


**15-826: Multimedia Databases
and Data Mining**


Lecture #23: DSP tools –
Fourier and Wavelets
C. Faloutsos



Must-read Material

- DFT/DCT: In [PTVF](#) ch. 12.1, 12.3, 12.4; in [Textbook](#) Appendix B.
- Wavelets: In [PTVF](#) ch. 13.10; in [MM Textbook](#) Appendix C

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Outline

Goal: ‘Find **similar / interesting** things’

- Intro to DB
- ➔ • Indexing - similarity search
- ➔ • Data Mining

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Indexing - Detailed outline

- primary key indexing
- ..
- multimedia
- ➔ Digital Signal Processing (DSP) tools
 - Discrete Fourier Transform (DFT)
 - Discrete Wavelet Transform (DWT)

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DSP - Detailed outline

- DFT
 - ➔ - what
 - why
 - how
 - Arithmetic examples
 - properties / observations
 - DCT
 - 2-d DFT
 - Fast Fourier Transform (FFT)

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Introduction

Goal: given a signal (eg., sales over time and/or space)


Find: patterns and/or compress

count

lynx caught per year


year

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 **What does DFT do?**


A: highlights the periodicities

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 **Why should we care?**

A: several real sequences are periodic
Q: Such as?

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 **Why should we care?**

A: several real sequences are periodic
Q: Such as?
A:
– sales patterns follow seasons;
– economy follows 50-year cycle
– temperature follows daily and yearly cycles
Many real signals follow (multiple) cycles

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Why should we care?

For example: human voice!

- Frequency analyzer
<http://www.relisoft.com/freeware/freq.html>
- speaker identification
- impulses/noise -> flat spectrum
- high pitch -> high frequency

Freq.exe

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DFT and stocks

- Dow Jones Industrial index, 6/18/2001-12/21/2001

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DFT and stocks

- Dow Jones Industrial index, 6/18/2001-12/21/2001
- just 3 DFT coefficients give very good approximation

Log(ampl)

freq

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DFT: definition

- Discrete Fourier Transform (n-point):

$$X_f = 1/\sqrt{n} \sum_{t=0}^{n-1} x_t * \exp(-j 2\pi f t / n)$$

$(j = \sqrt{-1})$ inverse DFT

$$x_t = 1/\sqrt{n} \sum_{f=0}^{n-1} X_f * \exp(+j 2\pi f t / n)$$

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Skip

How does it work?

Decomposes signal to a sum of sine (and cosine) waves.

Q: How to assess 'similarity' of \mathbf{x} with a wave?

value

time

$\mathbf{x} = \{x_0, x_1, \dots, x_{n-1}\}$

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How does it work? Skip

A: consider the waves with frequency 0, 1, ...;
use the inner-product (~cosine similarity)

value

freq. $f=0$

time

0 1 n-1

value

freq. $f=1 (\sin(t * 2 \pi/n))$

time

0 1 n-1

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How does it work? Skip

A: consider the waves with frequency 0, 1, ...;
use the inner-product (~cosine similarity)

value

freq. $f=2$

time

0 1 n-1

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How does it work? Skip

'basis' functions

0 1 n-1

sine, freq = 1

0 1 n-1

cosine, $f=1$

0 1 n-1

sine, freq = 2

0 1 n-1

cosine, $f=2$

0 1 n-1

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Skip

How does it work?

- Basis functions are actually n-dim vectors, **orthogonal** to each other
- ‘similarity’ of **x** with each of them: inner product
- DFT: ~ all the similarities of **x** with the basis functions

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Skip

How does it work?

Since $e^{jf} = \cos(f) + j \sin(f)$
 ($j = \text{sqrt}(-1)$),
 we finally have:

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DFT: definition

- Discrete Fourier Transform (n-point):

$$X_f = 1/\sqrt{n} \sum_{t=0}^{n-1} x_t * \exp(-j 2\pi f t / n)$$

$(j = \sqrt{-1})$

$$x_t = 1/\sqrt{n} \sum_{f=0}^{n-1} X_f * \exp(+j 2\pi f t / n)$$

inverse DFT

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DFT: definition

- **Good news:** Available in **all** symbolic math packages, eg., in 'mathematica'

```
x = [1,2,1,2];
X = Fourier[x];
Plot[ Abs[X] ];
```

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DFT: definition

(variations:

- 1/n instead of 1/sqrt(n)
- exp(-...) instead of exp(+...)

