Handling uncertainty over time: predicting, estimating, recognizing, learning

What is a “state”

- Everything you need to know to make the best prediction about what happens next.
- Depends how you define the “system” you care about.
- States are called x or s. Dependence on time can be indicated by x(t).
- States can be discrete or continuous.
- AI researchers tend to say “state” when they mean “some features derived from the state”. This should be discouraged.
- A “belief state” is your knowledge about the state, which is typically a probability density/distribution p(x).
- Processes with state are called Markov processes.

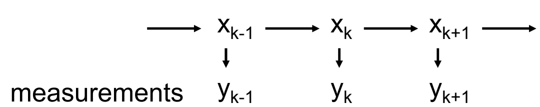
Dealing with time

- The concept of state gives us a handy way of thinking about how things evolve over time.
- We will use discrete time, for example 0.001, 0.002, 0.003, …
- State at time t_k, $x(t_k)$, will be written x_k or $x[k]$.
- Deterministic state transition function $x_{k+1} = f(x_k)$
- Stochastic state transition function $p(x_{k+1}|x_k)$
- Mildly stochastic state transition function

$x_{k+1} = f(x_k) + \varepsilon$, with ε being Gaussian.

Hidden state

- Sometimes the state is directly measurable/observable.
- Sometimes it isn’t. Then you have “hidden state” and a “hidden Markov model” or HMM.
- Examples: Do you have a disease? What am I thinking about? What is wrong with the Mars rover? Where is the Mars rover?



Measurements

- Measurements (y) are also called evidence (e) and observables (o).
- Measurements can be discrete or continuous.
- Deterministic measurement function $y_k = g(x_k)$
- Stochastic measurement function $p(y_k|x_k)$
- Mildly stochastic measurement function

$y_k = g(x_k) + \upsilon$, with υ being Gaussian.

Standard problems

- Predict the future.
- Estimate the current state (filtering).
- Estimate what happened in the past (smoothing).
- Find the most likely state trajectory (sequence/trajectory (speech) recognition).
- Learn about the process (learn state transition and measurement models).

Prediction, Case 0

- Deterministic state transition function $x_{k+1} = f(x_k)$ and known state x_k: Just apply f() n times to get x_{k+n}.
- When we worry about learning the state transition function and the fact that it will always have errors, the question will arise: To predict x_{k+n}, is it better to learn $x_{k+1} = f_1(x_k)$ and iterate, or learn $x_{k+n} = f_n(x_k)$ directly?

Prediction, Case 1

- Stochastic state transition function $p(x_{k+1}|x_k)$, discrete states, belief state $p(x_k)$
- Use tables to represent $p(x_k)$
- Propagate belief state: $p(x_{k+1}) = \Sigma p(x_{k+1}|x_k)p(x_k)$

Matrix notation:

Vector p_k, Transition matrix M, $M_{ij} = p(x_i|x_j)$; i, j, components, not time.

Propagate belief state: $p_{k+1} = Mp_k$

Stationary distribution $M^\infty = \lim(n\text{-}> \infty) M^n$

Mixing time: n for which $M^n \approx M^\infty$

Prediction, Case 2

- Stochastic state transition function $p(x_{k+1}|x_k)$, continuous states, belief state $p(x_k)$
- Propagate belief state analytically if possible

$p(x_{k+1}) = \int p(x_{k+1}|x_k)p(x_k)dx_k$

- Particle filtering (actually many ways to implement).
- Sample $p(x_k)$.
- For each sample, sample $p(x_{k+1}|x_k)$.
- Normalize/resample resulting samples to get $p(x_{k+1})$.
- Iterate to get $p(x_{k+n})$

Prediction, Case 3

- Mildly stochastic state transition function with $p(x_k)$ being $N(\mu,\Sigma_x)$, $x_{k+1} = f(x_k) + \varepsilon$, with ε being $N(0,\Sigma_\varepsilon)$, ε independent of process.
- $E(x_{k+1}) \approx f(\mu)$
- $A = \partial f/\partial x$
- $Var(x_{k+1}) \approx A\Sigma_x A^T + \Sigma_\varepsilon$
- $p(x_{k+1})$ is $N(E(x_{k+1}),Var(x_{k+1}))$.
- Exact if f() linear.
- Iterate to get $p(x_{k+n})$.
- Much simpler than particle filtering.

