CMU SCS

15-826: Multimedia Databases and Data Mining

Lecture #26: Graph mining - patterns

Christos Faloutsos

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Reading Material

- Michalis Faloutsos, Petros Faloutsos and Christos Faloutsos, *On Power-Law Relationships of the Internet Topology*, SIGCOMM 1999.
- R. Albert, H. Jeong, and A.-L. Barabási, *Diameter of the World Wide Web* Nature, **401**, 130-131 (1999).
- Réka Albert and Albert-László Barabási *Statistical mechanics of complex networks*, Reviews of Modern Physics, **74**, 47 (2002).
- Jure Leskovec, Jon Kleinberg, Christos Faloutsos *Graphs over Time: Densification Laws, Shrinking Diameters and Possible Explanations*, KDD 2005, Chicago, IL, USA, 2005.
- D. Chakrabarti and C. Faloutsos, *Graph Mining: Laws, Generators and Algorithms*, in ACM Computing Surveys, 38(1), 2006 (pdf draft, internal to CMU)

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Thanks

- Deepayan Chakrabarti (CMU)

- Soumen Chakrabarti (*IIT-Bombay*)

- Michalis Faloutsos (UCR)

- George Siganos (UCR)

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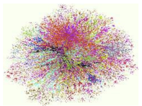
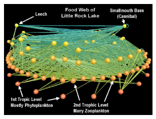
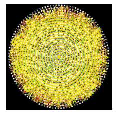
Introduction



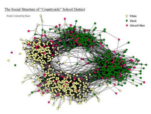
Internet Map
[lumeta.com]



Food Web
[Martinez '91]



Protein Interactions
[genomebiology.com]



Friendship Network
[Moody '01]

Graphs are everywhere!

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Graph structures

- Physical networks
- Physical Internet
- Telephone lines
- Commodity distribution networks

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Networks derived from "behavior"

- Telephone call patterns
- Email, Blogs, Web, Databases, XML
- Language processing
- Web of trust, epinions

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Outline

- Topology & 'laws'
- Generators
- Discussion and tools

Motivating questions:

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Motivating questions

- What do real graphs look like?
  - What properties of nodes, edges are important to model?
  - What local and global properties are important to measure?
- How to generate realistic graphs?

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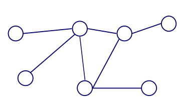
Motivating questions

Given a graph:



- Are there un-natural sub-graphs? (criminals' rings or terrorist cells)?
- How do P2P networks evolve?

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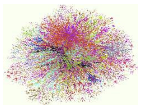
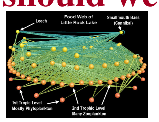
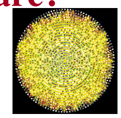
Why should we care?



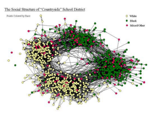
Internet Map [lumeta.com]



Food Web [Martinez '91]



Protein Interactions [genomebiology.com]



Friendship Network [Moody '01]

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Why should we care?

- **A1: extrapolations**: how will the Internet/Web look like next year?
- **A2: algorithm design**: what is a realistic network topology,
  - to try a new routing protocol?
  - to study virus/rumor propagation, and immunization?

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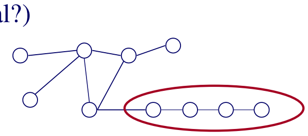
Why should we care? (cont'd)

- **A3: Sampling**: How to get a 'good' sample of a network?
- **A4: Abnormalities**: is this sub-graph / sub-community / sub-network 'normal'? (what **is** normal?)



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Outline

- Topology & 'laws'
  - Static graphs
  - Evolving graphs
- Generators
- Discussion and tools

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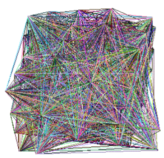
Topology

How does the Internet look like? Any rules?



(Looks random – right?)

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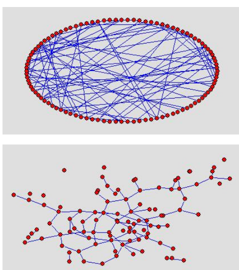
Are real graphs random?

- random (Erdos-Renyi) graph – 100 nodes, avg degree = 2
- before layout
- after layout
- No obvious patterns

(generated with: pajek http://vlado.fmf.uni-lj.si/pub/networks/pajek/)



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Laws and patterns

Real graphs are NOT random!!

