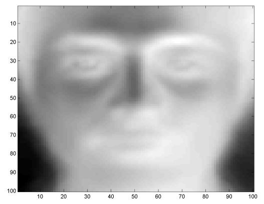
- Hint: only changing weights assigned to Eigen faces..

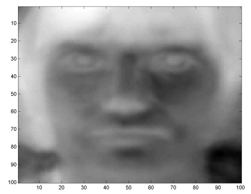
Class specificity

eigenface1



eigenface2





eigenface3

- The Eigenimages (bases) are very specific to the class of data they are trained on
  - Faces here
- They will not be useful for other classes

Class specificity

- Eigen bases are class specific

- Composing a fishbowl from Eigenfaces

Class specificity

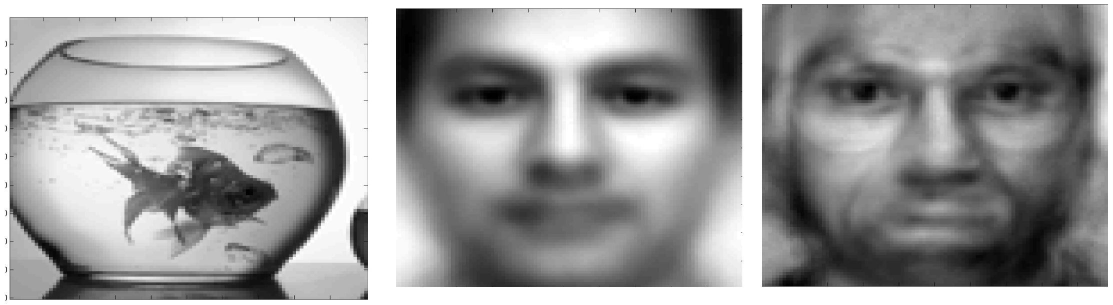
- Eigen bases are class specific

- Composing a fishbowl from Eigenfaces
- With 1 basis

$$f = w_1 \mathbf{v}_1$$

Class specificity

- Eigen bases are class specific

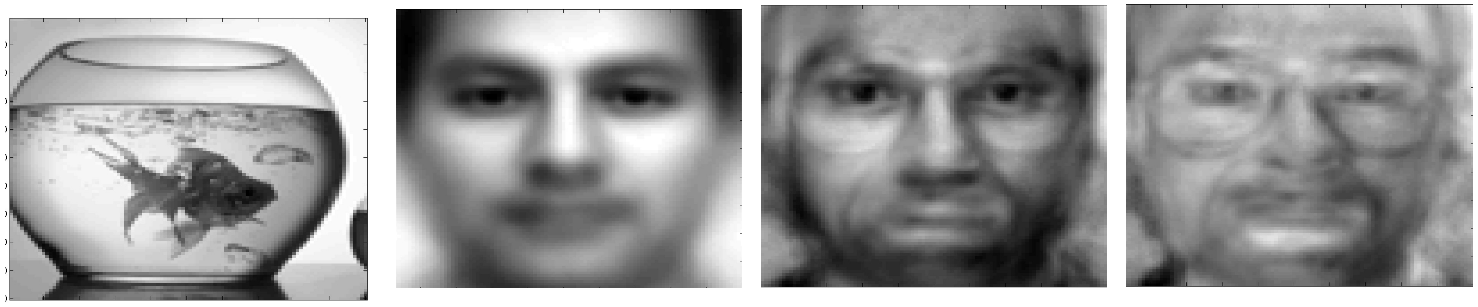


- Composing a fishbowl from Eigenfaces
- With 10 bases

$$f = w_1\mathbf{v}_1 + w_2\mathbf{v}_2 + \ldots + w_{10}\mathbf{v}_{10}$$

Class specificity

- Eigen bases are class specific

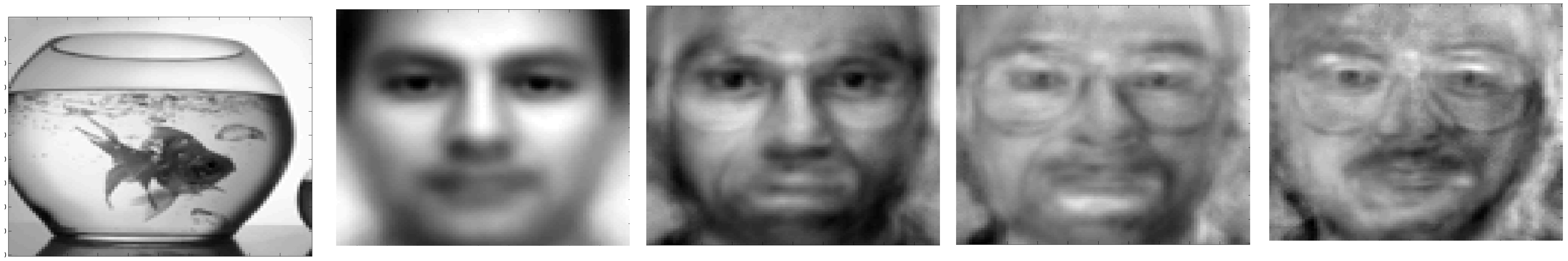


- Composing a fishbowl from Eigenfaces
- With 30 bases

$$f = w_1\mathbf{v}_1 + w_2\mathbf{v}_2 + \ldots + w_{10}\mathbf{v}_{10} + \ldots + w_{30}\mathbf{v}_{30}$$

