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15-826: Multimedia Databases and Data Mining

Lecture #21: DSP tools –

DFT – Discrete Fourier Transform

C. Faloutsos

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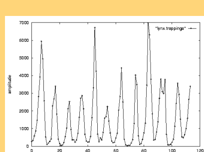
Problem

Goal: given a signal (eg., sales over time and/or space)

Q: Find patterns and/or compress

count



year

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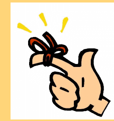
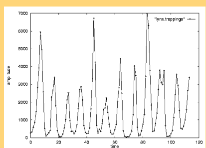


Solutions:

Goal: given a signal (eg., sales over time and/or space)

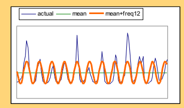
Q: Find patterns and/or compress

count

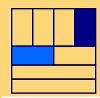


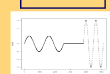
year

A1: Fourier (DFT)



A2: Wavelets (DWT)





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Must-read Material

- DFT/DCT: In PTVF ch. 12.1, 12.3, 12.4; in Textbook Appendix B.
- Wavelets: In PTVF ch. 13.10; in MM Textbook Appendix C

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Outline

Goal: 'Find similar / interesting things'

- Intro to DB
- Indexing - similarity search
- Data Mining

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Indexing - Detailed outline

- primary key indexing
- ..
- Multimedia –
  - Digital Signal Processing (DSP) tools
    - Discrete Fourier Transform (DFT)
    - Discrete Wavelet Transform (DWT)

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DSP - Detailed outline

- DFT
  - what
  - why
  - how
  - Arithmetic examples
  - properties / observations
  - DCT
  - 2-d DFT
  - Fast Fourier Transform (FFT)

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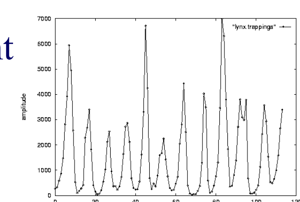
Introduction

Goal: given a signal (eg., sales over time and/or space)

Find: patterns and/or compress

count



lynx caught per year

year

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What does DFT do?

A: highlights the periodicities

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Why should we care?

A: several real sequences are periodic

Q: Such as?

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Why should we care?

A: several real sequences are periodic

Q: Such as?

A:

- sales patterns follow seasons;
- economy follows 50-year cycle
- temperature follows daily and yearly cycles

Many real signals follow (multiple) cycles

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Why should we care?

For example: human voice!

- Frequency analyzer
  http://www.relisoft.com/freeware/freq.html
- speaker identification
- impulses/noise -> flat spectrum
- high pitch -> high frequency

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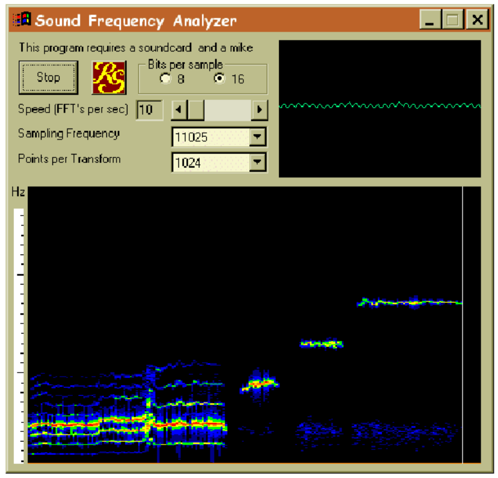
'Frequency Analyzer'



frequency ⬆

time ➡

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DFT: definition

- Discrete Fourier Transform (n-point):

$$X_f = 1/\sqrt{n} \sum_{t=0}^{n-1} x_t * \exp(-j 2\pi f t / n) \qquad f = 0, \ldots, n-1$$

$$(j = \sqrt{-1})$$

inverse DFT

$$x_t = 1/\sqrt{n} \sum_{f=0}^{n-1} X_f * \exp(+j 2\pi f t / n)$$

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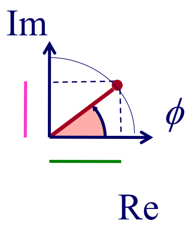
(Reminder)

$$\exp(\phi * j) = \cos(\phi) + j * \sin(\phi)$$



(fun fact: the equation with the 5 most important numbers:

$$e^{j\pi} + 1 = 0$$

)

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DFT: definition

- **Good** news: Available in **all** symbolic math packages, eg., in ‘mathematica’

  x = [1,2,1,2];

  X = Fourier[x];

  Plot[Abs[X]];

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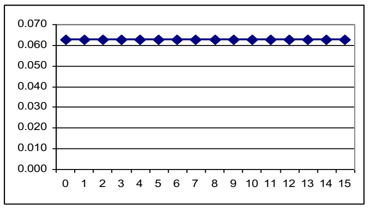
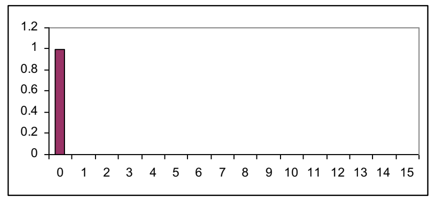
DFT: examples

Plot[Abs[Fourier[x]]];

flat

Amplitude





time freq

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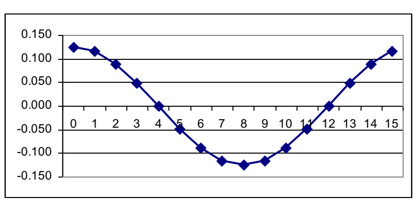
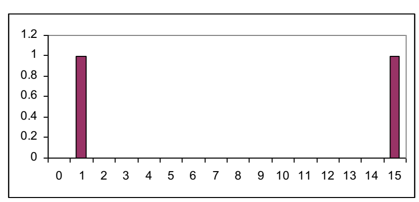
DFT: examples

Low frequency sinusoid





time freq

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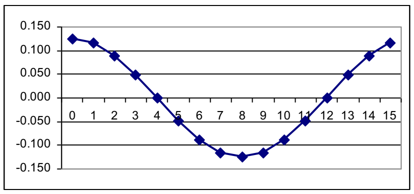
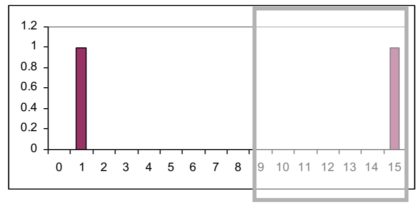
DFT: examples

- Sinusoid - symmetry property: $X_f = X^*_{n-f}$





time freq

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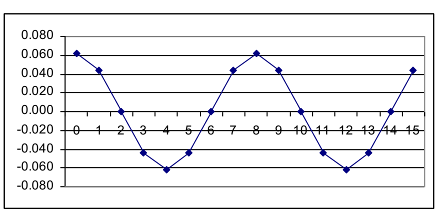
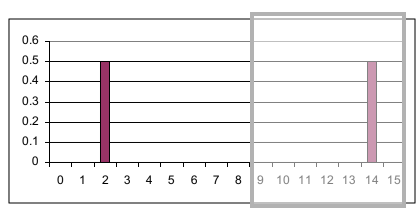
DFT: examples