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DFT: definition

(variations:

...
or even

$$a_f = \sum_{i=0}^{N-1} x_i \cos(2\pi f i/N)$$

$$b_f = \sum_{i=0}^{N-1} x_i \sin(2\pi f i/N)$$

$$f = 0, 1, \dots, N/2$$

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DFT: definition

Observations:

- X_f : are complex numbers except $-X_0$, who is real
- $\text{Im}(X_f)$: ~ amplitude of sine wave of frequency f
- $\text{Re}(X_f)$: ~ amplitude of cosine wave of frequency f
- \mathbf{x} : is the sum of the above sine/cosine waves

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DFT: definition

Skip

Observation - SYMMETRY property:

$X_f = (X_{n-f})^*$

(“*” : complex conjugate: $(a + bj)^* = a - bj$)

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DFT: definition

Definitions

- $A_f = |X_f|$: amplitude of frequency f
- $|X_f|^2 = \text{Re}(X_f)^2 + \text{Im}(X_f)^2 = \text{energy of frequency } f$
- phase ϕ_f at frequency f

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DFT: definition

Amplitude spectrum: $|X_f|$ vs $f (f=0, 1, \dots, n-1)$

SYMMETRIC (Thus, we plot the first half only)

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DFT: definition

Phase spectrum $|\phi_f|$ vs $f (f=0, 1, \dots, n-1)$:

Anti-symmetric

(Rarely used)

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DFT: examples

flat

Amplitude

time freq

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DFT: examples

Low frequency sinusoid

The time-domain plot shows a sine wave with an amplitude of approximately 0.15 and a period of 16 samples. The frequency-domain plot shows two peaks at frequencies 1 and 15, with a magnitude of approximately 1.0.

time freq

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DFT: examples

- Sinusoid - symmetry property: $X_f = X_{n-f}^*$

The time-domain plot shows a sine wave with an amplitude of approximately 0.15 and a period of 16 samples. The frequency-domain plot shows two peaks at frequencies 1 and 15, with a magnitude of approximately 1.0, illustrating the symmetry property.

time freq

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DFT: examples

- Higher freq. sinusoid

The time-domain plot shows a sine wave with an amplitude of approximately 0.08 and a period of 8 samples. The frequency-domain plot shows two peaks at frequencies 2 and 14, with a magnitude of approximately 0.5.

time freq

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DFT: examples

examples

The diagram shows a signal on the left, which is a sum of three components on the right. The first component is a constant value (green line). The second component is a cosine wave (magenta line). The third component is a sine wave (orange line). The x-axis for all plots is labeled from 0 to 15.

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DFT: examples

examples

The diagram shows a signal on the left and its amplitude spectrum on the right. The signal is a sum of a constant, a cosine wave, and a sine wave. The amplitude spectrum shows three peaks: one at frequency 0 (the constant), one at frequency 12 (the cosine wave), and one at frequency 12 (the sine wave). The x-axis for the spectrum is labeled from 0 to 15.

Ampl.

Freq.

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DFT: Amplitude spectrum

Amplitude: $A_f^2 = \text{Re}^2(X_f) + \text{Im}^2(X_f)$

count

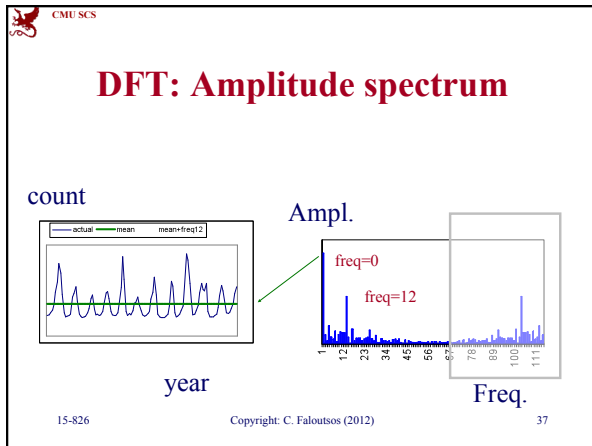
The diagram shows a time series plot on the left and its amplitude spectrum on the right. The time series plot shows a signal over time (year). The amplitude spectrum shows peaks at frequency 0 and frequency 12. The x-axis for the spectrum is labeled from 1 to 111.