Class specificity

- Eigen bases are class specific

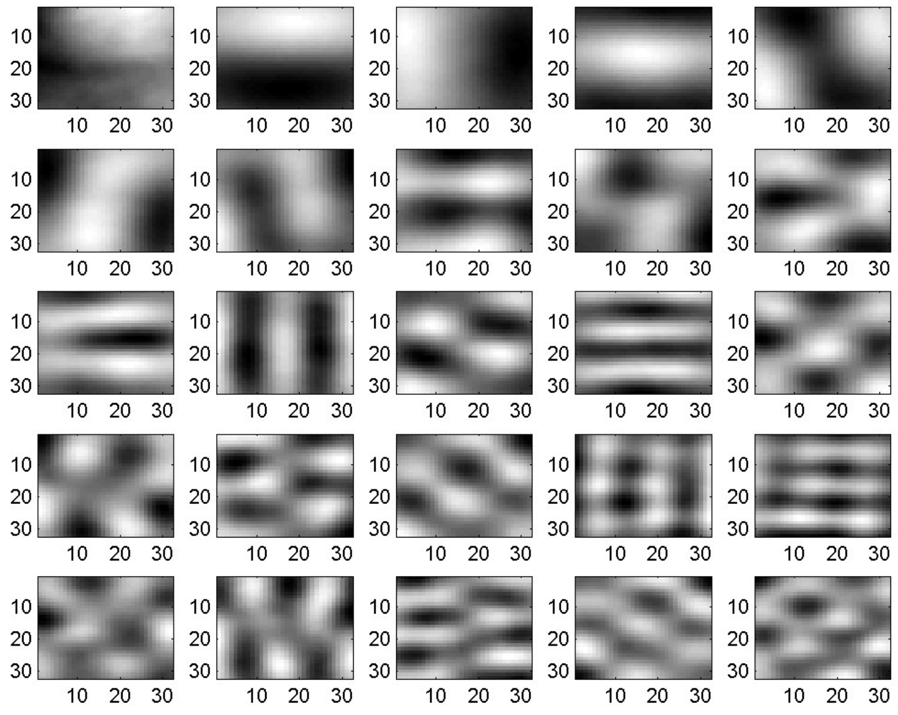


- Composing a fishbowl from Eigenfaces
- With 100 bases

$$f = w_1\mathbf{v}_1 + w_2\mathbf{v}_2 + \ldots + w_{10}\mathbf{v}_{10} + \ldots + w_{30}\mathbf{v}_{30} + \ldots + w_{100}\mathbf{v}_{100}$$

Universal bases

- Universal bases..



- End up looking a lot like *discrete cosine transforms!!!!*
- *DCTs are the best "universal" bases*
  - *If you don't know what your data are, use the DCT*

Relation of Eigen decomposition to SVD

Eigen Decomposition of the Correlation Matrix

$$\mathbf{XX}^T = \mathbf{R} = \mathbf{EDE}^T$$

SVD of the Data Matrix

$$\mathbf{X} = \mathbf{USV}^T$$

$$\mathbf{XX}^T = \mathbf{USV}^T\ \mathbf{VSU}^T = \mathbf{US}^2\mathbf{U}^T$$

Comparing $\mathbf{E} = \mathbf{U}$ $\mathbf{D} = \mathbf{S}^2$

- Eigen decomposition of the correlation matrix gives you left singular vectors of data matrix

Dimensionality Reduction

- $\mathbf{R} = \mathbf{EDE}^T$
  - The columns of $\mathbf{E}$ are our "Eigen" bases
- We can express any vector X as a combination of these bases

$$X = w_D^X E_1 + w_D^X E_D + \cdots + w_D^X E_D$$

- Using only the "top" K bases
  - Corresponding to the top K Eigen values

$$X \approx w_D^X E_1 + w_D^X E_D + \cdots + w_K^X E_K$$

Dimensionality Reduction

- Using only the "top" K bases
  - Corresponding to the top K Eigen values

$$X \approx w_D^X E_1 + w_D^X E_D + \cdots + w_K^X E_K$$

- In vector form:

$$X \approx \boldsymbol{E}_{1:K} \mathbf{w}_K^X$$
$$\mathbf{w}_K^X = Pinv(\boldsymbol{E}_{1:K})X = \boldsymbol{E}_{1:K}^T X$$
$$\mathbf{W}_K^X = \boldsymbol{E}_{1:K}^T \mathbf{X}$$

- If "$\boldsymbol{E}$" is agreed upon, knowing $\mathbf{W}_K^X$ is sufficient to reconstruct $\mathbf{X}$
  - Store only K numbers per vector instead of D without losing too much information
  - **Dimensionality Reduction**

Lets give it a name

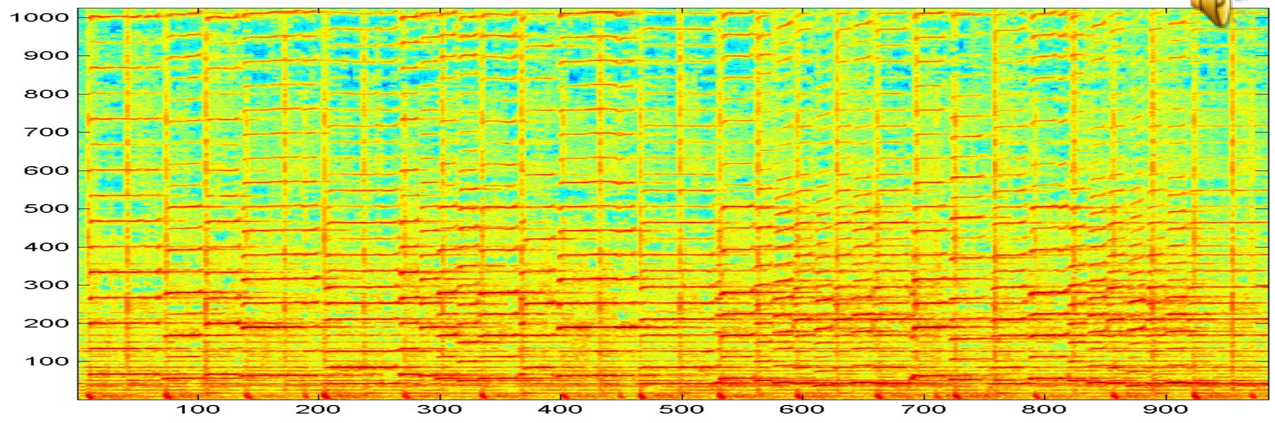
$$\mathbf{R} = \mathbf{E}\mathbf{D}\mathbf{E}^T$$