- Higher freq. sinusoid





time freq

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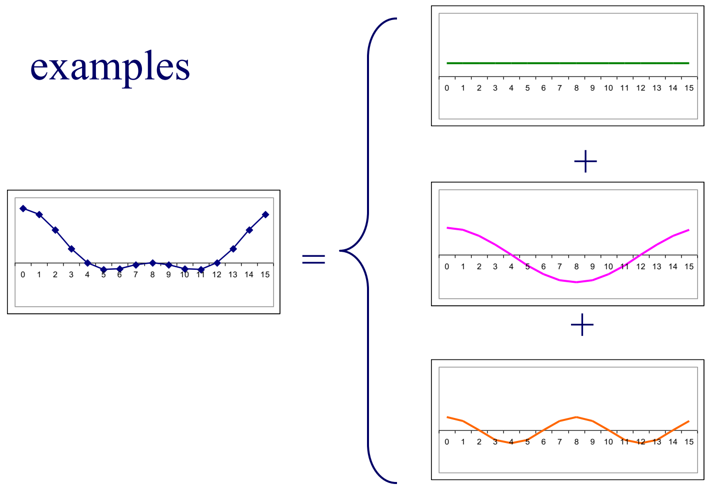
DFT: examples

examples



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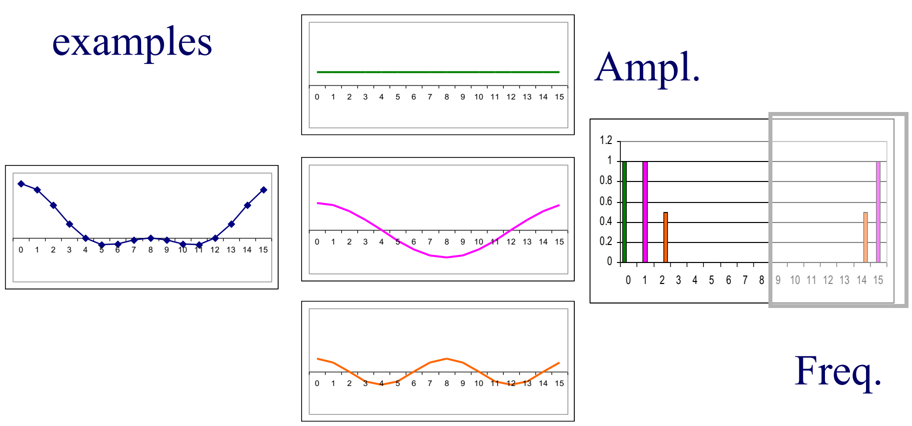
DFT: examples

examples



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DSP - Detailed outline

- DFT
  - what
  - why
  - how
  - Arithmetic examples
  - properties / observations
  - DCT
  - 2-d DFT
  - Fast Fourier Transform (FFT)

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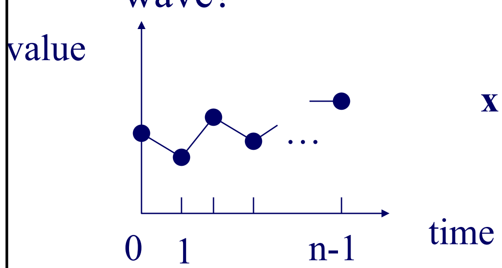
details

How does it work?

Decomposes signal to a sum of sine (and cosine) waves.

Q:How to assess 'similarity' of **x** with a wave?



$\mathbf{x} = \{x_0, x_1, \ldots x_{n-1}\}$

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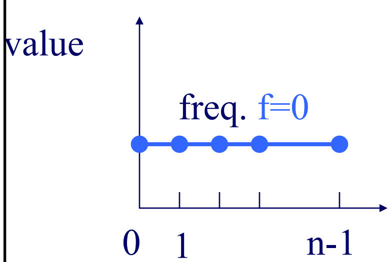
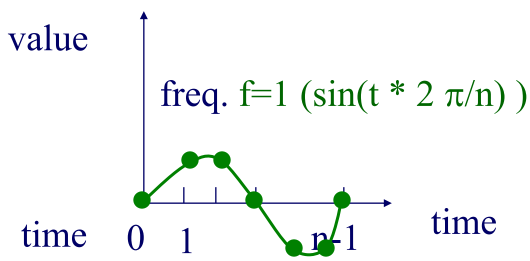
How does it work?

A: consider the waves with frequency 0, 1, ...;
use the inner-product (~cosine similarity)





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How does it work?

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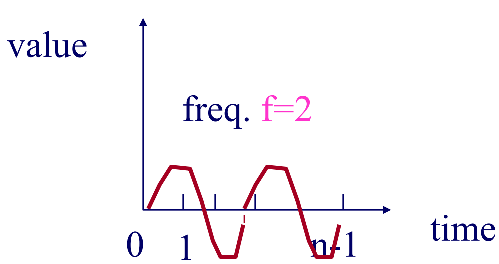

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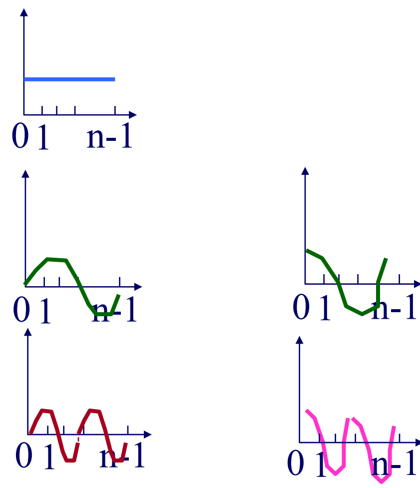
How does it work?

'basis' functions (vectors)



sine, freq =1

cosine, f=1

sine, freq = 2

cosine, f=2

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How does it work?

- Basis functions are actually n-dim vectors, **orthogonal** to each other
- 'similarity' of **x** with each of them: inner product
- DFT: ~ all the similarities of **x** with the basis functions

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DFT: definition

- **Good** news: Available in **all** symbolic math packages, eg., in 'mathematica'

```
x = [1,2,1,2];
X = Fourier[x];
Plot[ Abs[X] ];
```


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DFT: definition

(variations:

- 1/n instead of 1/sqrt(n)
- exp(-...) instead of exp(+...)