- Diameter
- in- and out- degree distributions
- other (surprising) patterns

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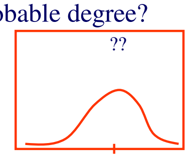
Laws – degree distributions

- Q: avg degree is ~2 - what is the most probable degree?



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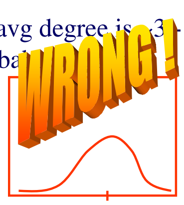
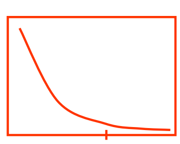
Laws – degree distributions

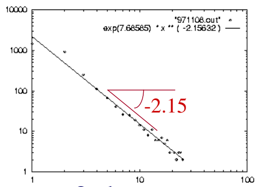
- Q: avg degree is ~3 - what is the most proba





I.Power-law: outdegree *O*

Frequency



Exponent = slope

O = -2.15

Nov'97

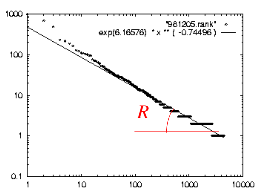
Outdegree

The plot is linear in log-log scale [FFF'99]

$freq = degree^{(-2.15)}$

II.Power-law: rank *R*

outdegree



Exponent = slope

R = -0.74

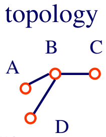
Dec'98

Rank: nodes in decreasing outdegree order

- The plot is a line in log-log scale

III. Eigenvalues

- Let *A* be the adjacency matrix of graph
- The eigenvalue λ is:
  - $A \underline{v} = \lambda \underline{v}$, where $\underline{v}$ some vector
- Eigenvalues are strongly related to graph topology



	A	B	C	D
A		1		
B	1		1	1
C		1		
D		1		

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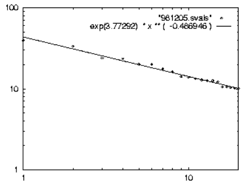
III.Power-law: eigen *E*

Eigenvalue



Exponent = slope

E = -0.48

Dec'98

Rank of decreasing eigenvalue

- Eigenvalues in decreasing order (first 20)
- [Mihail+, 02]: *R = 2 * E*

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IV. The Node Neighborhood

- *N(h)* = # of pairs of nodes within *h* hops

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IV. The Node Neighborhood

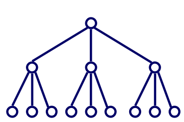
- Q: average degree = 3 - how many neighbors should I expect within 1,2,… *h* hops?
- Potential answer:

  1 hop -> 3 neighbors

  2 hops -> 3 * 3

  …

  h hops -> 3^h



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IV. The Node Neighborhood

- Q: average degree = 3 - how many neighbors should I expect within 1,2,… *h* hops?
- Potential answer:
  1 hop -> 3 neighbors
  2 hops -> 3 * 3
  …
  h hops -> 3^h

WE HAVE DUPLICATES!

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IV. The Node Neighborhood

- Q: average degree = 3 - how many neighbors should I expect within 1,2,… *h* hops?
- Potential answer:
  1 hop -> 3 neighbors
  2 hops -> 3 * 3
  …
  h hops -> 3^h

'avg' degree: meaningless!

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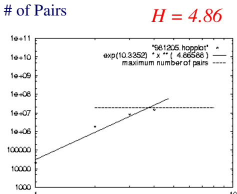
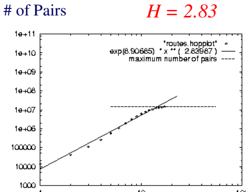
IV. Power-law: hopplot *H*

of Pairs $H = 4.86$



Hops Dec 98

# of Pairs $H = 2.83$



Hops Router level '95

Pairs of nodes as a function of hops $N(h) = h^H$

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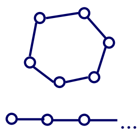
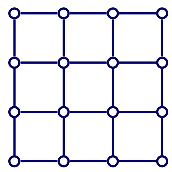
Observation

- Q: Intuition behind 'hop exponent'?
- A: 'intrinsic=fractal dimensionality' of the network



$N(h) \sim h^1$



$N(h) \sim h^2$

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But:

- Q1: How about graphs from other domains?
- Q2: How about temporal evolution?