Filtering, in general

- Start with $p(x_{k-1}^+)$
- Predict $p(x_k^-)$
- Apply measurement using Bayes' Rule to get $p(x_k^+) = p(x_k|y_k)$
- $p(x_k|y_k) = p(y_k|x_k)p(x_k)/p(y_k)$
- Sometimes we ignore $p(y_k)$ and just renormalize as necessary, so all we have to do is $p(x_k|y_k) = \alpha p(y_k|x_k)p(x_k)$

Filtering, Case 1

- Stochastic state transition function $p(x_{k+1}|x_k)$, discrete states, belief state $p(x_k)$, $p(y_k|x_k)$
- Use tables to represent $p(x_k)$
- Propagate belief state:

$p(x_{k+1}^-) = \Sigma\, p(x_{k+1}|x_k)p(x_k^+)$

- Weight each entry by $p(y_k|x_k)$:

$p(x_{k+1}^+) \propto p(y_k|x_k)p(x_{k+1}^-)$

- Normalize so sum of p() = 1
- This is called a Discrete Bayes Filter

Filtering, Case 2

- Stochastic state transition function $p(x_{k+1}|x_k)$, continuous states, belief state $p(x_k)$
- Particle filtering (actually many ways to implement).
- Sample $p(x_k)$.
- For each sample, sample $p(x_{k+1}|x_k)$.
- Weight each sample by $p(y_k|x_k)$.
- Normalize/resample resulting samples to get $p(x_{k+1})$.
- Iterate to get $p(x_{k+n})$

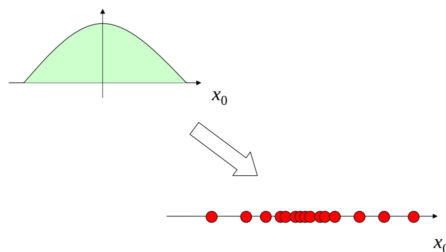
EEE 581 Lecture 16

Particle Filters:
a Gentle Introduction
http://www.fulton.asu.edu/~morrell/581/

Particle Filter Algorithm

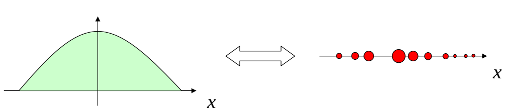
- Create particles as samples from the initial state distribution $p(\mathbf{x}_0)$.
- For k going from 1 to K
  - Update each particle using the state update equation.
  - Compute weights for each particle using the observation value.
  - (Optionally) resample particles.

Initial State Distribution: Samples Only



Samples and Weights

- Each particle has a value and a weight

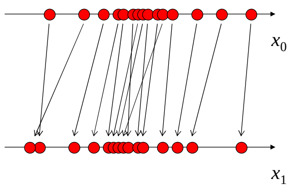


Importance Sampling

- Ideally, the particles would represent samples drawn from the distribution $p(x)$.
  - In practice, we usually cannot get $p(x)$ in closed form; in any case, it would usually be difficult to draw samples from $p(x)$.
- We use importance sampling:
  - Particles are drawn from an *importance distribution*.
  - Particles are weighted by *importance weights*.

State Update

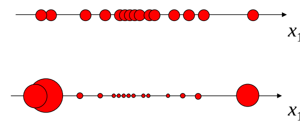
$\mathbf{x}_{k+1} = \mathbf{f}_0(\mathbf{x}_k) + \text{noise}$



Things are more complicated if have multimodal $p(x_{k+1}|x_k)$

Compute Weights

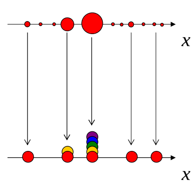
Use $p(y|x)$ to alter weights



Can also draw samples with replacement using $p(y|x)$*weight as p(selection)

Resampling

- In inference problems, most weights tend to zero except a few (from particles that closely match observations), which become large.
- We resample to concentrate particles in regions where $p(x|y)$ is larger.