$\mathbf{E}$ are the “Eigen Bases”

$$\mathbf{W}_K^X = \boldsymbol{E}_{1:K}^T\mathbf{X}$$

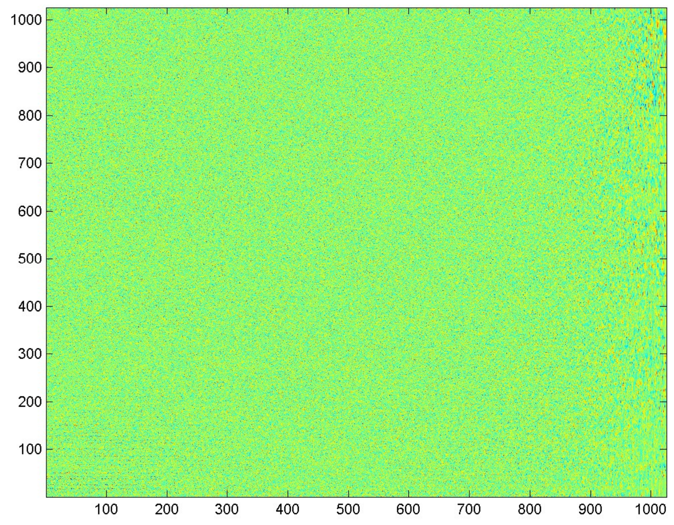
- Retaining only the top *K* weights for every data vector
  - Computed by multiplying the data matrix by the transpose of the top *K* Eigen vectors of ***R***
- This is called the *Karhunen Loeve Transform*
  - ***Not PCA!***

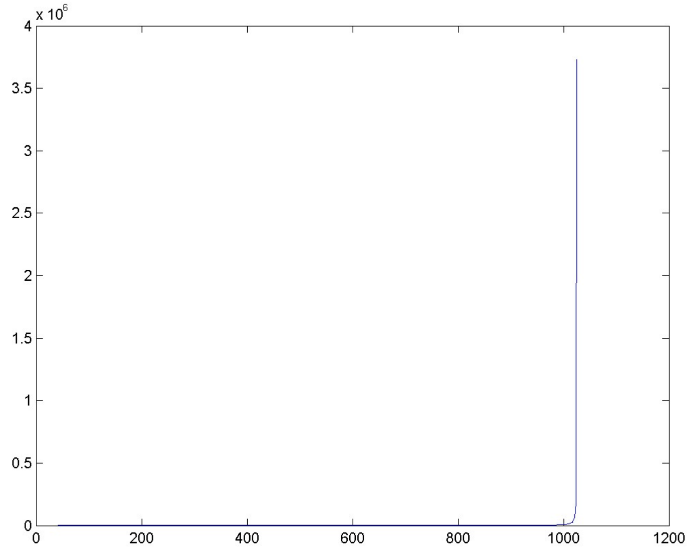
An audio example



- The spectrogram has 974 vectors of dimension 1025
- The covariance matrix is size 1025 x 1025
- There are 1025 eigenvectors

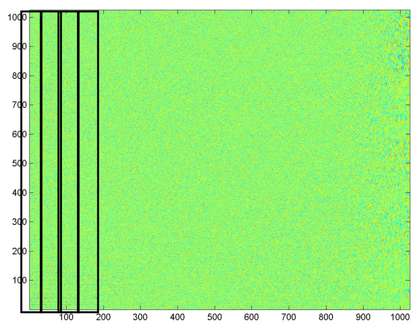
Eigenvalues and Eigenvectors

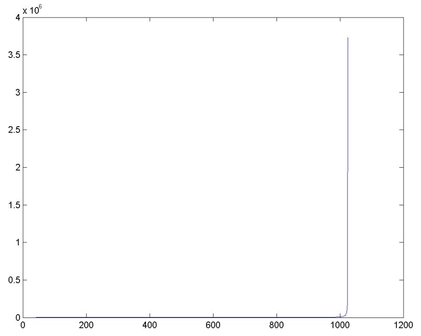


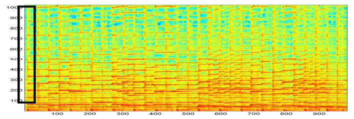


- Left panel: Matrix with 1025 eigen vectors
- Right panel: Corresponding eigen values
  - Most Eigen values are close to zero
    - The corresponding eigenvectors are “unimportant”