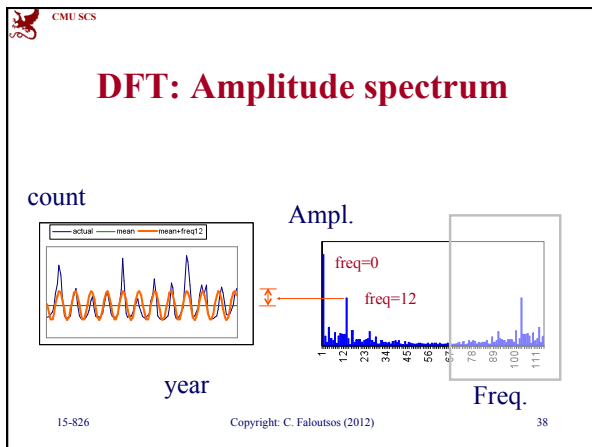
year

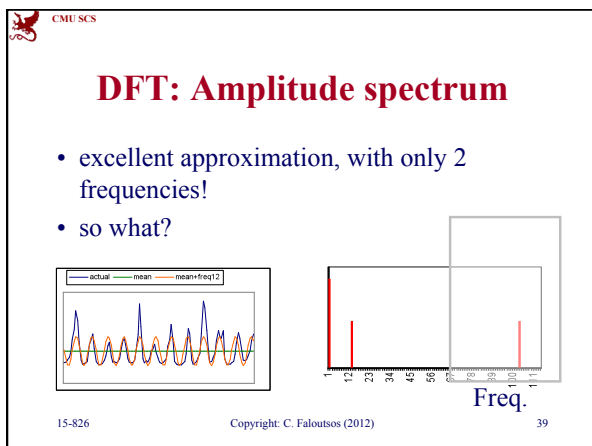
Ampl.

Freq.

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DFT: Amplitude spectrum

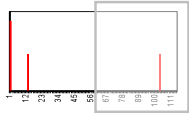
- excellent approximation, with only 2 frequencies!
- so what?
- A1: compression
- A2: pattern discovery
- (A3: forecasting)

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DFT: Amplitude spectrum

- excellent approximation, with only 2 frequencies!
- so what?
- A1: **(lossy) compression**
- A2: pattern discovery

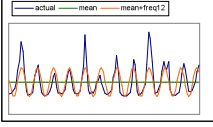


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DFT: Amplitude spectrum

- excellent approximation, with only 2 frequencies!
- so what?
- A1: (lossy) compression
- A2: **pattern discovery**



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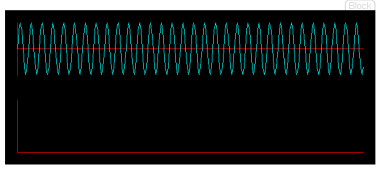
DFT: Amplitude spectrum

- Let's see it in action (defunct now...)
- (http://www.dsptutor.freeuk.com/jsanalyser/FFT_SpectrumAnalyser.html)
- plain sine
- phase shift
- two sine waves
- the 'chirp' function
- <http://ion.researchsystems.com/>

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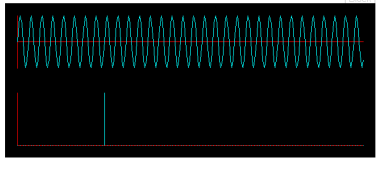
Plain sine



Number of samples: 256
Sampling rate: 8000 samples / s
Signal waveform expression: sin(2000*pi*t)
Plot signal Plot spectrum

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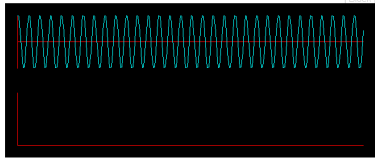
Plain sine



Number of samples: 256
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Signal waveform expression: sin(2000*pi*t)
Plot signal Plot spectrum

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Plain sine – phase shift

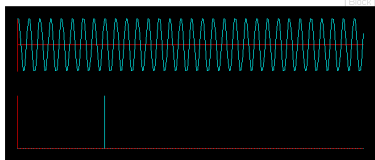


Number of samples: 256
Sampling rate: 8000 samples / s
Signal waveform expression: $\sin(2000\pi t + 1.2)$

Plot signal Plot spectrum

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Plain sine – phase shift

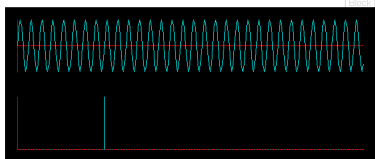


Number of samples: 256
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Signal waveform expression: $\sin(2000\pi t + 1.2)$

Plot signal Plot spectrum

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Plain sine

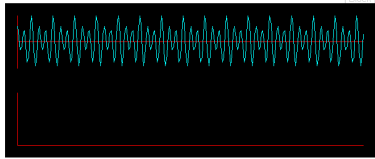


Number of samples: 256
Sampling rate: 8000 samples / s
Signal waveform expression: $\sin(2000\pi t)$

Plot signal Plot spectrum

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Two sines

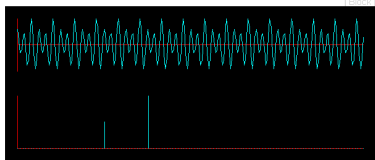


Number of samples: 256
Sampling rate: 8000 samples / s
Signal waveform expression: $\sin(2000\pi t) + 2\cos(3000\pi t + 0.5)$

Plot signal Plot spectrum

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Two sines

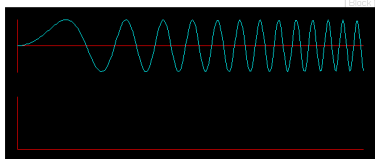


Number of samples: 256
Sampling rate: 8000 samples / s
Signal waveform expression: $\sin(2000\pi t) + 2\cos(3000\pi t + 0.5)$