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DFT: definition

Observations:

- X_f : are complex numbers except
  - X_0 , who is real
- Im (X_f): ~ amplitude of sine wave of frequency f
- Re (X_f): ~ amplitude of cosine wave of frequency f
- **x:** is the sum of the above sine/cosine waves

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more details

Intuition behind X_f - 'phasors'

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more details

DFT: definition

- Discrete Fourier Transform (n-point):

$$X_f = 1/\sqrt{n} \sum_{t=0}^{n-1} x_t * \exp(-j 2\pi f t / n) \qquad f = 0, \ldots, n-1$$

$$(j = \sqrt{-1})$$

inverse DFT

$$x_t = 1/\sqrt{n} \sum_{f=0}^{n-1} X_f * \exp(+j 2\pi f t / n)$$

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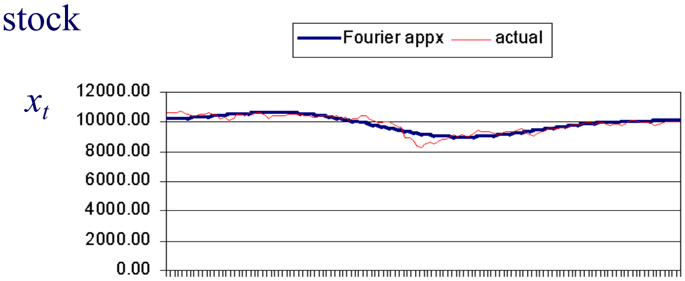
Phasors – intuition behind X_f

IBM stock



0 1 179

n=180

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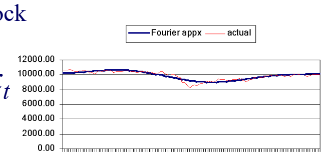
Phasors – intuition behind X_f

IBM stock

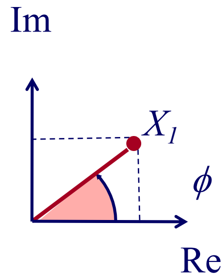
x_t



$$X_f = 1/\sqrt{n} \sum_{t=0}^{n-1} x_t * \exp(-j 2\pi f t / n) \qquad f = 0, \ldots, n-1$$

$$(j = \sqrt{-1})$$

$$x_t = 1/\sqrt{n} \sum_{f=0}^{n-1} X_f * \exp(+j 2\pi f t / n)$$



X_1: $A_1 \exp(j\,\phi)$

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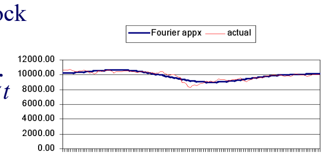
more details

Phasors – intuition behind X_f

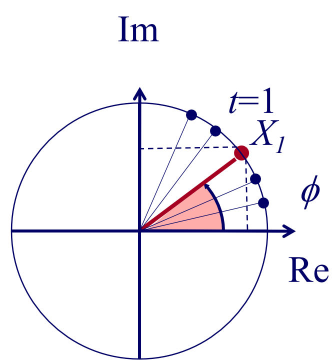
IBM stock

x_t



Rotation
as t =0, …

$$x_t = 1/\sqrt{n} \sum_{f=0}^{n-1} X_f * \exp(+j 2\pi f t / n)$$



X_1: $A_1 \exp(j\,\phi)$

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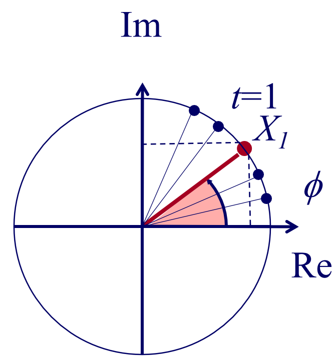
more details

Phasors – intuition behind X_f

Each X_f: **phasor**, ie
rotating complex number ->
Generates a wave of

- Amplitude A_f
- Phase ϕ_f
- Frequency f



Rotation
as t =0, …

$$x_t = 1/\sqrt{n} \sum_{f=0}^{n-1} X_f * \exp(+j 2\pi f t / n)$$

X_1: A_1 exp(j ϕ)

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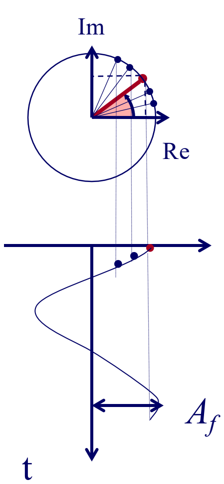
Phasors – intuition behind X_f



Each X_f: **phasor**, ie
rotating complex number ->
Generates a wave of

- Amplitude A_f
- Phase ϕ_f
- Frequency f
- By its projection on Re
- (and another, imaginary, on Im
  - But those cancel out

See en.wikipedia.org/wiki/Phasor

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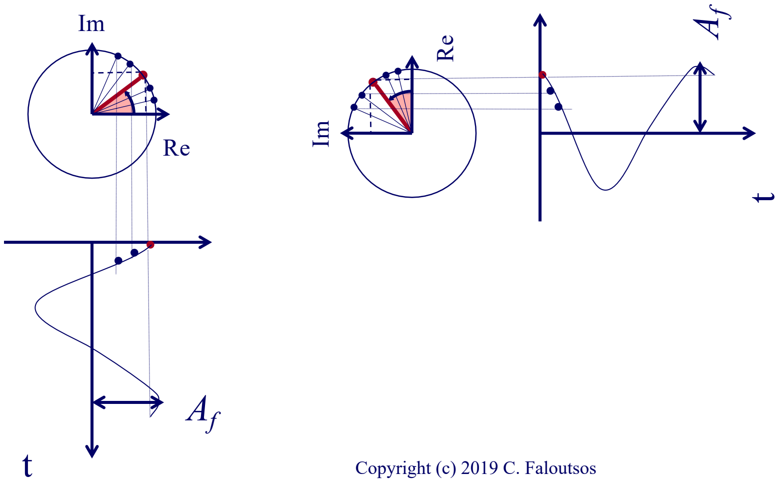
Phasors – intuition behind X_f



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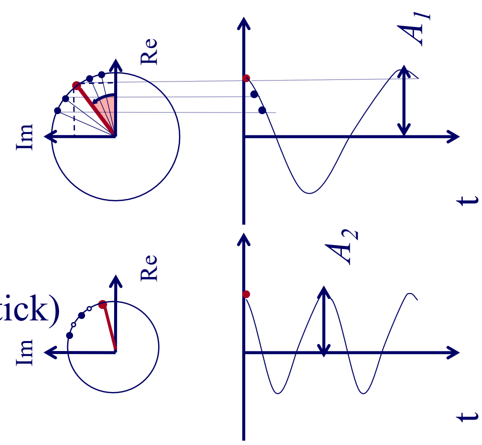
Phasors – intuition behind X_f

X_1

X_2 (every 2[nd] tick)

....



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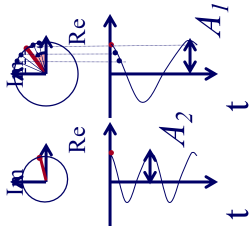
Phasors – intuition behind X_f

X_1

X_2

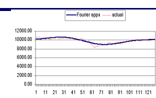


….