Eigenvalues and Eigenvectors





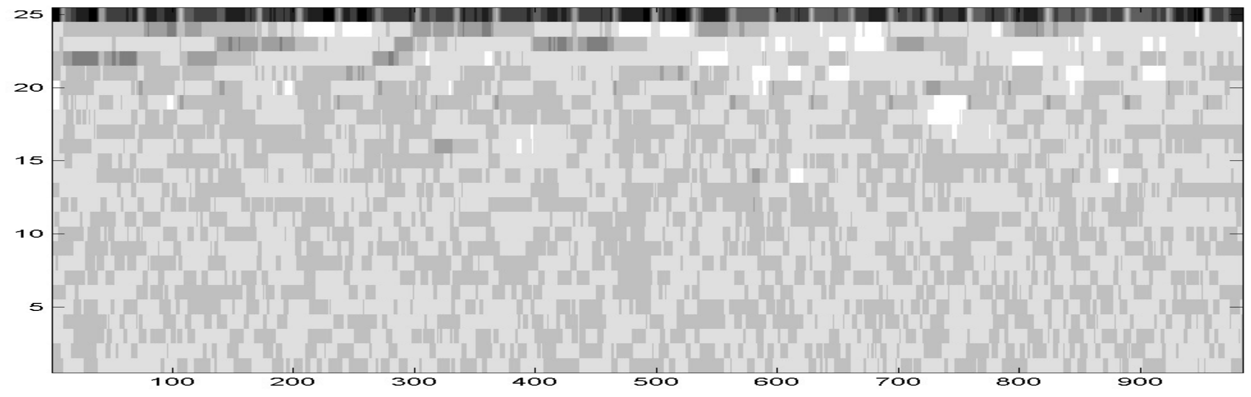


Vec = a1 *eigenvec1 + a2 * eigenvec2 + a3 * eigenvec3 …

- The vectors in the spectrogram are linear combinations of all 1025 Eigen vectors
- The Eigen vectors with low Eigen values contribute very little
  - The average value of a_i is proportional to the square root of the Eigenvalue
  - Ignoring these will not affect the composition of the spectrogram

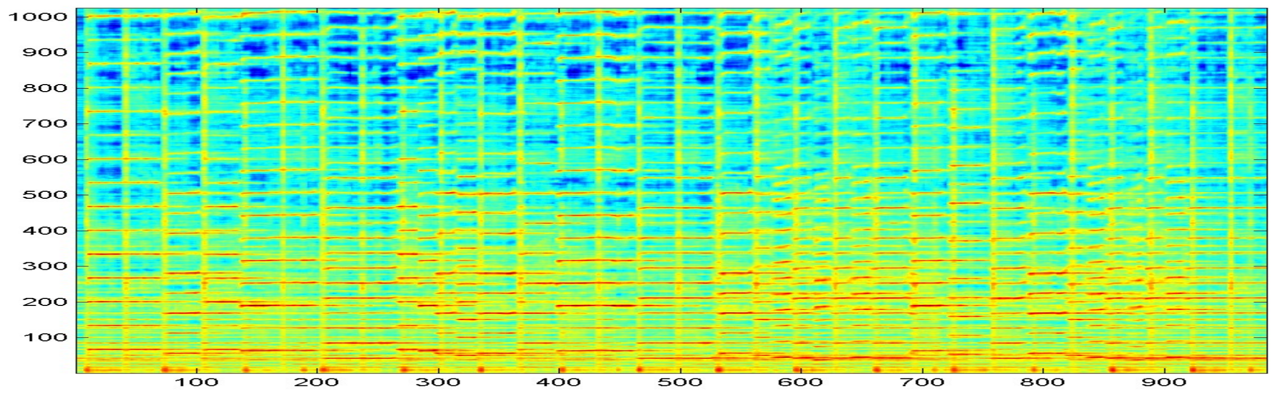
An audio example

$$V_{reduced} = [V_1 \quad . \quad . \quad V_{25}]$$
$$M_{lowdim} = Pinv(V_{reduced})M$$



- The same spectrogram projected down to the 25 eigen vectors with the highest eigen values
  – Only the 25-dimensional weights are shown
    - The weights with which the 25 eigen vectors must be added to compose a least squares approximation to the spectrogram

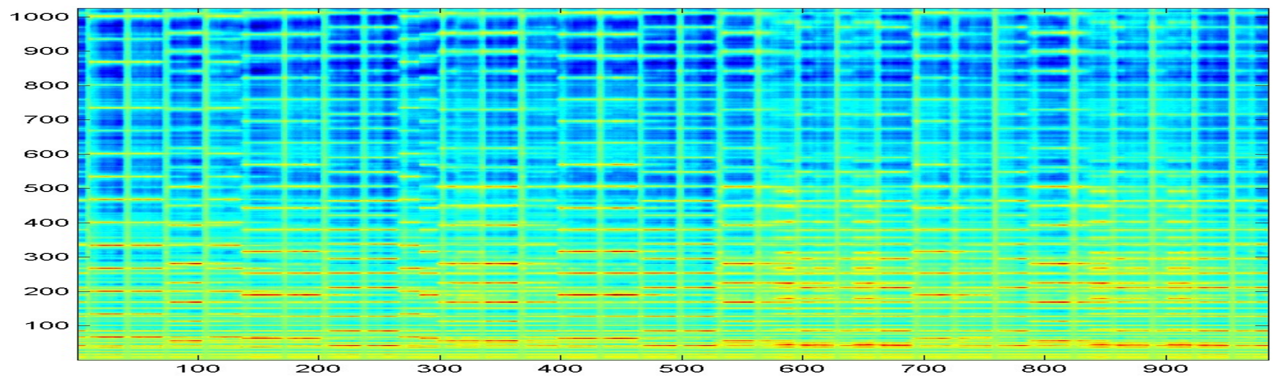
An audio example



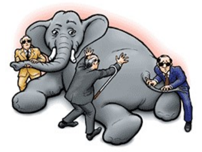
$$M_{reconstructed} = V_{reduced} M_{low\dim}$$