Advantages of Particle Filters

- Under general conditions, the particle filter estimate becomes asymptotically optimal as the number of particles goes to infinity.
- Non-linear, non-Gaussian state update and observation equations can be used.
- Multi-modal distributions are not a problem.
- Particle filter solutions to inference problems are often easy to formulate.

Disadvantages of Particle Filters

- Naïve formulations of problems usually result in significant computation times.
- It is hard to tell if you have enough particles.
- The best importance distribution and/or resampling methods may be very problem specific.

Conclusions

Particle filters (and other Monte Carlo methods) are a powerful tool to solve difficult inference problems.

- Formulating a filter is now a tractable exercise for many previously difficult or impossible problems.
- Implementing a filter effectively may require significant creativity and expertise to keep the computational requirements tractable.

Particle Filtering Comments

- Reinvented many times in many fields: sequential Monte Carlo, condensation, bootstrap filtering, interacting particle approximations, survival of the fittest, …
- Do you need R^d samples to cover space? R is crude measure of linear resolution, d is dimensionality.
- You maintain a belief state p(x). How do you answer the question "Where is the robot now?" mean, best sample, robust mean, max likelihood, … What happens if p(x) really is multimodal?

Return to our regularly scheduled programming …

- Filtering …

Filtering, Case 3

- Mildly stochastic state transition function with $p(x_k)$ being $N(\mu,\Sigma_x)$, $x_{k+1} = f(x_k) + \varepsilon$, with ε being $N(0,\Sigma_\varepsilon)$ and independent of process.
- Mildly stochastic measurement function

$y_k = g(x_k) + \upsilon$, with υ being $N(0,\Sigma_\upsilon)$ and independent of everything else.

- This will lead to Kalman Filtering
- Nonlinear f() or g() means you are doing Extended Kalman Filtering (EKF).

Combining Measurements: 1D

- True value x
- Measurements m_1, m2: $E(m_1-x) = 0$, $Var(m_1) = \sigma_1^2$, $E(m_2-x) = 0$, $Var(m_2) = \sigma_2^2$, independent
- Linear estimate $\mathit{x} = k_1 m_1 + k_2 m_2$
- Unbiased estimate means $k_2 = 1 - k_1$ so $E(\mathit{x}) = x$
- Minimize $Var(\mathit{x}) = k_1^2 \sigma_1^2 + (1 - k_1)^2 \sigma_2^2$
- So $\partial Var(\mathit{x})/\partial k_1 = 0$ -> $2k_1(\sigma_1^2 + \sigma_2^2) - 2\sigma_2^2 = 0$
- So $k_1 = \sigma_2^2/(\sigma_1^2 + \sigma_2^2)$, $k_2 = \sigma_1^2/(\sigma_1^2 + \sigma_2^2)$
- So $Var(\mathit{x}) = \sigma_1^2 \sigma_2^2/(\sigma_1^2 + \sigma_2^2)$
- What happens when $\sigma_2^2 = 0$? σ_2^2 = infinity?
- BLUE: Best Linear Unbiased Estimator

Filtering, Case 3

- Mildly stochastic state transition function with $p(x_k)$ being $N(\mu,\Sigma_x)$, $x_{k+1} = f(x_k) + \varepsilon$, with ε being $N(0,\Sigma_\varepsilon)$ and independent of process.
- Mildly stochastic measurement function

$y_k = g(x_k) + \upsilon$, with υ being $N(0,\Sigma_\upsilon)$ and independent of everything else.

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Filtering, Case 3 Prediction Step

- $E(x_{k+1}^-) \approx f(\mu)$
- $A = \partial f/\partial x$
- $Var(x_{k+1}^-) \approx A\Sigma_x A^T + \Sigma_\varepsilon$
- $p(x_{k+1}^-)$ is $N(E(x_{k+1}^-),Var(x_{k+1}^-))$

Filtering, Case 3
Measurement Update Step

- $E(x_k^+) \approx E(x_k^-) + K_k(y_k - g(E(x_k^-)))$
- $C = \partial g/\partial x$
- $\Sigma_k^- = Var(x_k^-)$
- $Var(x_k^+) \approx \Sigma_k^- - K_k C \Sigma_k^-$
- $S_k = C\Sigma_k^- C^T + \Sigma_\upsilon$
- $K_k = \Sigma_k^- C^T S_k^{-1}$
- $p(x_k^+)$ is $N(E(x_k^+), Var(x_k^+))$

Unscented Filter

- Numerically find best fit Gaussian instead of analytical computation.
- Good if f() or g() strongly nonlinear.