X_{179}

X_0

Sum:



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DFT: definition

Observation - SYMMETRY property:

$$X_f = (X_{n-f})^*$$

(“*”: complex conjugate: *(a + b j)* = a - b j)*

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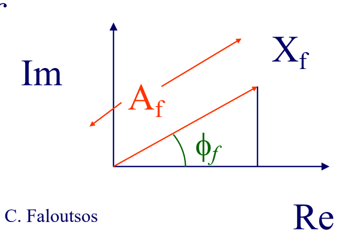
DFT: definition

Definitions

- $A_f = |X_f|$: amplitude of frequency *f*
- $|X_f|^2 = Re(X_f)^2 + Im(X_f)^2$ = energy of frequency *f*
- phase ϕ_f at frequency *f*



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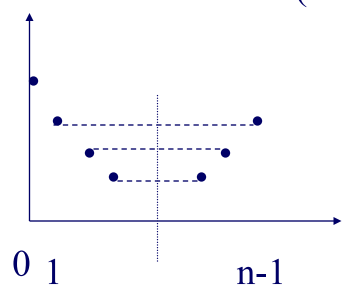
DFT: definition

Amplitude spectrum: $| X_f |$ vs f $(f=0, 1, \ldots n-1)$

SYMMETRIC (Thus, we plot the first half only)



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DFT: definition

Phase spectrum $| \phi_f |$ vs f $(f=0, 1, \ldots n-1)$: Anti-symmetric

(Rarely used)

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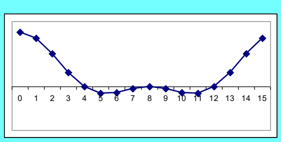
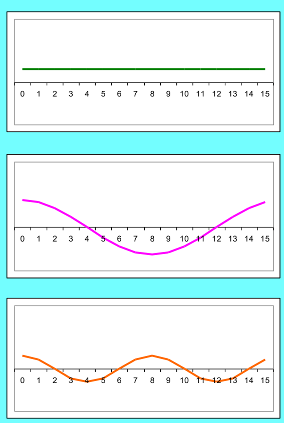
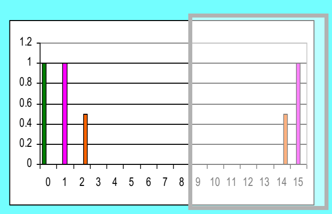
DFT: examples

examples





Ampl.



Freq.

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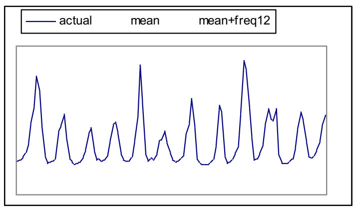
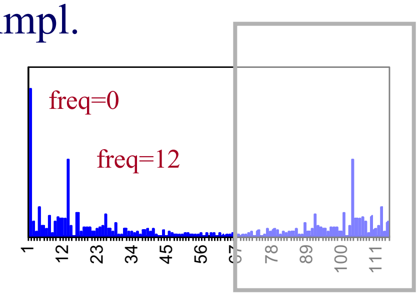
DFT: Amplitude spectrum

Amplitude: $A_f^{\;2} = \mathrm{Re}^2(X_f) + \mathrm{Im}^2(X_f)$

count



Ampl.



year

Freq.

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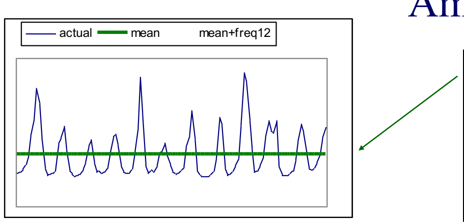
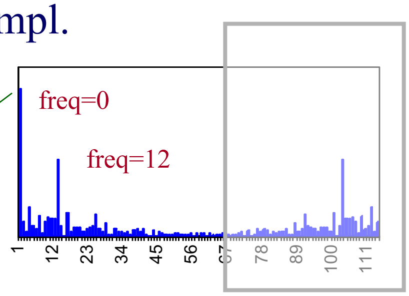
DFT: Amplitude spectrum

count



Ampl.



year

Freq.

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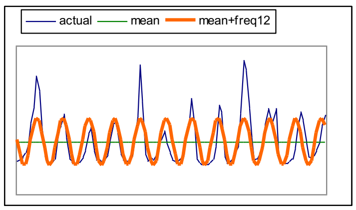
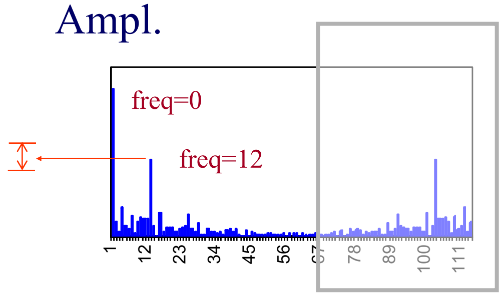
DFT: Amplitude spectrum

count



Ampl.



year

Freq.

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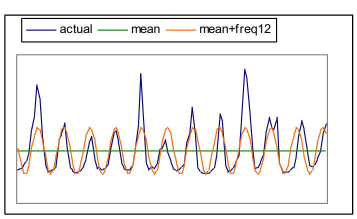
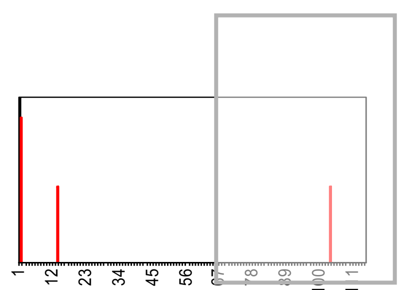
DFT: Amplitude spectrum

- excellent approximation, with only 2 frequencies!
- so what?





Freq.

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DFT: Amplitude spectrum

- excellent approximation, with only 2 frequencies!
- so what?
- A1: compression
- A2: pattern discovery
- (A3: forecasting)

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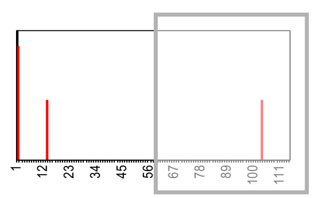
DFT: Amplitude spectrum

- excellent approximation, with only 2 frequencies!
- so what?
- A1: **(lossy) compression**
- A2: pattern discovery



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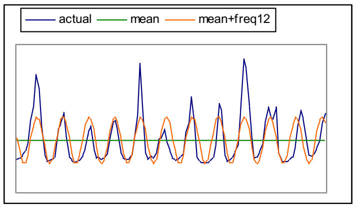
DFT: Amplitude spectrum

- excellent approximation, with only 2 frequencies!
- so what?
- A1: (lossy) compression
- A2: **pattern discovery**