Plot signal Plot spectrum

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Chirp



Number of samples: 256
Sampling rate: 8000 samples / s
Signal waveform expression: $\sin(25000\pi t^2)$

Plot signal Plot spectrum

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Chirp

Number of samples: 256
 Sampling rate: 8000 samples / s
 Signal waveform expression: `sin(25000*pi*t)`

Plot signal Plot spectrum

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Another applet

<http://www.falstad.com/fourier/>
 (seems virus-free – but scan, before you install)
[FFT applet/index.html](#)

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Properties

- Time shift sounds the same
 - Changes only phase, not amplitudes
- Sawtooth has almost all frequencies
 - With decreasing amplitude
- Spike has all frequencies

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DFT: Parseval's theorem

$$\sum(x_t^2) = \sum(|X_f|^2)$$

Ie., DFT preserves the 'energy'
or, alternatively: it does an axis rotation:

$x = \{x_0, x_1\}$

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DFT: Parseval's theorem

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Ie., DFT preserves the 'energy'
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DSP - Detailed outline

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 - ➔ - Arithmetic examples
 - properties / observations
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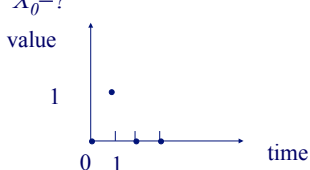
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Arithmetic examples

- Impulse function: $\mathbf{x} = \{0, 1, 0, 0\}$ ($n = 4$)
- $X_0 = ?$

value



time

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Arithmetic examples

- Impulse function: $\mathbf{x} = \{0, 1, 0, 0\}$ ($n = 4$)
- $X_0 = ?$
- A: $X_0 = 1/\sqrt{4} * 1 * \exp(-j 2 \pi 0 / n) = 1/2$
- $X_1 = ?$
- $X_2 = ?$
- $X_3 = ?$

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Arithmetic examples

- Impulse function: $\mathbf{x} = \{0, 1, 0, 0\}$ ($n = 4$)
- $X_0 = ?$
- A: $X_0 = 1/\sqrt{4} * 1 * \exp(-j 2 \pi 0 / n) = 1/2$
- $X_1 = -1/2 j$
- $X_2 = -1/2$
- $X_3 = +1/2 j$
- Q: does the 'symmetry' property hold?

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Arithmetic examples

- Impulse function: $\mathbf{x} = \{ 0, 1, 0, 0 \}$ ($n = 4$)
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- $X_3 = +1/2 j$

} ←

- Q: does the 'symmetry' property hold?
- A: Yes (of course)

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Arithmetic examples

- Impulse function: $\mathbf{x} = \{ 0, 1, 0, 0 \}$ ($n = 4$)
- $X_0 = ?$
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- $X_1 = -1/2 j$
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- $X_3 = +1/2 j$
- Q: check Parseval's theorem

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Arithmetic examples

- Impulse function: $\mathbf{x} = \{ 0, 1, 0, 0 \}$ ($n = 4$)
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- Q: (Amplitude) spectrum?

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Arithmetic examples

- Impulse function: $x = \{ 0, 1, 0, 0 \}$ ($n = 4$)
- $X_0 = ?$
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- $X_1 = -1/2 j$
- $X_2 = -1/2$
- $X_3 = +1/2 j$
- Q: (Amplitude) spectrum?
- A: FLAT!

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Arithmetic examples

- Q: What does this mean?

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Arithmetic examples

- Q: What does this mean?
- A: All frequencies are equally important ->
 - we need n numbers in the frequency domain to represent just one non-zero number in the time domain!
 - “frequency leak”

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DSP - Detailed outline

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Observations

- DFT of 'step' function:
 $x = \{ 0, 0, \dots, 0, 1, 1, \dots, 1 \}$

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Observations

- DFT of 'step' function:
 $x = \{ 0, 0, \dots, 0, 1, 1, \dots, 1 \}$

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Observations

- DFT of 'step' function:
 $x = \{ 0, 0, \dots, 0, 1, 1, \dots, 1 \}$

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Observations

- DFT of 'step' function:
 $x = \{ 0, 0, \dots, 0, 1, 1, \dots, 1 \}$

- the more frequencies,
the better the approx.
- 'ringing' becomes worse
- reason: discontinuities; trends

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Observations

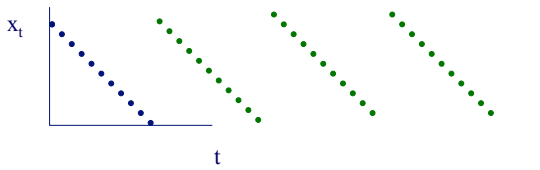
- Ringing for trends: because DFT 'sub-consciously' replicates the signal