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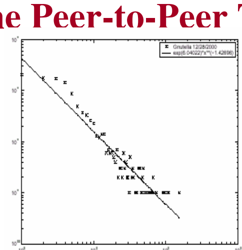
The Peer-to-Peer Topology



(a) Gnutella snapshot from Dec. 28, 2000 (|r|=0.94)

[Jovanovic+]

- Frequency versus degree
- Number of adjacent peers follows a power-law

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More Power laws

- Also hold for other web graphs [Barabasi+, ‘99], [Kumar+, ‘99] with additional ‘rules’ (bi-partite cores follow power laws)

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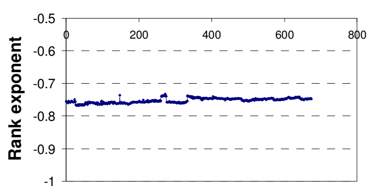
Time Evolution: rank *R*



Domain level

#days since Nov. ‘97

The rank exponent has not changed! [Siganos+, ‘03]

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Outline

- Topology & ‘laws’
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Any other 'laws'?

Yes!

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Any other 'laws'?

Yes!

- Small diameter (~ constant!) –
  - six degrees of separation / 'Kevin Bacon'
  - small worlds [Watts and Strogatz]

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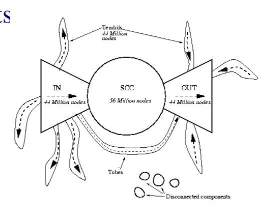
Any other 'laws'?

- Bow-tie, for the web [Kumar+ '99]
- IN, SCC, OUT, 'tendrils'
- disconnected components



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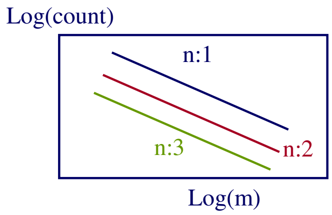
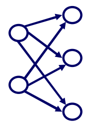
Any other ‘laws’?

- power-laws in communities (bi-partite cores) [Kumar+, ‘99]





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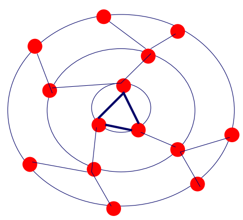
Any other ‘laws’?

- “Jellyfish” for Internet [Tauro+ ’01]
- core: ~clique
- ~5 concentric layers
- many 1-degree nodes



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Summary of ‘laws’

- Power laws for degree distributions
- …………... for eigenvalues, bi-partite cores
- Small diameter (‘6 degrees’)
- ‘Bow-tie’ for web; ‘jelly-fish’ for internet

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Outline

- Topology & 'laws'
  - Static graphs
  - ➡ Evolving graphs
- Generators
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Evolution of diameter?

- Prior analysis, on power-law-like graphs, hints that

  diameter ~ O(log(N)) or

  diameter ~ O(log(log(N)))
- i.e.., slowly increasing with network size
- Q: What is happening, in reality?

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Evolution of diameter?

- Prior analysis, on power-law-like graphs, hints that

  diameter ~[illegible](log([illegible])) or

  diameter [illegible] O([illegible]g(l[illegible](N)))
- i.e.., slowly increa[illegible] w[illegible] network size
- Q: What is happening, in reality?
- A: It **shrinks**(!!), towards a constant value

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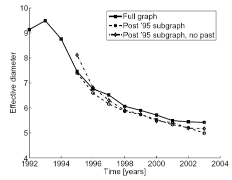
Shrinking diameter

[Leskovec+05a]

- Citations among physics papers
- 11yrs; @ 2003:
  - 29,555 papers
  - 352,807 citations
- For each month *M*, create a graph of all citations up to month *M*

diameter



(a) arXiv citation graph

time

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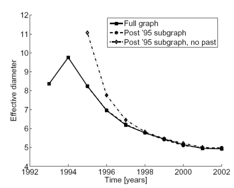
Shrinking diameter

- Authors & publications
- 1992
  - 318 nodes
  - 272 edges
- 2002
  - 60,000 nodes
    - 20,000 authors
    - 38,000 papers
  - 133,000 edges



(b) Affiliation network

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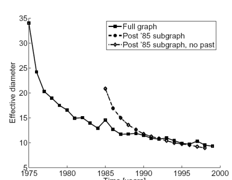
Shrinking diameter

- Patents & citations
- 1975
  - 334,000 nodes
  - 676,000 edges
- 1999
  - 2.9 million nodes
  - 16.5 million edges
- Each year is a datapoint