- The same spectrogram constructed from only the 25 Eigen vectors with the highest Eigen values
  - Looks similar
    - With 100 Eigenvectors, it would be indistinguishable from the original
  - Sounds pretty close
  - But now sufficient to store 25 numbers per vector (instead of 1024)

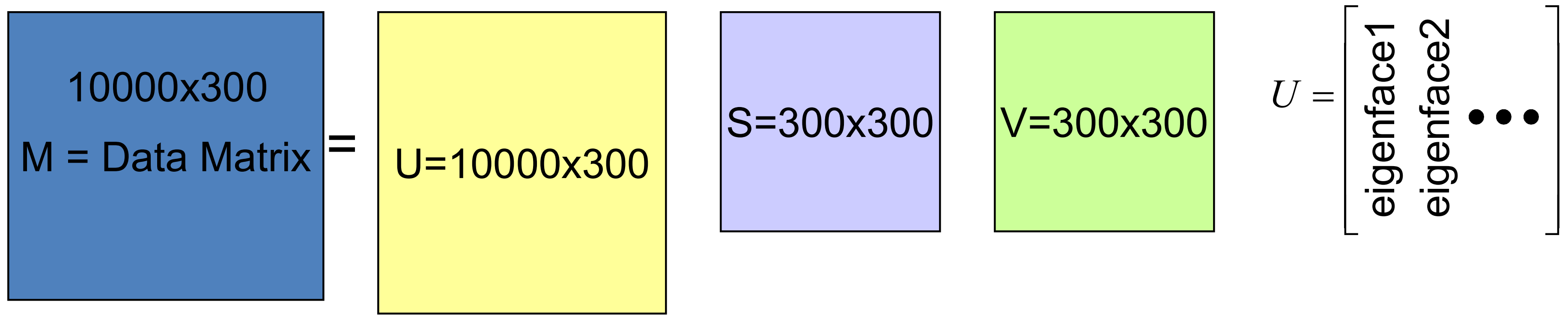
With only 5 eigenvectors



- The same spectrogram constructed from only the 5 Eigen vectors with the highest Eigen values
  - Highly recognizable



SVD instead of Eigen



- Do we need to compute a 10000 x 10000 correlation matrix and then perform Eigen analysis?
  - Will take a very long time on your laptop
- SVD
  - Only need to perform “Thin” SVD. Very fast
    - U = 10000 x 300
      - The columns of U are the eigen faces!
      - The Us corresponding to the “zero” eigen values are not computed
    - S = 300 x 300
    - V = 300 x 300

Using SVD to compute Eigenbases

[U, S, V] = SVD(X)

- U will have the Eigenvectors
- Thin SVD for 100 bases:

[U,S,V] = svds(X, 100)

- Much more efficient

Eigen Decomposition of data

- Nothing magical about faces or sound – can be applied to any data.
  - Eigen analysis is one of the key components of data compression and representation
  - Represent N-dimensional data by the weights of the K leading Eigen vectors
    - Reduces effective dimension of the data from N to K
    - But requires knowledge of Eigen vectors

What kind of representation?

- What we just saw: *Karhunen Loeve Expansion*
- What you may be familiar with: *Principal Component Analysis*

- The two are similar, but not the same!!

Linear vs. Affine

- The model we saw (KLE)
  - Approximate **every** face f as
    $f = w_{f,1} V_1 + w_{f,2} V_2 + \ldots + w_{f,k} V_k$
  - Linear combination of bases

- If you add a constant (PCA)

$$f = w_{f,1} V_1 + w_{f,2} V_2 + \ldots + w_{f,k} V_k + m$$

  - *Affine* combination of bases

Affine expansion

- Estimate

$$f = m + w_{f,1} V_1 + w_{f,2} V_2 + \ldots + w_{f,k} V_k$$

- Using the *energy compaction* principle leads to the usual incremental estimation rule
  - m must explain most of the energy
  - Each new basis must explain most of the residual energy

Estimation with the constant

- Estimate

$$f = w_{f,1} V_1 + w_{f,2} V_2 + \ldots + w_{f,k} V_k + m$$

- Lets do this incrementally first:
- $f \approx m$
  - For every face
  - Find *m* to optimize the approximation

Estimation with the constant

- Estimate

  $$\mathrm{f} \approx \mathrm{m}$$

  - for *every f!*
- Error over all faces $E = \sum_f ||f - m||^2$
- Minimizing the error with respect to *m*, we simply get
  - $m = \frac{1}{N}\sum_f f$
- The *mean* of the data

Estimation the remaining

- Same procedure as before:
  - Remaining "typical faces" must model what the constant *m* could not

- Subtract the constant from every data point
  - $\hat{f} = f - m$
- Now apply the model:
  - $\hat{f} = \mathrm{w}_{f,1}\,\mathrm{V}_1 + \mathrm{w}_{f,2}\,\mathrm{V}_2 + \ldots + \mathrm{w}_{f,k}\,\mathrm{V}_k$

- This is just Eigen analysis of the "mean-normalized" data
  - Also called the "centered" data

Estimating the Affine model

$$f = w_{f,1} V_1 + w_{f,2} V_2 + \ldots + w_{f,k} V_k + m$$

- First estimate the mean *m*

$$m = \frac{1}{N}\sum_{f} f$$

- Compute the correlation matrix of the "centered" data $\hat{f} = f - m$
  - $\boldsymbol{C} = \sum_f \hat{f}\hat{f}^T = \sum_f (f - m)(f - m)^T$
  - This is the *covariance* matrix of the set of f