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Observations

- Ringing for trends: because DFT ‘sub-consciously’ replicates the signal

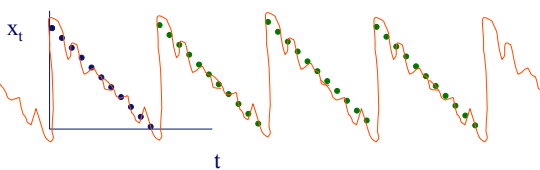


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Observations

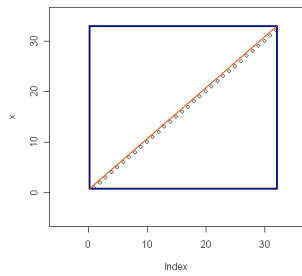
- Ringing for trends: because DFT ‘sub-consciously’ replicates the signal



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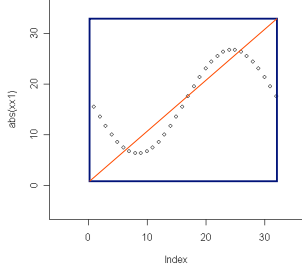
original



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DC and 1st



abs(x1)

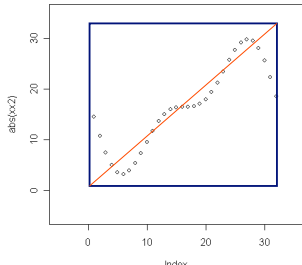
Index

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The plot shows a parabolic trend of data points (open circles) with a red linear regression line. The y-axis is labeled 'abs(x1)' and ranges from 0 to 30. The x-axis is labeled 'Index' and ranges from 0 to 30. The data points are roughly symmetric around Index 15, with values increasing from 0 to 15 and then decreasing back to 0.

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DC and 1st
And 2nd



abs(x2)

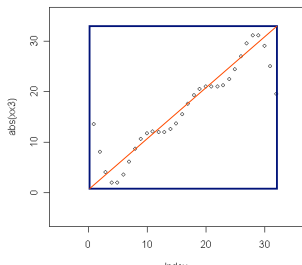
Index

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The plot shows a linear trend of data points (open circles) with a red linear regression line. The y-axis is labeled 'abs(x2)' and ranges from 0 to 30. The x-axis is labeled 'Index' and ranges from 0 to 30. The data points are roughly symmetric around Index 15, with values increasing from 0 to 15 and then decreasing back to 0.

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DC and 1st
And 2nd
And 3rd



abs(x3)

Index

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The plot shows a linear trend of data points (open circles) with a red linear regression line. The y-axis is labeled 'abs(x3)' and ranges from 0 to 30. The x-axis is labeled 'Index' and ranges from 0 to 30. The data points are roughly symmetric around Index 15, with values increasing from 0 to 15 and then decreasing back to 0.

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DC and 1st
And 2nd
And 3rd
And 4th

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Observations

- Q: DFT of a sinusoid, eg.
 $x_t = 3 \sin(2 \pi / 4 t)$
($t = 0, \dots, 3$)
- Q: $X_0 = ?$
- Q: $X_1 = ?$
- Q: $X_2 = ?$
- Q: $X_3 = ?$

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Observations

- Q: DFT of a sinusoid, eg.
 $x_t = 3 \sin(2 \pi / 4 t)$
($t = 0, \dots, 3$)
- Q: $X_0 = 0$
- Q: $X_1 = -3j$
- Q: $X_2 = 0$
- Q: $X_3 = 3j$

- check 'symmetry'
- check Parseval

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Observations

- Q: DFT of a sinusoid, eg.

$$x_t = 3 \sin(2 \pi / 4 t)$$
 $(t = 0, \dots, 3)$
 - Q: $X_0 = 0$
 - Q: $X_1 = -3j$
 - Q: $X_2 = 0$
 - Q: $X_3 = 3j$

•Does this make sense?

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Property

- Shifting x in time does NOT change the amplitude spectrum
- eg., $x = \{0\ 0\ 0\ 1\}$ and $x' = \{0\ 1\ 0\ 0\}$: same (flat) amplitude spectrum
- (only the phase spectrum changes)
- Useful property when we search for patterns that may 'slide'

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DSP - Detailed outline

- DFT
 - what
 - why
 - how
 - Arithmetic examples
 - properties / observations
- ➔ DCT
- 2-d DFT
- Fast Fourier Transform (FFT)

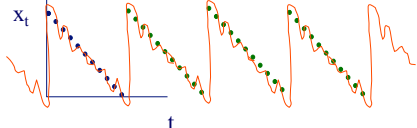
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DCT

Discrete Cosine Transform

- motivation#1: DFT gives complex numbers
- motivation#2: how to avoid the 'frequency leak' of DFT on trends?

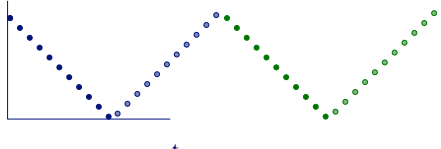


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DCT

- brilliant solution to both problems: mirror the sequence, do DFT, and drop the redundant entries!