(c) Patents

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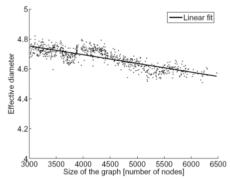
Shrinking diameter

- Autonomous systems
- 1997
  - 3,000 nodes
  - 10,000 edges
- 2000
  - 6,000 nodes
  - 26,000 edges
- One graph per day

diameter



(d) AS

N

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Temporal evolution of graphs

- N(t) nodes; E(t) edges at time t
- suppose that

  N(t+1) = 2 * N(t)
- Q: what is your guess for

  E(t+1) =? 2 * E(t)

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Temporal evolution of graphs

- N(t) nodes; E(t) edges at time t
- suppose that

  N(t+1) = 2 * N(t)
- Q: what is your guess for

  E(t+1) =? ~~2~~ * E(t)
- A: over-doubled!

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Temporal evolution of graphs

- A: over-doubled - but obeying:

$E(t) \sim N(t)^a$ for all t

where $1<a<2$

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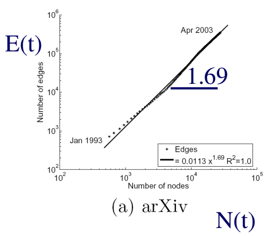
Densification Power Law

ArXiv: Physics papers and their citations



(a) arXiv

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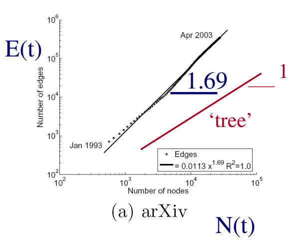
Densification Power Law

ArXiv: Physics papers and their citations



(a) arXiv

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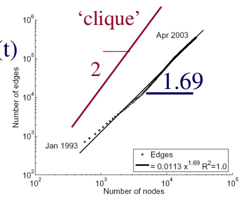
Densification Power Law

ArXiv: Physics papers and their citations

E(t)



(a) arXiv

N(t)

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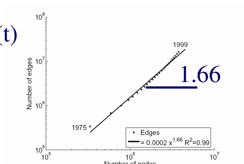
Densification Power Law

U.S. Patents, citing each other

E(t)



(b) Patents

N(t)

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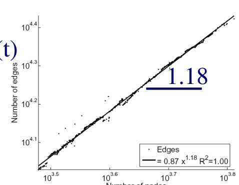
Densification Power Law

Autonomous Systems

E(t)



(c) Autonomous Systems

N(t)

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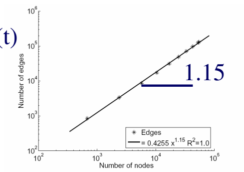
Densification Power Law

ArXiv: authors & papers

E(t)



(d) Affiliation network

N(t)

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Summary of ‘laws’

- Power laws for degree distributions
- …………... for eigenvalues, bi-partite cores
- Small & **shrinking** diameter (‘6 degrees’)
- ‘Bow-tie’ for web; ‘jelly-fish’ for internet
- ``Densification Power Law'', over time

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Outline

- Topology & ‘laws’
- ➡ Generators
  - Erdos – Renyi
  - Degree-based
  - Process-based
  - Recursive generators
- Discussion and tools

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Generators

- How to generate random, realistic graphs?
  - Erdos-Renyi model: beautiful, but unrealistic
  - degree-based generators
  - process-based generators
  - recursive/self-similar generators

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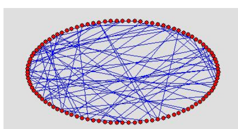
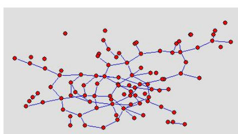
Erdos-Renyi

- random graph – 100 nodes, avg degree = 2
- Fascinating properties (phase transition)
- But: unrealistic (Poisson degree distribution != power law)





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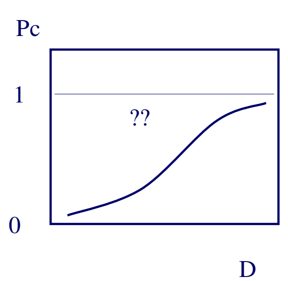
skip

E-R model & Phase transition

- vary avg degree D
- watch Pc =
  Prob(there is a giant connected component)
- How do you expect it to be?