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DSP - Detailed outline

- DFT
  - what
  - why
  - how
  - Arithmetic examples
  - properties / observations
  - DCT
  - 2-d DFT
  - Fast Fourier Transform (FFT)

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DFT: Amplitude spectrum

- Let's see it in action (defunct now…)
- (http://www.dsptutor.freeuk.com/jsanalyser/FFTSpectrumAnalyser.html)
- plain sine
- phase shift
- two sine waves
- the 'chirp' function
- http://ion.researchsystems.com/

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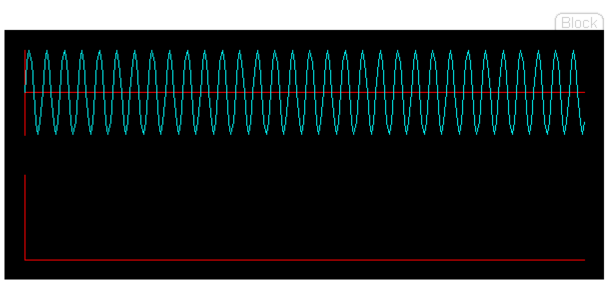
Plain sine



Number of samples: 256

Sampling rate: 8000 samples / s

Signal waveform expression: sin(2000*pi*t)

Plot signal | Plot spectrum

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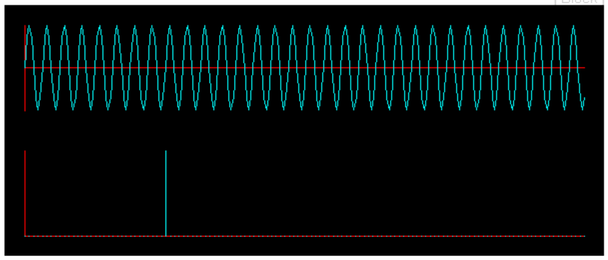
Plain sine



Number of samples: 256

Sampling rate: 8000 samples / s

Signal waveform expression: sin(2000*pi*t)

Plot signal | Plot spectrum

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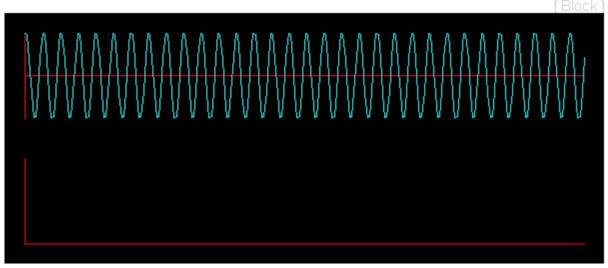
Plain sine – phase shift



Number of samples: 256

Sampling rate: 8000 samples / s

Signal waveform expression: sin(2000*pi*t+1.2)

Plot signal | Plot spectrum

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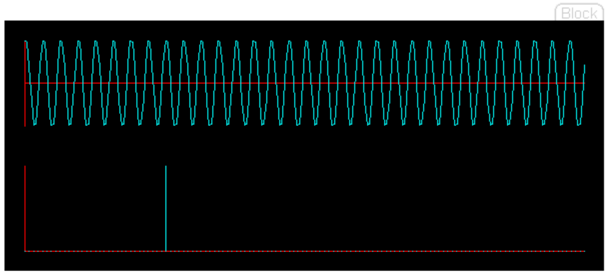
Plain sine – phase shift



Number of samples: 256

Sampling rate: 8000 samples / s

Signal waveform expression: sin(2000*pi*t+1.2)

Plot signal | Plot spectrum

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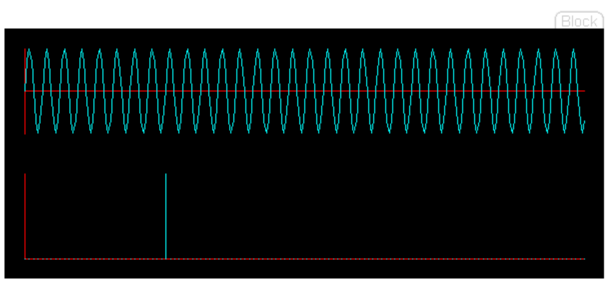
Plain sine



Number of samples: 256

Sampling rate: 8000 samples / s

Signal waveform expression: sin(2000*pi*t)

Plot signal | Plot spectrum

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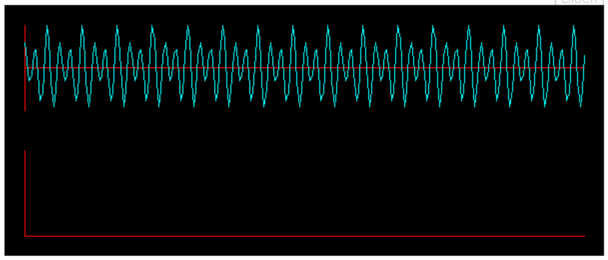
Two sines



Number of samples: 256

Sampling rate: 8000 samples / s

Signal waveform expression: sin(2000*pi*t)+2*cos(3000*pi*t+0.5)

Plot signal | Plot spectrum

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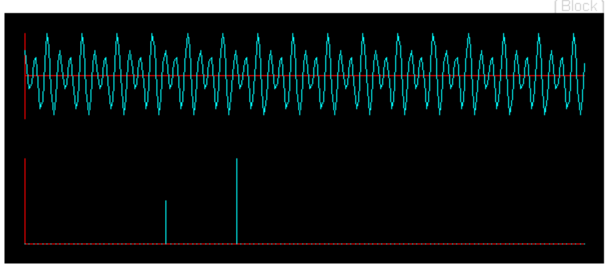
Two sines



Number of samples: 256

Sampling rate: 8000 samples / s

Signal waveform expression: sin(2000*pi*t)+2*cos(3000*pi*t+0.5)

Plot signal | Plot spectrum

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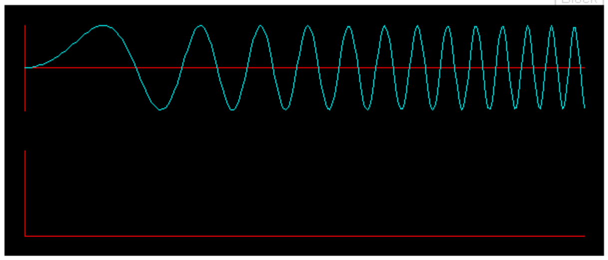
Chirp



Number of samples: 256

Sampling rate: 8000 samples / s

Signal waveform expression: sin(25000*pi*t*t)

Plot signal | Plot spectrum

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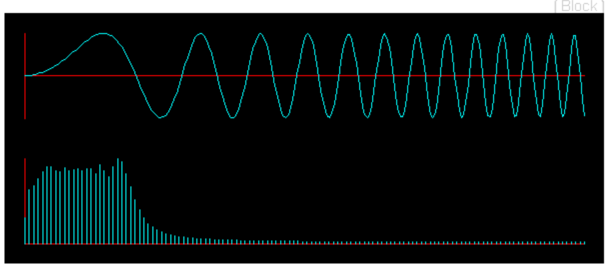
Chirp