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DCT

- (see Numerical Recipes for exact formulas)

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DCT - properties

- it gives real numbers as the result
- it has no problems with trends
- it is very good when x_t and $x_{(t+1)}$ are correlated

(thus, is used in JPEG, for image compression)

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DSP - Detailed outline

- DFT
 - what
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 - ➔ - 2-d DFT
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2-d DFT

- Definition:

$$X_{f_1, f_2} = \frac{1}{\sqrt{n_1}} \frac{1}{\sqrt{n_2}} \sum_{i_1=0}^{n_1-1} \sum_{i_2=0}^{n_2-1} x_{i_1, i_2} \exp(-2\pi j i_1 f_1 / n_1) \exp(-2\pi j i_2 f_2 / n_2)$$

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2-d DFT

- Intuition:

do 1-d DFT on each row

and then 1-d DFT on each column

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2-d DFT

- Quiz: how do the basis functions look like?
- for $f_1 = f_2 = 0$
- for $f_1 = 1, f_2 = 0$
- for $f_1 = 1, f_2 = 1$

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2-d DFT

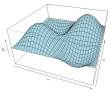
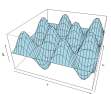
- Quiz: how do the basis functions look like?
- for $f_1 = f_2 = 0$ flat
- for $f_1 = 1, f_2 = 0$ wave on x; flat on y
- for $f_1 = 1, f_2 = 1$ ~ egg-carton

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2-d DFT

- Quiz: how do the basis functions look like?
- for $f_1 = f_2 = 0$ flat
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DSP - Detailed outline

- DFT
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FFT

- What is the complexity of DFT?

$$X_f = 1/\sqrt{n} \sum_{t=0}^{n-1} x_t * \exp(-j2\pi tf/n)$$

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FFT

- What is the complexity of DFT?

$$X_f = 1/\sqrt{n} \sum_{t=0}^{n-1} x_t * \exp(-j2\pi tf/n)$$

- A: Naively, $O(n^2)$

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FFT

- However, if n is a power of 2 (or a number with many divisors), we can make it $O(n \log n)$

Main idea: if we know the DFT of the odd time-ticks, and of the even time-ticks, we can quickly compute the whole DFT

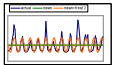
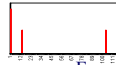
Details: in Num. Recipes

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DFT - Conclusions

- It spots periodicities (with the ‘**amplitude spectrum**’)
- can be quickly computed ($O(n \log n)$), thanks to the FFT algorithm.
- **standard** tool in signal processing (speech, image etc signals)

Freq.

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Detailed outline

- primary key indexing
- ..
- multimedia
- Digital Signal Processing (DSP) tools
 - Discrete Fourier Transform (DFT)
 - ➔ – Discrete Wavelet Transform (DWT)

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Reminder: Problem:

Goal: given a signal (eg., #packets over time)
 Find: patterns, periodicities, and/or **compress**

count

year

lynx caught per year
 (packets per day;
 virus infections per month)

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Wavelets - DWT

- DFT is great - but, how about compressing a spike?

value

time

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Wavelets - DWT

- DFT is great - but, how about compressing a spike?
- A: Terrible - all DFT coefficients needed!

value

time

Ampl

Freq

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Wavelets - DWT

- DFT is great - but, how about compressing a spike?
- A: Terrible - all DFT coefficients needed!

value

time

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Wavelets - DWT

- Similarly, DFT suffers on short-duration waves (eg., baritone, soprano)

value

time

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Wavelets - DWT

- Solution#1: Short window Fourier transform (SWFT)
- But: how short should be the window?

The diagram shows a 4x4 grid with the top-right and bottom-left cells shaded blue. To the left is a small spectrogram. To the right is a plot of a signal 'value' over 'time' with a dashed line indicating a window. The x-axis is labeled 'time' and has markers at 0, 100, 200, 300, 400, 500.

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Wavelets - DWT

- Answer: **multiple** window sizes! -> DWT

Time domain

The diagram compares three methods: DFT (represented by a grid of horizontal lines), SWFT (represented by a grid of squares), and DWT (represented by a grid of rectangles of varying sizes). The y-axis is labeled 'freq' and the x-axis is labeled 'time'.

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Haar Wavelets

- subtract sum of left half from right half
- repeat recursively for quarters, eight-ths, ...

The diagram shows the recursive construction of Haar wavelets. It starts with a constant function, then shows the first level of decomposition (a step function), and then shows the second level (a function with three levels of detail). The functions are shown in a grid-like arrangement.