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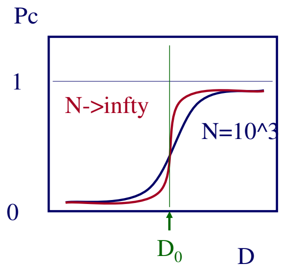
skip

E-R model & Phase transition

- vary avg degree D
- watch Pc =
  Prob(there is a giant connected component)
- How do you expect it to be?



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Degree-based

- Figure out the degree distribution (eg., ‘Zipf’)
- Assign degrees to nodes
- Put edges, so that they match the original degree distribution

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Process-based

- Barabasi; Barabasi-Albert: Preferential attachment -> power-law tails!
  – ‘rich get richer’
- [Kumar+]: preferential attachment + mimick
  – Create ‘communities’

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Process-based (cont'd)

- [Fabrikant+, '02]: H.O.T.: connect to closest, high connectivity neighbor
- [Pennock+, '02]: Winner does NOT take all

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Outline

- Topology & 'laws'
- Generators
  - Erdos – Renyi
  - Degree-based
  - Process-based
  - ➡ Recursive generators
- Discussion and tools

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Recursive generators

- (RMAT [Chakrabarti+,'04])
- Kronecker product

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Wish list for a generator:

- Power-law-tail in- and out-degrees
- Power-law-tail scree plots
- **shrinking/constant** diameter
- Densification Power Law
- communities-within-communities

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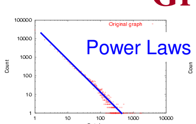
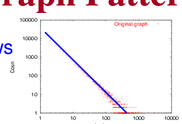
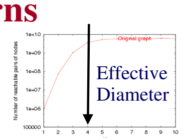
Graph Patterns



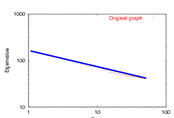
Count vs Indegree



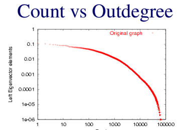
Count vs Outdegree



Hop-plot



Eigenvalue vs Rank



"Network values" vs Rank

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Wish list for a generator:

- Power-law-tail in- and out-degrees
- Power-law-tail scree plots
- **shrinking/constant** diameter
- Densification Power Law
- communities-within-communities

Q: how to achieve all of them?

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Wish list for a generator:

- Power-law-tail in- and out-degrees
- Power-law-tail scree plots
- **shrinking/constant** diameter
- Densification Power Law
- communities-within-communities

Q: how to achieve all of them?

A: Kronecker matrix product [Leskovec+05b]

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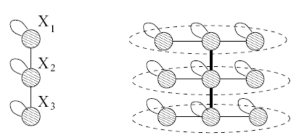
Kronecker product



(a) Graph G_1 (b) Intermediate stage

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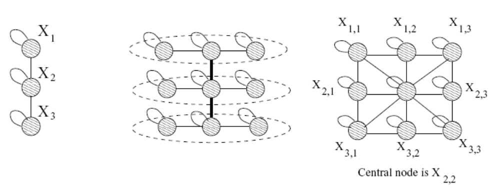
Kronecker product



(a) Graph G_1 (b) Intermediate stage (c) Graph $G_2 = G_1 \otimes G_1$

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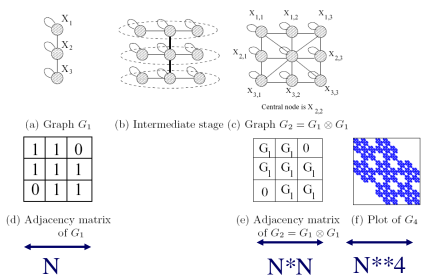
Kronecker product



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Properties of Kronecker graphs:

- ✓ Power-law-tail in- and out-degrees
- ✓ Power-law-tail scree plots
- ✓ **constant** diameter
- ✓ perfect Densification Power Law
- ✓ communities-within-communities

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Properties of Kronecker graphs:

- ✓ Power-law-tail in- and out-degrees
- ✓ Power-law-tail scree plots
- ✓ **constant** diameter
- ✓ perfect Densification Power Law
- ✓ communities-within-communities

and we can <u>prove all</u> of the above

(**first and only** generator that does that)