Number of samples: 256

Sampling rate: 8000 samples / s

Signal waveform expression: sin(25000*pi*t*t)

Plot signal | Plot spectrum

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DSP - Detailed outline

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Properties

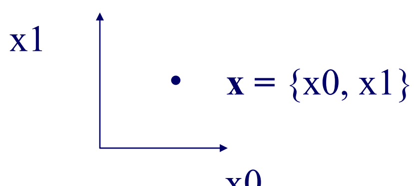
- Time shift sounds the same
  - Changes only phase, not amplitudes
- Sawtooth has almost all frequencies
  - With decreasing amplitude
- Spike has all frequencies

DFT: Parseval's theorem

$$\text{sum}(x_t^{\,2}) = \text{sum} (| X_f |^2)$$

Ie., DFT preserves the 'energy'

or, alternatively: it does an axis rotation:



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DFT: Parseval's theorem

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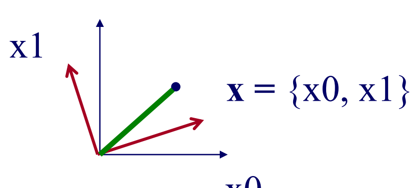

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DFT: Parseval's theorem

sum($x_t^{\,2}$) = sum (| X_f | ^{2})

… equivalently,

matrix **F** (= $\left[\frac{1}{\sqrt{n}} e^{-j2\pi f t}\right]$)

is row-orthonormal

Row: *f*

Column: *t*

F

$$\begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_f \\ \vdots \end{bmatrix} = \begin{bmatrix} e^{-j2\pi f t} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_t \\ \vdots \end{bmatrix}$$

DSP - Detailed outline

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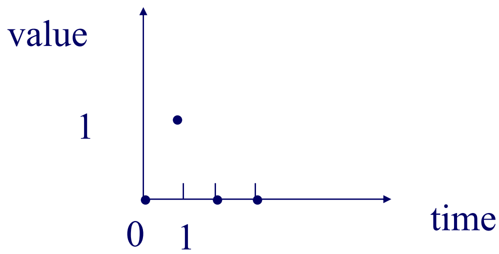
Arithmetic examples

- Impulse function: $\mathbf{x} = \{ 0, 1, 0, 0\}$ ($n = 4$)
- X_0=?



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Arithmetic examples

- Impulse function: $\mathbf{x} = \{ 0, 1, 0, 0\}$ ($n = 4$)
- X_0=?
- A: X_0 = 1/sqrt(4) * 1* exp(-j 2 π 0 / n) = 1/2
- X_1=?
- X_2=?
- X_3=?

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Arithmetic examples

- Impulse function: $\mathbf{x} = \{ 0, 1, 0, 0\}$ ($n = 4$)
- X_0=?
- A: X_0 = 1/sqrt(4) * 1* exp(-j 2 π 0 / n) = 1/2
- X_1= -1/2 j
- X_2= - 1/2
- X_3= + 1/2 j
- Q: does the 'symmetry' property hold?

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Arithmetic examples

- Impulse function: $\mathbf{x} = \{ 0, 1, 0, 0\}$ ($n = 4$)
- X_0=?
- A: X_0 = 1/sqrt(4) * 1* exp(-j 2 π 0 / n) = 1/2
- X_1= -1/2 j
- X_2= - 1/2
- X_3= + 1/2 j
- Q: does the 'symmetry' property hold?
- A: Yes (of course)

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Arithmetic examples

- Impulse function: **x** = { 0, 1, 0, 0} (n = 4)
- X_0=?
- A: X_0 = 1/sqrt(4) * 1* exp(-j 2 π 0 / n) = 1/2
- X_1= -1/2 j
- X_2= - 1/2
- X_3= + 1/2 j
- Q: check Parseval's theorem

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Arithmetic examples

- Impulse function: **x** = { 0, 1, 0, 0} (n = 4)
- X_0=?
- A: X_0 = 1/sqrt(4) * 1* exp(-j 2 π 0 / n) = 1/2
- X_1= -1/2 j
- X_2= - 1/2
- X_3= + 1/2 j
- Q: (Amplitude) spectrum?

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Arithmetic examples

- Impulse function: **x** = { 0, 1, 0, 0} (n = 4)
- X_0=?
- A: X_0 = 1/sqrt(4) * 1* exp(-j 2 π 0 / n) = 1/2
- X_1= -1/2 j
- X_2= - 1/2
- X_3= + 1/2 j
- Q: (Amplitude) spectrum?
- A: FLAT!

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Arithmetic examples

- Q: What does this mean?

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Arithmetic examples

- Q: What does this mean?
- A: All frequencies are equally important ->
  - we need *n* numbers in the frequency domain to represent just one non-zero number in the time domain!
  - "*frequency leak*"

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DSP - Detailed outline

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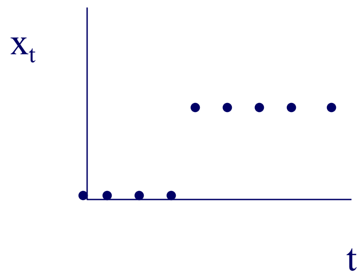
Observations

- DFT of 'step' function:
  x = { 0, 0, ..., 0, 1, 1, ... 1}



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Observations

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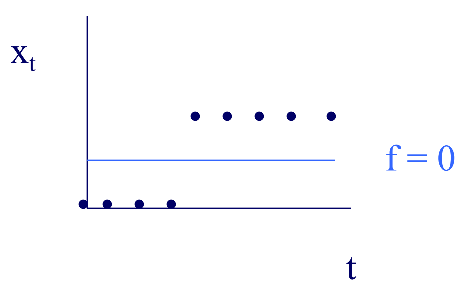

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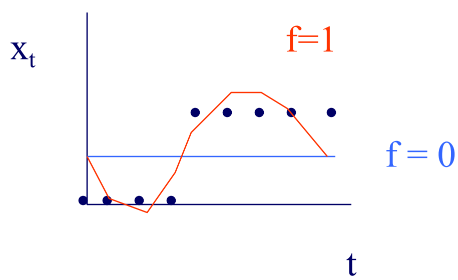
  x = { 0, 0, ..., 0, 1, 1, ... 1}



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Observations

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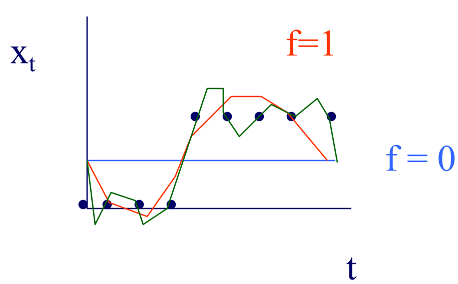
  x = { 0, 0, ..., 0, 1, 1, ... 1}