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Wavelets - construction

$x_0 \ x_1 \ x_2 \ x_3 \ x_4 \ x_5 \ x_6 \ x_7$

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Wavelets - construction

level 1 $d_{1,0}$ $s_{1,0}$ $d_{1,1}$ $s_{1,1}$

$x_0 \ x_1 \ x_2 \ x_3 \ x_4 \ x_5 \ x_6 \ x_7$

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Wavelets - construction

level 2 $d_{2,0}$ $s_{2,0}$

$x_0 \ x_1 \ x_2 \ x_3 \ x_4 \ x_5 \ x_6 \ x_7$

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Wavelets - construction

etc ...

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Wavelets - construction

Q: map each coefficient on the time-freq. plane

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Wavelets - construction

Q: map each coefficient on the time-freq. plane

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Haar wavelets - code

```

#!/usr/bin/perl
# expects a file with numbers
# and prints the dot transform
# The number of time-ticks should be a power of 2
# USAGE
# haar.pl <fname>

my @vals=();
my @smooth; # the smooth component of the signal
my @diff; # the high-freq. component

# collect the values into the array @val
while(<){
    @vals = (@vals, split);
}

my $len = scalar(@vals);
my $half = int($len/2);
while($half >= 1){
    for(my $i=0; $i< $half; $i++){
        $diff[$i] = ($vals[2*$i] - $vals[2*$i + 1]) / sqrt(2);
        print "d", $diff[$i];
        $smooth[$i] = ($vals[2*$i] + $vals[2*$i + 1]) / sqrt(2);
    }
    print "n";
    @vals = @smooth;
    $half = int($half/2);
}
print "t", $vals[0], "n"; # the final, smooth component
  
```

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Wavelets - construction

Observation 1:

- ‘+’ can be some weighted addition
- ‘-’ is the corresponding weighted difference (‘Quadrature mirror filters’)

Observation 2: unlike DFT/DCT, there are *many* wavelet bases: Haar, Daubechies-4, Daubechies-6, Coifman, Morlet, Gabor, ...

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Wavelets - how do they look like?

- E.g., Daubechies-4

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Wavelets - how do they look like?

- E.g., Daubechies-4

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Wavelets - how do they look like?

- E.g., Daubechies-4

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Wavelets - Drill#1:

- Q: baritone/silence/soprano - DWT?

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Wavelets - Drill#1:

- Q: baritone/silence/soprano - DWT?

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Wavelets - Drill#2:

- Q: spike - DWT?

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Wavelets - Drill#2:

- Q: spike - DWT?

0.00	0.00	0.71	0.00
0.00	0.50	-0.35	0.35

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Wavelets - Drill#3:

- Q: weekly + daily periodicity, + spike - DWT?

f

t

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Wavelets - Drill#3:

- Q: **weekly** + daily periodicity, + spike - DWT?

f

t

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Wavelets - Drill#3:

- Q: weekly + **daily** periodicity, + spike - DWT?

f

t

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Wavelets - Drill#3:

- Q: weekly + daily periodicity, + spike - DWT?

f t

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Wavelets - Drill#3:

- Q: weekly + daily periodicity, + spike - DWT?

f t

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Wavelets - Drill#3:

- Q: DFT?

DWT DFT

f t f t

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Wavelets - Drill:

Let's see it live:

<http://dsp.rice.edu/software/dsp-teaching-tools>

delta; cosine; cosine2; chirp

- Haar vs Daubechies-4, -6, etc

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Delta?

$x(0)=1; x(t)=0$ elsewhere

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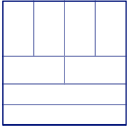
Delta?

$x(0)=1; x(t)=0$ elsewhere

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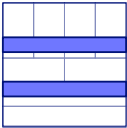
2 cosines?

$$x(t) = \cos(2 * \pi * 4 * t / 1024) + 5 * \cos(2 * \pi * 8 * t / 1024)$$


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2 cosines?

$$x(t) = \cos(2 * \pi * 4 * t / 1024) + 5 * \cos(2 * \pi * 8 * t / 1024)$$


Which one is for freq.=4?

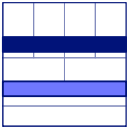
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2 cosines?

$$x(t) = \cos(2 * \pi * 4 * t / 1024) + 5 * \cos(2 * \pi * 8 * t / 1024)$$

f~8 → f



f~4 →

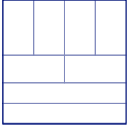
Which one is for freq.=4?

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Chirp?

$$x(t) = \cos(2 * \pi * t * t / 1024)$$


f

t

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Chirp?

$$x(t) = \cos(2 * \pi * t * t / 1024)$$

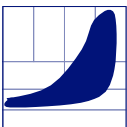
f

t

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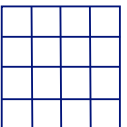
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Chirp?

$$x(t) = \cos(2 * \pi * t * t / 1024)$$

f

t

SWFT?