Estimating the Affine model

$$f = w_{f,1} V_1 + w_{f,2} V_2 + \ldots + w_{f,k} V_k + m$$

- First estimate the mean m

$$m = \frac{1}{N} \sum_f f$$

- Compute the covariance matrix
  - $\boldsymbol{C} = \sum_f (f - m)(f - m)^T$

- Eigen decompose!

$$\mathbf{CV} = \mathbf{\Lambda V}$$

- The Eigen vectors corresponding to the top *k* Eigen values give us the bases V_k

Linear vs. Affine

- The model we saw
  - Approximate **every** face f as
    $f = w_{f,1} V_1 + w_{f,2} V_2 + \ldots + w_{f,k} V_k$
  - The ***Karhunen Loeve Expansion***
  - Retains maximum ***Energy*** for any order k

- If you add a constant

$$f = w_{f,1} V_1 + w_{f,2} V_2 + \ldots + w_{f,k} V_k + m$$

  - ***Principal Component Analysis***
  - Retains maximum ***Variance*** for any order k

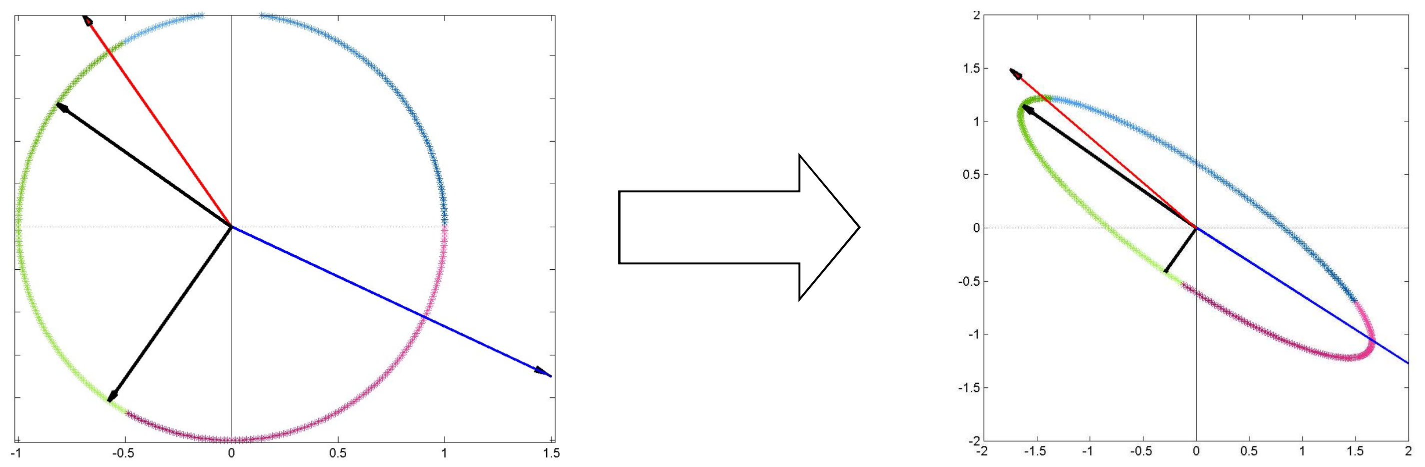
How do they relate

- Relationship between correlation matrix and covariance matrix

$$\mathbf{R} = \mathbf{C} + mm^{\mathrm{T}}$$

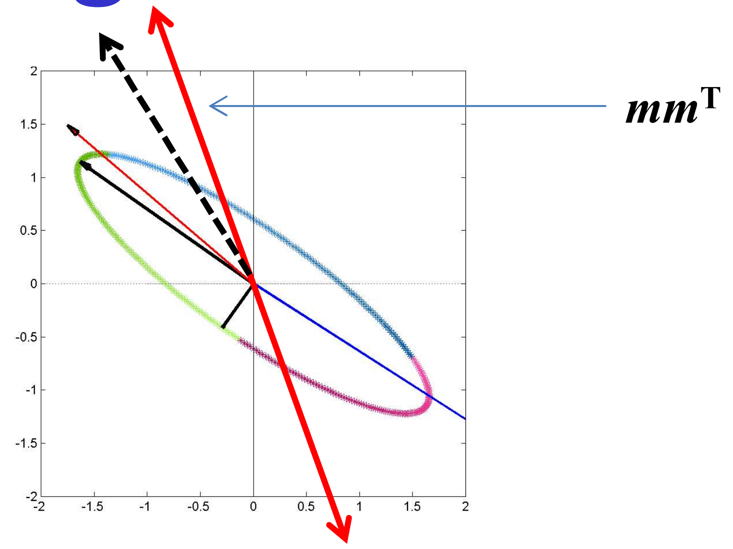
- *Karhunen Loeve* bases are Eigen vectors of **R**
- *PCA* bases are Eigen vectors of **C**
- How do they relate
  - Not easy to say..