- the more frequencies, the better the approx.
- 'ringing' becomes worse
- reason: discontinuities; trends

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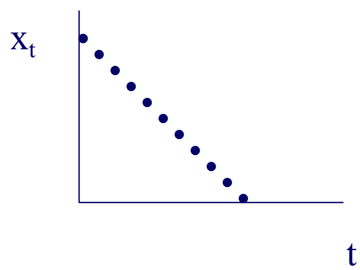
Observations

- Ringing for trends: because DFT 'sub-consciously' replicates the signal



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Observations

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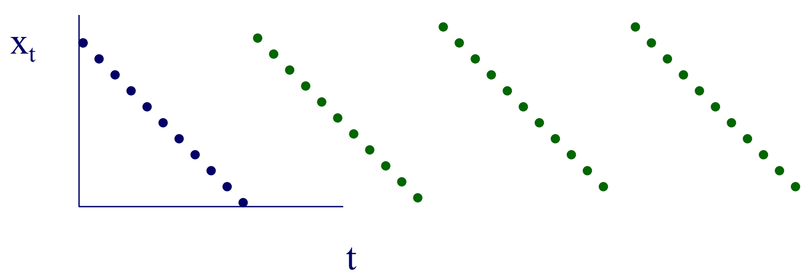

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Observations

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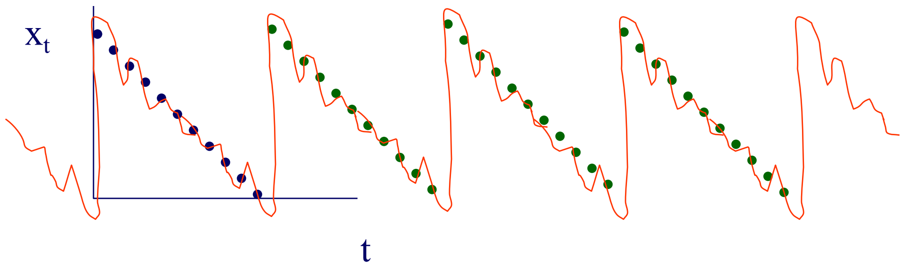

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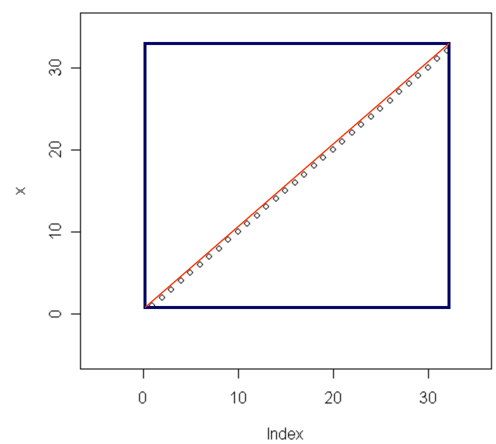
original



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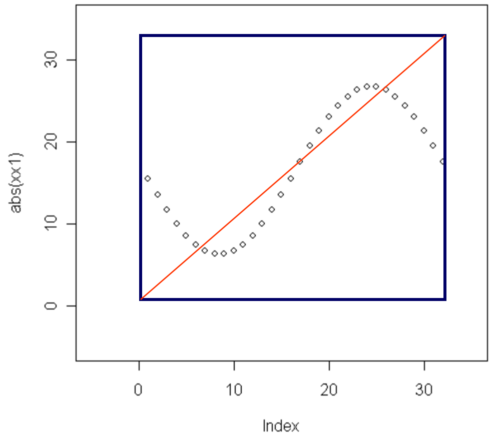
DC and 1st



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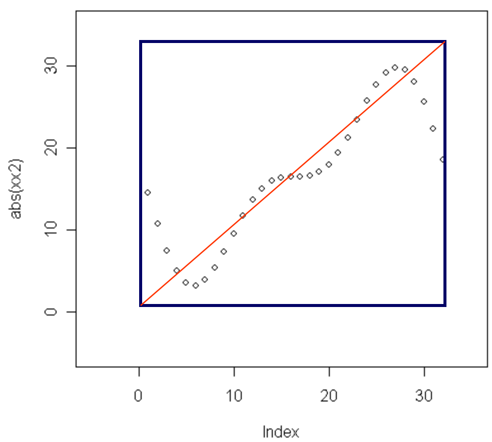
DC and 1st

And 2nd



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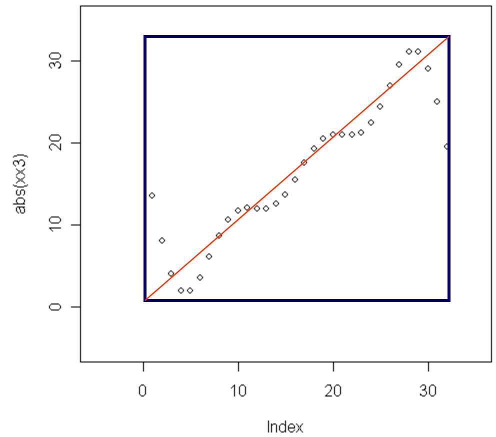
DC and 1st

And 2nd

And 3rd



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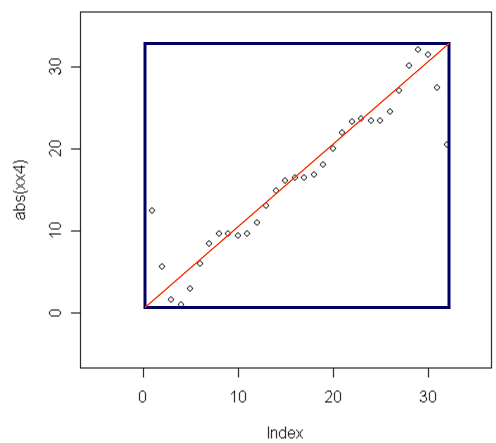
DC and 1st

And 2nd

And 3rd

And 4th



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Observations

- Q: DFT of a sinusoid, eg.

$$x_t = 3 \sin(2 \pi / 4 t)$$

(t = 0, ... , 3)

- Q: X_0 = ?
- Q: X_1 = ?
- Q: X_2 = ?
- Q: X_3 = ?

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Observations

- Q: DFT of a sinusoid, eg.

$$x_t = 3 \sin(2 \pi / 4 t)$$

(t = 0, ... , 3)

- Q: $X_0 = 0$
- Q: $X_1 = -3 j$
- Q: $X_2 = 0$
- Q: $X_3 = 3j$

•check 'symmetry'
•check Parseval

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Observations

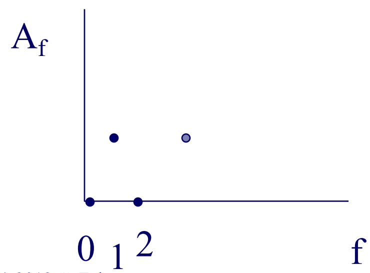
- Q: DFT of a sinusoid, eg.

$$x_t = 3 \sin(2 \pi / 4 t)$$

(t = 0, ... , 3)

- Q: $X_0 = 0$
- Q: $X_1 = -3 j$
- Q: $X_2 = 0$
- Q: $X_3 = 3j$

•Does this make sense?