Smoothing, in general

- Have $y_{1:N}$, want $p(x_k|y_{1:N})$
- Know how to compute $p(x_k|y_{1:k})$ from filtering slides
- $p(x_k|y_{1:N}) = p(x_k|y_{1:k}, y_{k+1:N})$
- $p(x_k|y_{1:N}) \propto p(x_k|y_{1:k})p(y_{k+1:N}|y_{1:k}, x_k)$
- $p(x_k|y_{1:N}) \propto p(x_k|y_{1:k})p(y_{k+1:N}|x_k)$
- $p(y_{k+1:N}|x_k) = \Sigma/\int p(y_{k+1:N}|x_k, x_{k+1})p(x_{k+1}|, x_k)\, dx_{k+1}$
- $= \Sigma/\int p(y_{k+1:N}|x_{k+1})p(x_{k+1}|, x_k)\, dx_{k+1}$
- $= \Sigma/\int p(y_{k+1}|x_{k+1})p(y_{k+2:N}|x_{k+1})p(x_{k+1}|, x_k)\, dx_{k+1}$
- Note recursion implied by $p(y_{k+i+1:N}|x_{k+i})$

Smoothing, general comments

- Need to maintain distributional information at all time steps from forward filter.
- Case 1: discrete states: forward/backward algorithm.
- Case 2: continuous states, nasty dynamics or noise: particle smoothing (expensive).
- Case 3: continuous states, Gaussian noise: Kalman smoother.

Finding most likely state trajectory

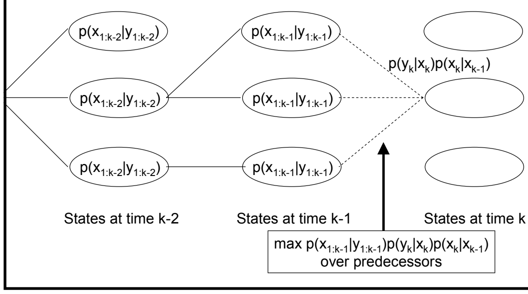
- Goal in speech recognition
- $p(x_1, x_2, \ldots, x_N|y_{1:N}) \neq p(x_1|y_{1:N})p(x_2|y_{1:N})\ldots p(x_N|y_{1:N})$
- Are we screwed? Computing joint probability is hard!

Viterbi Algorithm

- $\max p(x_{1:k}|y_{1:k})$
- $= \max p(y_{1:k}|x_{1:k})p(x_{1:k})$
- $= \max p(y_{1:k-1}, y_k|x_{1:k})p(x_{1:k})$
- $= \max p(y_{1:k-1}|x_{1:k-1})p(y_k|x_k)p(x_{1:k})$
- $= \max p(y_{1:k-1}|x_{1:k-1})p(y_k|x_k)p(x_k|x_{1:k-1})p(x_{1:k-1})$
- $= \max [p(y_{1:k-1}|x_{1:k-1})\, p(x_{1:k-1})]p(y_k|x_k)p(x_k|x_{1:k-1})$
- $= \max p(x_{1:k-1}|y_{1:k-1})p(y_k|x_k)p(x_k|x_{k-1})$
- Note recursion
- Do we evaluate this over all possible sequences?

Viterbi Algorithm (2)

- Use dynamic programming



Viterbi Algorithm (3)

- Well, this still only really works for discrete states.
- Continuous states have too many possible states at each step.
- D dimensions, R resolution in each dimension implies R^D states at each time step.
- Ask me about local sequence maximization.

Learning

- Given data, want to learn dynamic/transition and sensor models.
- Smooth, choose most likely state at each time, learn models, iterate.
- This is known as the EM algorithm.
- Discrete case: Baum-Welch Algorithm