The Eigen vectors



- The Eigen vectors of ***C*** are the major axes of the ellipsoid ***Cv***, where ***v*** are the vectors on the unit sphere

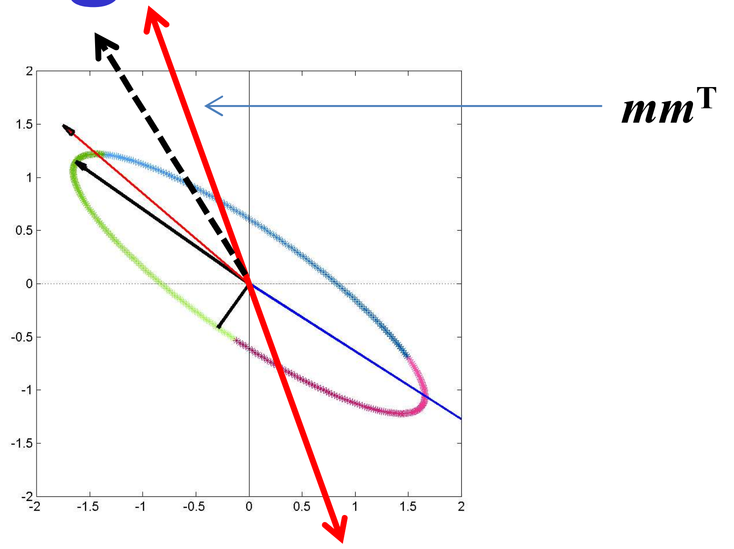
The Eigen vectors



- The Eigen vectors of ***R*** are the major axes of the ellipsoid ***Cv*** + ***mm***^{T}***v***

- Note that ***mm***^{T} has rank 1 and ***mm***^{T}***v*** is a line

The Eigen vectors



- The principal Eigenvector of ***R*** lies between the principal Eigen vector of ***C*** and ***m***

$$\mathbf{e}_R = \alpha \mathbf{e}_C + (1-\alpha)\frac{\mathbf{m}}{\|\mathbf{m}\|}$$

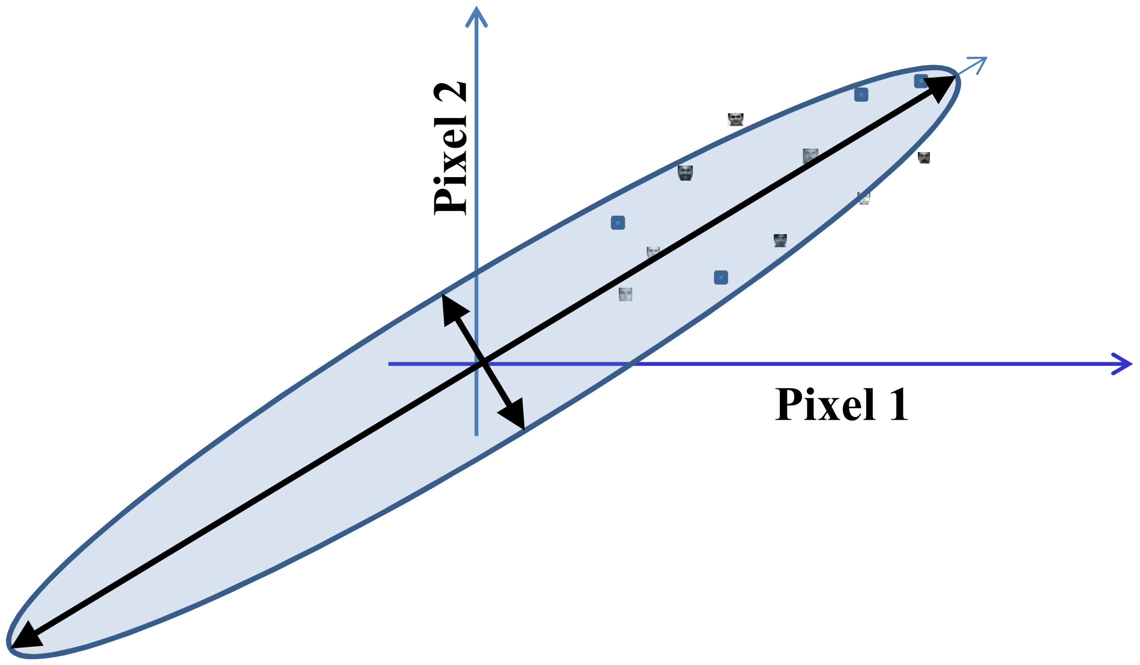
$$0 \le \alpha \le 1$$

- Similarly the principal Eigen *value*

$$\lambda_R = \alpha\lambda_C + (1-\alpha)\|\mathbf{m}\|^2$$

- Similar logic is not easily extendable to the other Eigenvectors, however

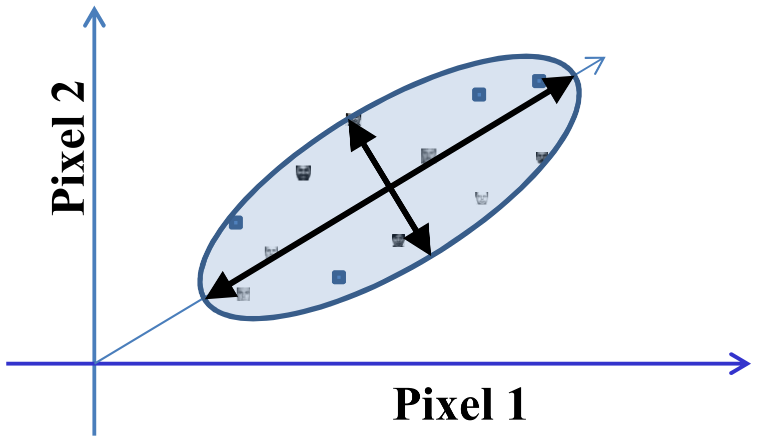
Eigenvectors



- Turns out: Eigenvectors of the *correlation* matrix represent the major and minor axes of an ellipse centered at the origin which encloses the data most compactly