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Property

- Shifting **x** in time does NOT change the amplitude spectrum
- eg., **x** = { 0 0 0 1} and **x'** = { 0 1 0 0 }: same (flat) amplitude spectrum
- (only the phase spectrum changes)
- Useful property when we search for patterns that may 'slide'

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Summary of properties

- Spike in time: -> all frequencies
- Step/Trend: -> ringing (~ all frequencies)
- Single/dominant sinusoid: -> spike in spectrum
- Time shift -> same amplitude spectrum

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DSP - Detailed outline

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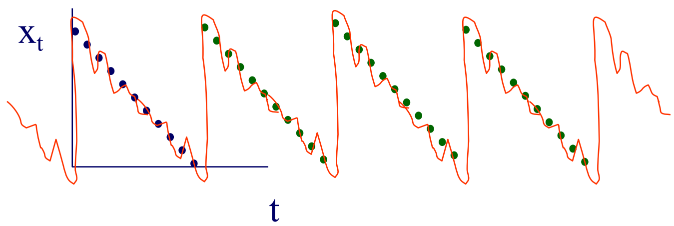
details

DCT

Discrete Cosine Transform

- motivation#1: DFT gives complex numbers
- motivation#2: how to avoid the 'frequency leak' of DFT on trends?



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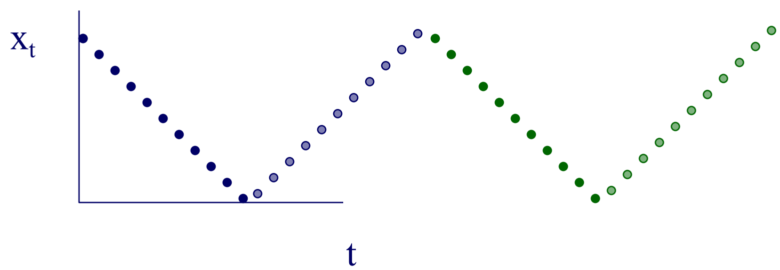
DCT

- brilliant solution to both problems: mirror the sequence, do DFT, and drop the redundant entries!



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DCT

- (see Numerical Recipes for exact formulas)

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DCT - properties

- it gives real numbers as the result
- it has no problems with trends
- it is very good when x_t and $x_{(t+1)}$ are correlated

(thus, is used in JPEG, for image compression)

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2-d DFT

- Definition:

$$X_{f_1,f_2} = \frac{1}{\sqrt{n_1}} \frac{1}{\sqrt{n_2}} \sum_{i_1=0}^{n_1-1} \sum_{i_1=0}^{n_2-1} x_{i_1,i_2} \exp(-2\pi j i_1 f_1 / n_1) \exp(-2\pi j i_2 f_2 / n_2)$$

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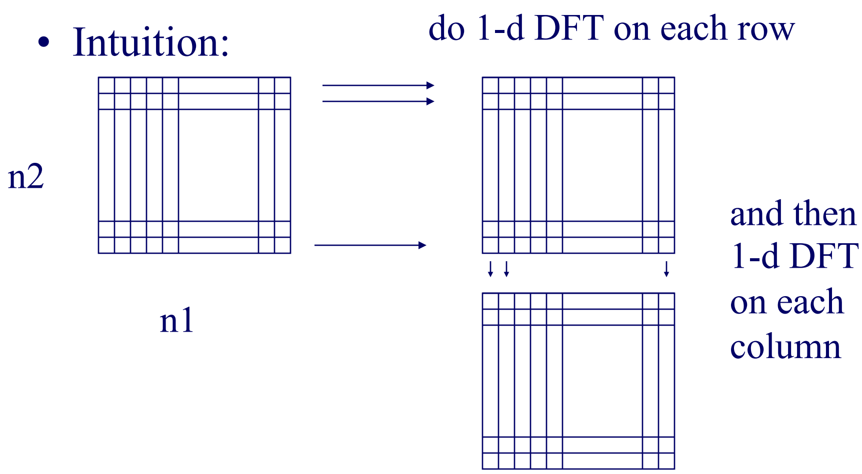
2-d DFT

- Intuition:



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2-d DFT

- Quiz: how do the basis functions look like?
- for f1= f2 =0
- for f1=1, f2=0
- for f1=1, f2=1

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2-d DFT

- Quiz: how do the basis functions look like?
- for f1= f2 =0 flat
- for f1=1, f2=0 wave on x; flat on y
- for f1=1, f2=1 ~ egg-carton

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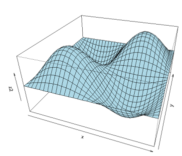
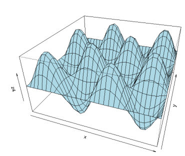
2-d DFT

- Quiz: how do the basis functions look like?
- for f1= f2 =0 flat
- for f1=1, f2=0 wave on x; flat on y
- for f1=1, f2=1 ~ egg-carton





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FFT

- What is the complexity of DFT?

$$X_f = 1/\sqrt{n} \sum_{t=0}^{n-1} x_t * \exp(-j2\pi\, tf / n)$$

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FFT

- What is the complexity of DFT?

$$X_f = 1/\sqrt{n} \sum_{t=0}^{n-1} x_t * \exp(-j2\pi\, tf / n)$$

- A: Naively, O(n^2)

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FFT

- However, if n is a power of 2 (or a number with many divisors), we can make it

  O(n log n)

Main idea: if we know the DFT of the odd time-ticks, and of the even time-ticks, we can quickly compute the whole DFT

Details: in Num. Recipes

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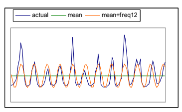
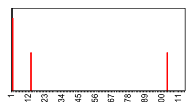
DFT - Conclusions

- It spots periodicities (with the '**amplitude spectrum**')
- can be quickly computed (O(n log n)), thanks to the FFT algorithm.
- **standard** tool in signal processing (speech, image etc signals)

Freq.

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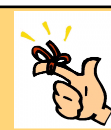
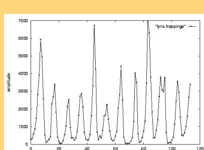


Solutions:

Goal: given a signal (eg., sales over time and/or space)

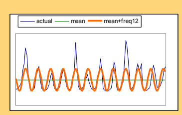
Q: Find patterns and/or compress

count

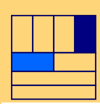
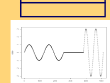


year

✔A1: Fourier (DFT)



A2: Wavelets (DWT)

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