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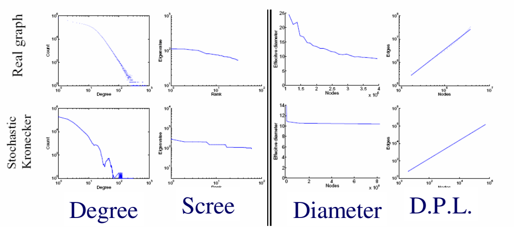
Kronecker - patents



15-826 #76

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Outline

- Topology & 'laws'
- Generators
- ➡ Discussion - tools

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Power laws

- Q1: Why so many?
- A1:

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Power laws

- Q1: Why so many?
- A1: self-similarity; 'rich-get-richer', etc - see Newman's paper
  http://arxiv.org/abs/cond-mat/0412004v3

Other settings with power laws?

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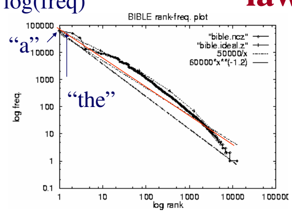
A famous power law: Zipf's law

log(freq)



log(rank)

- Bible - **rank** vs **frequency** (log-log)

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Power laws, cont'ed

- web hit counts [Huberman]
- Click-stream data [Montgomery+01]

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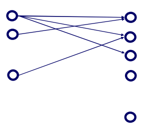
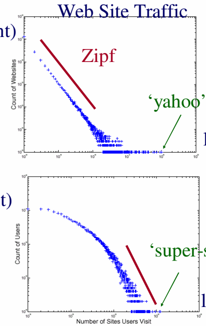
Click-stream data





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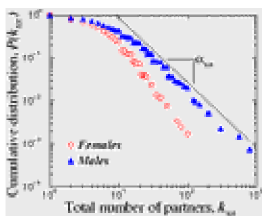
Swedish sex-web

Nodes: people (Females; Males)
Links: sexual relationships

Albert Laszlo Barabasi
http://www.nd.edu/~networks/Publication%20Categories/04%20Talks/2005-norway-3hours.ppt



4781 Swedes; 18-74;
59% response rate.

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Liljeros et al. *Nature* 2001

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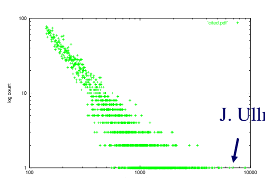
Lotka's law

(Lotka's law of publication count); and
citation counts: (*citeseer.nj.nec.com* 6/2001)



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Conclusions - Laws

'Laws' and patterns:

- Power laws for degrees, eigenvalues, 'communities'/cores
- Small diameter and shrinking diameter
- Bow-tie; jelly-fish
- densification power law

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Conclusions - Generators

- Preferential attachment (Barabasi)
- Variations
- Recursion – Kronecker graphs

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Conclusions - Tools

- Power laws –
  - rank/frequency plots ~ log-log NCDF
  - log-log PDF
- Self-similarity / recursion / fractals
- 'correlation integral' = hop-plot

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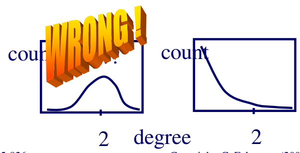
Conclusions - Tools (cont'd)

- Real settings/graphs: skewed distributions
  - 'mean' is meaningless
  - slope of power law, instead



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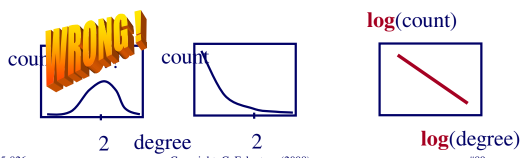
Conclusions - Tools (cont'd)

- Real settings/graphs: skewed distributions
  - 'mean' is meaningless
  - slope of power law, instead



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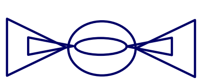
Conclusions - Tools (cont'd)

- Recursion/self-similarity
  - May reveal non-obvious patterns (e.g., bow-ties within bow-ties within bow-ties) [Dill+, '01]



"To iterate is human, to recurse is divine"

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Resources

Generators:

- Kronecker (christos@cs.cmu.edu)
- BRITE http://www.cs.bu.edu/brite/
- INET: http://topology.eecs.umich.edu/inet

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Other resources

Visualization - graph algo's:

- Graphviz: http://www.graphviz.org/
- pajek: http://vlado.fmf.uni-lj.si/pub/networks/pajek/

Kevin Bacon web site:
http://www.cs.virginia.edu/oracle/

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