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Chirp?

$$x(t) = \cos(2 * \pi * t * t / 1024)$$

f

t

SWFT

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More examples (BGP updates)

BGP-lens: Patterns and Anomalies in Internet Routing Updates B. Aditya Prakash et al, SIGKDD 2009

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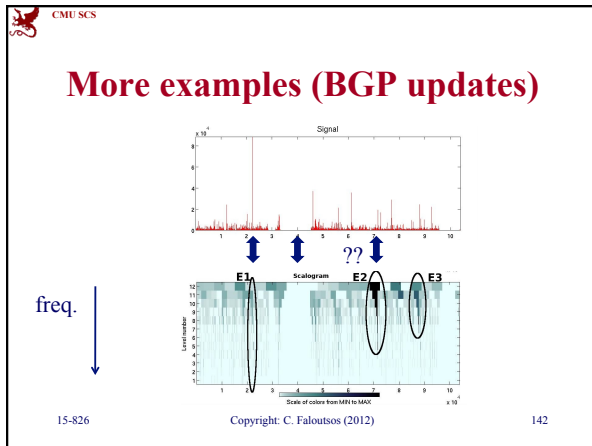
More examples (BGP updates)

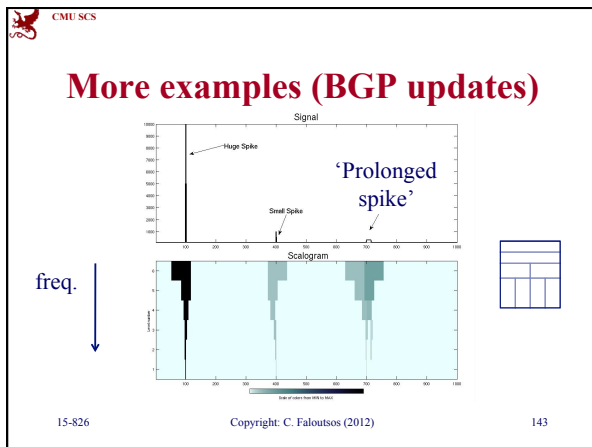
Low freq.:
omitted

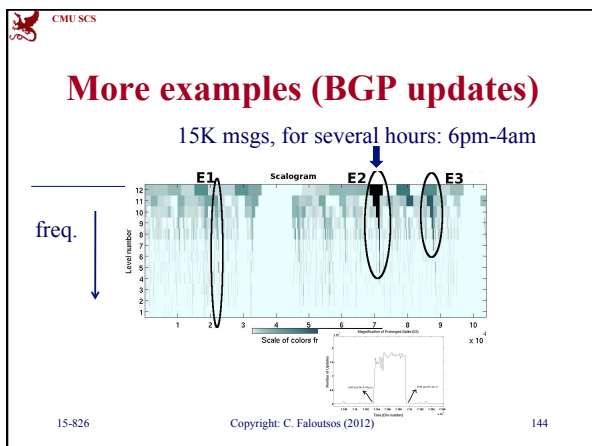
freq.

↓

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Wavelets - Drill

- Or use 'R', 'octave' or 'matlab' – R:

```
install.packages("wavelets")
library("wavelets")
X1<-c(1,2,3,4,5,6,7,8)
dwt(X1, n.levels=3, filter="d4")
mra(X1, n.levels=3, filter="d4")
```

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Wavelets - k-dimensions?





- easily defined for any dimensionality (like DFT, DCT)

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Wavelets - example

<http://grail.cs.washington.edu/projects/query/>
Wavelets achieve *great* compression:

			
20	100	400	16,000
# coefficients			

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Wavelets - intuition

- Edges (horizontal; vertical; diagonal)

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Wavelets - intuition

- Edges (horizontal; vertical; diagonal)
- recurse

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Wavelets - intuition


- Edges (horizontal; vertical; diagonal)
- <http://www331.jpl.nasa.gov/public/wave.html>

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Advantages of Wavelets

- Better compression (better RMSE with same number of coefficients)
- closely related to the processing of the mammalian eye and ear
- Good for progressive transmission
- handle spikes well
- usually, fast to compute ($O(n)$)



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Overall Conclusions

- DFT, DCT spot periodicities
- DWT : multi-resolution - matches processing of mammalian ear/eye better
- All three: powerful tools for compression, pattern detection in real signals
- All three: included in math packages (matlab, R, mathematica, ...)

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Resources

- Numerical Recipes in C: great description, intuition and code for all three tools
- *xwpl*: open source wavelet package from Yale, with excellent GUI.

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Resources (cont'd)

- (defunct?)
<http://www.dsptutor.freeuk.com/jsanalyser/FFTSpectrumAnalyser.html> : Nice java applets
- <http://www.relisoft.com/freeware/freq.html> : voice frequency analyzer (needs microphone – MSwindows only)

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Resources (cont'd)

- www-dsp.rice.edu/software/EDU/mra.shtml (wavelets and other demos)
- R ('install.packages("wavelets")')

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