- The SVD of data matrix X uncovers these vectors
  - **KLT**

Eigenvectors

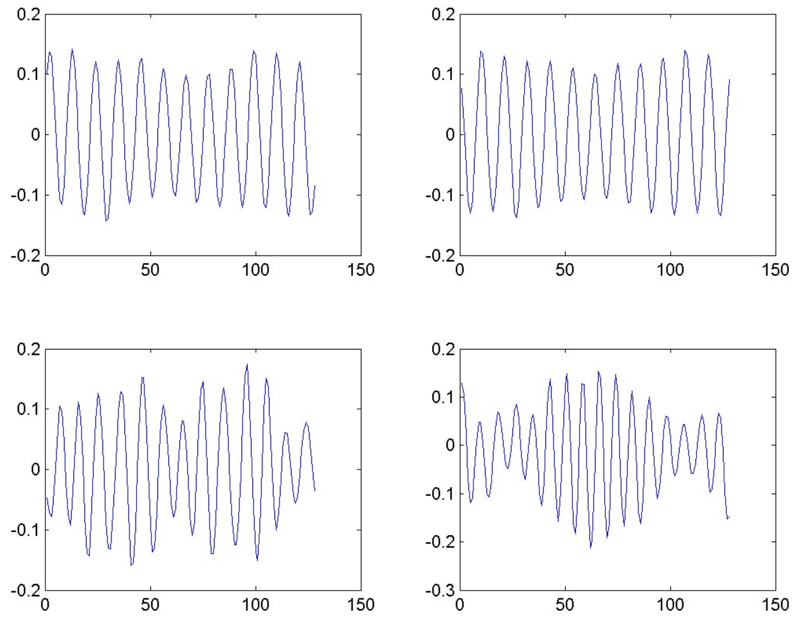


- Turns out: Eigenvectors of the *covariance* represent the major and minor axes of an ellipse centered at the *mean* which encloses the data most compactly

- PCA uncovers these vectors
- In practice, “Eigen faces” refers to *PCA* faces, and not KLT faces

What about sound?

- Finding Eigen bases for speech signals:



- Look like DFT/DCT
- Or wavelets

- DFTs are pretty good most of the time

Eigen Analysis

- Can often find surprising features in your data
- Trends, relationships, more
- Commonly used in recommender systems

- An interesting example..

Eigen Analysis

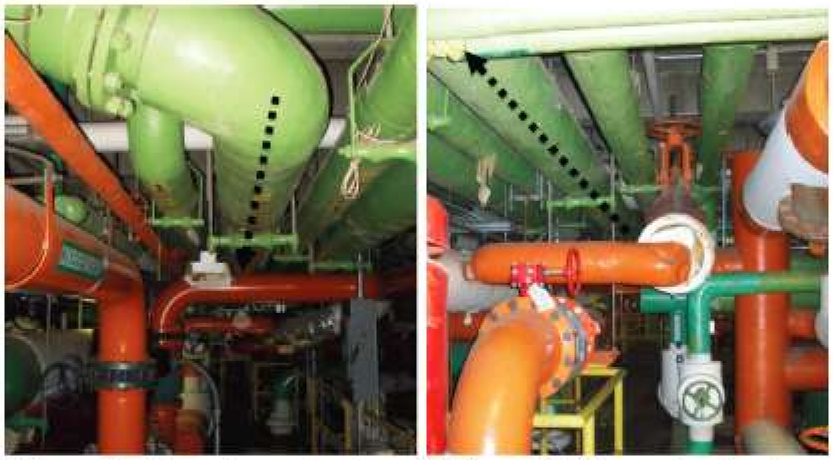


Figure1. Experiment setup @Wean Hall mechanical space. Pipe with arrow indicates a 10" diameter hot water pipe carrying pressurized hot water flow, on which piezoelectric sensors are installed every 10 ft. A National instruments data acquisition system is used to acquire and store the data for later processing.

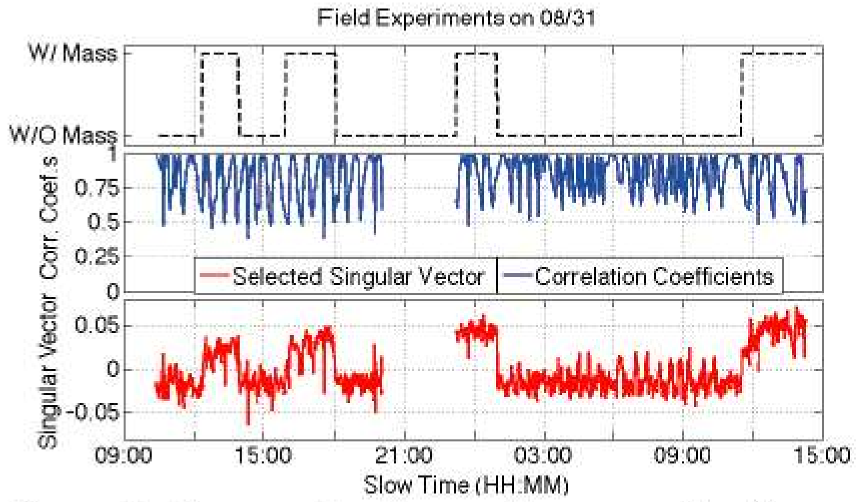


Figure 2. Damage detection results compared with conventional methods. **Top**: Ground truth of whether the pipe is damaged or not. **Middle**: Conventional method only captures temperature variations, and shows no indication of the presence of damage. **Bottom**: The SVD method clearly picks up the steps where damage are introduced and removed.

- Cheng Liu's research on pipes..
- SVD automatically separates useful and uninformative features