Binary Search Trees & Trees in General

02-201 / 02-601

Dictionary Abstract Data Type (ADT)

- Most basic and most useful ADT:
  - `insert(key, value)`
  - `delete(key)`
  - `value = find(key)`

- Many languages have it built in like Go's **map**:

awk:	D["AAPL"] = 130	# associative array
perl:	`my %D; $D["AAPL"] = 130;`	# hash
python:	`D = {}; D["AAPL"] = 130`	# dictionary
C++:	`map<string,string> D = new map<string, string>();`	
	`D["AAPL"] = 130;`	// map

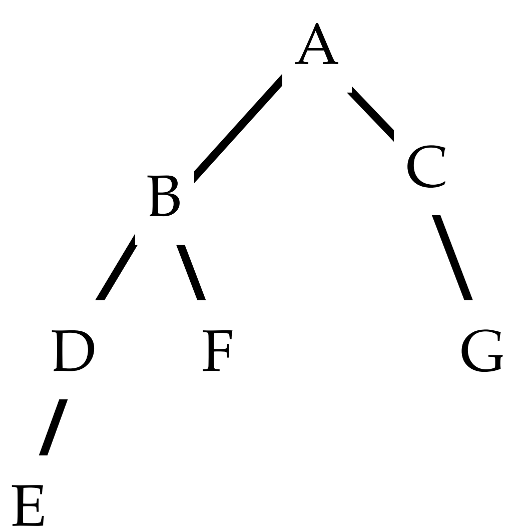
- **Insert**, **delete, find** each either ≈ log n steps [C++] or expected constant # of steps [perl, python]

- How can such dictionaries are implemented? — There are a number of ways; we'll see one next.

Trees

Hierarchies

Many ways to represent tree-like information:



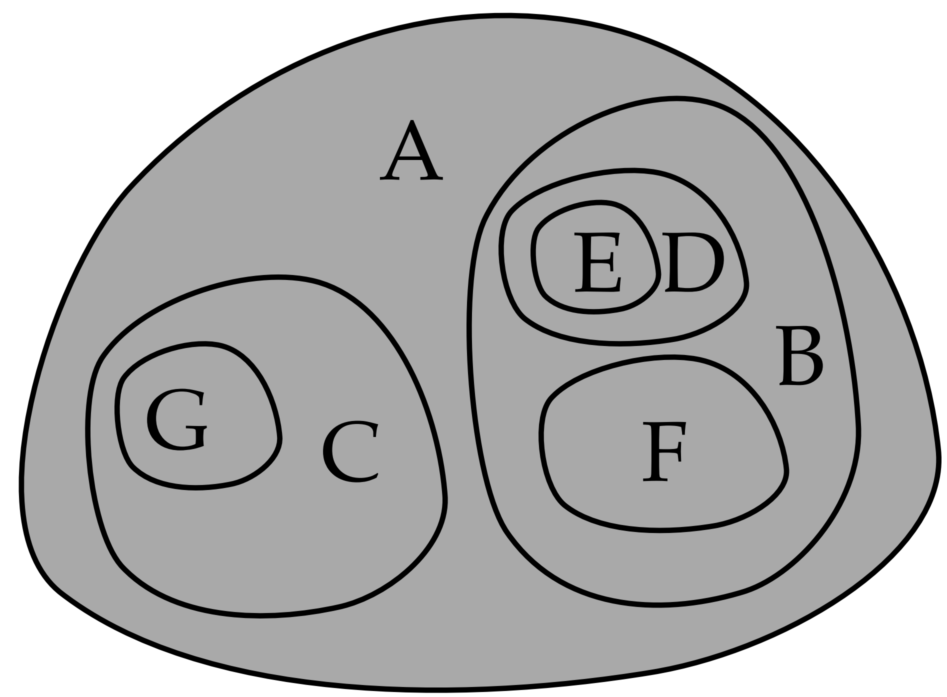
linked hierarchy

I. A
  1. B
    a. D
      i. E
    b. F
  2. C
    a. G

outlines, indentations

(((E):D), F):B, (G):C):A

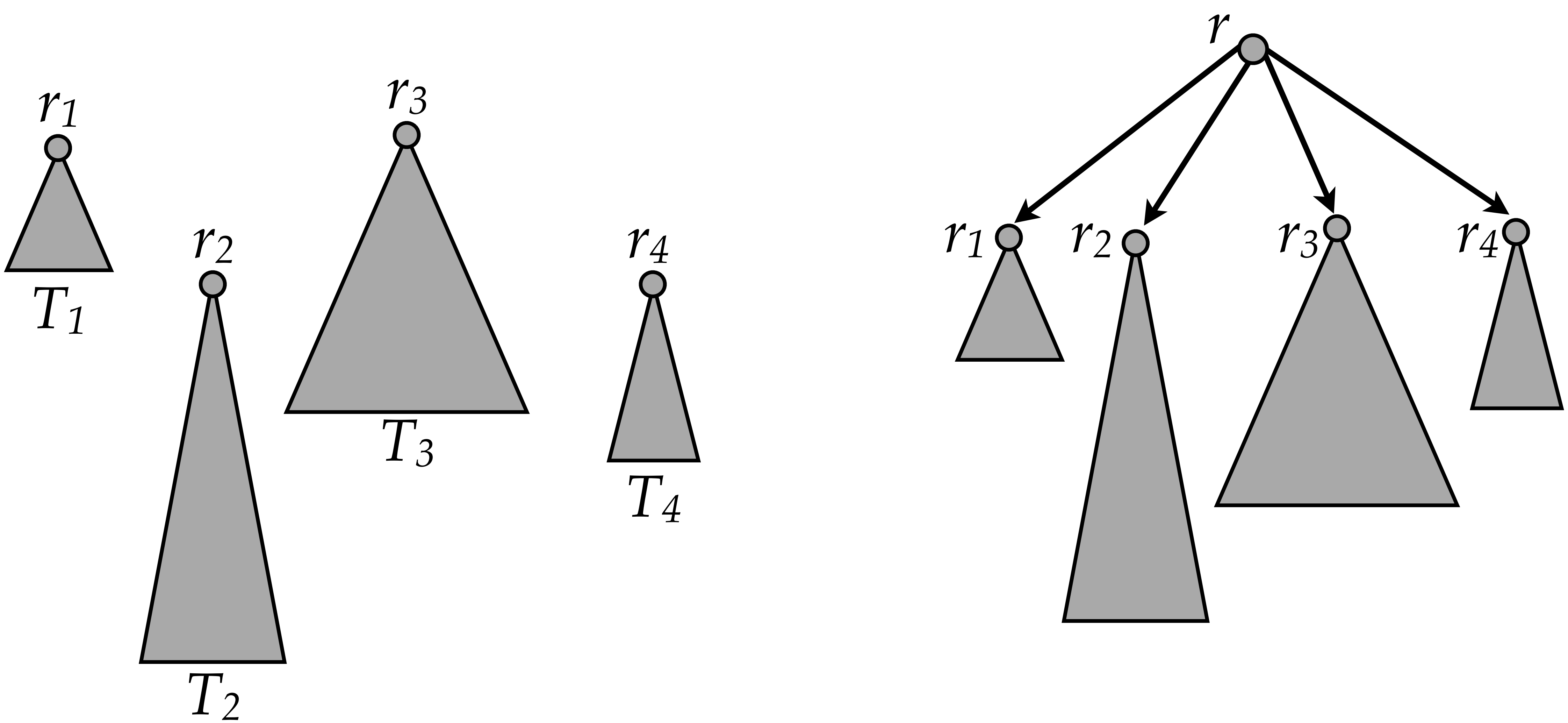
nested, labeled parenthesis



nested sets

Definition – Rooted Tree

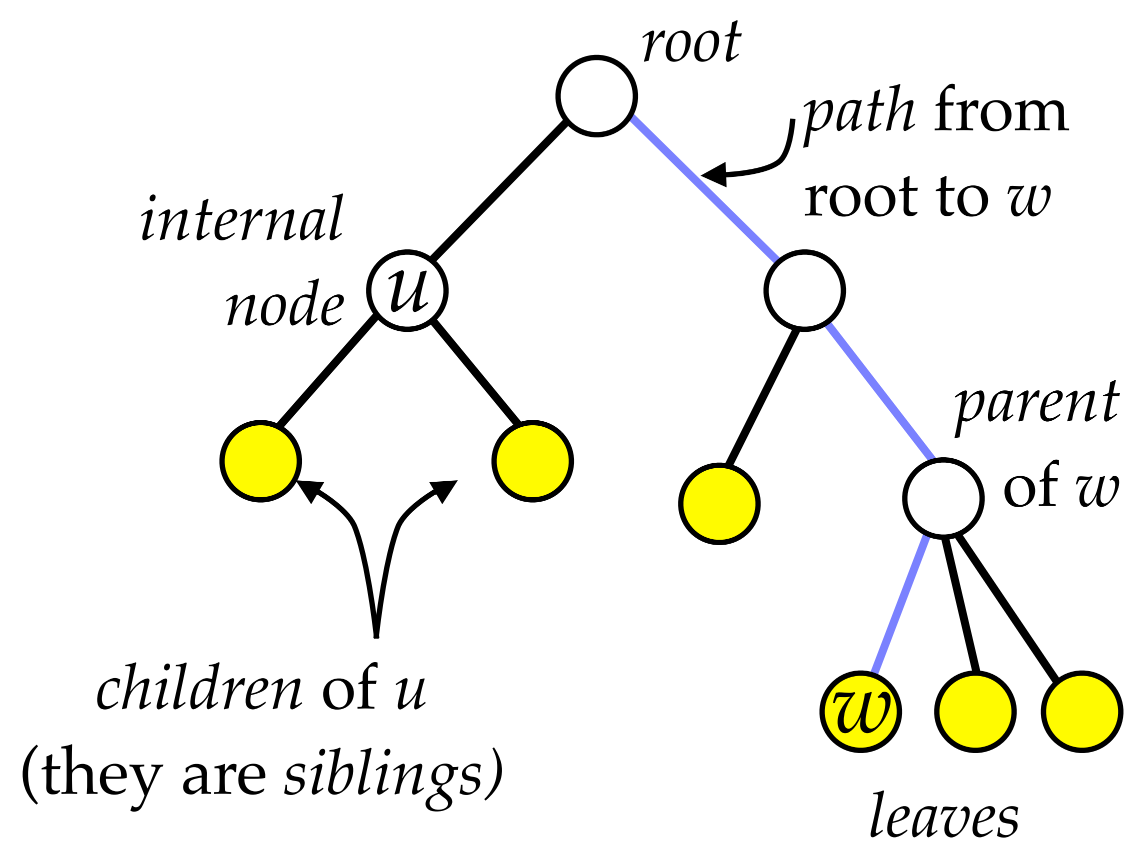
- **nil** is a tree
- If $T_1, T_2, \ldots, T_k$ are trees with roots $r_1, r_2, \ldots, r_k$ and r is a node $\notin$ any T_i, then the structure that consists of the T_i, node r, and edges (r, r_i) is also a tree.



Terminology

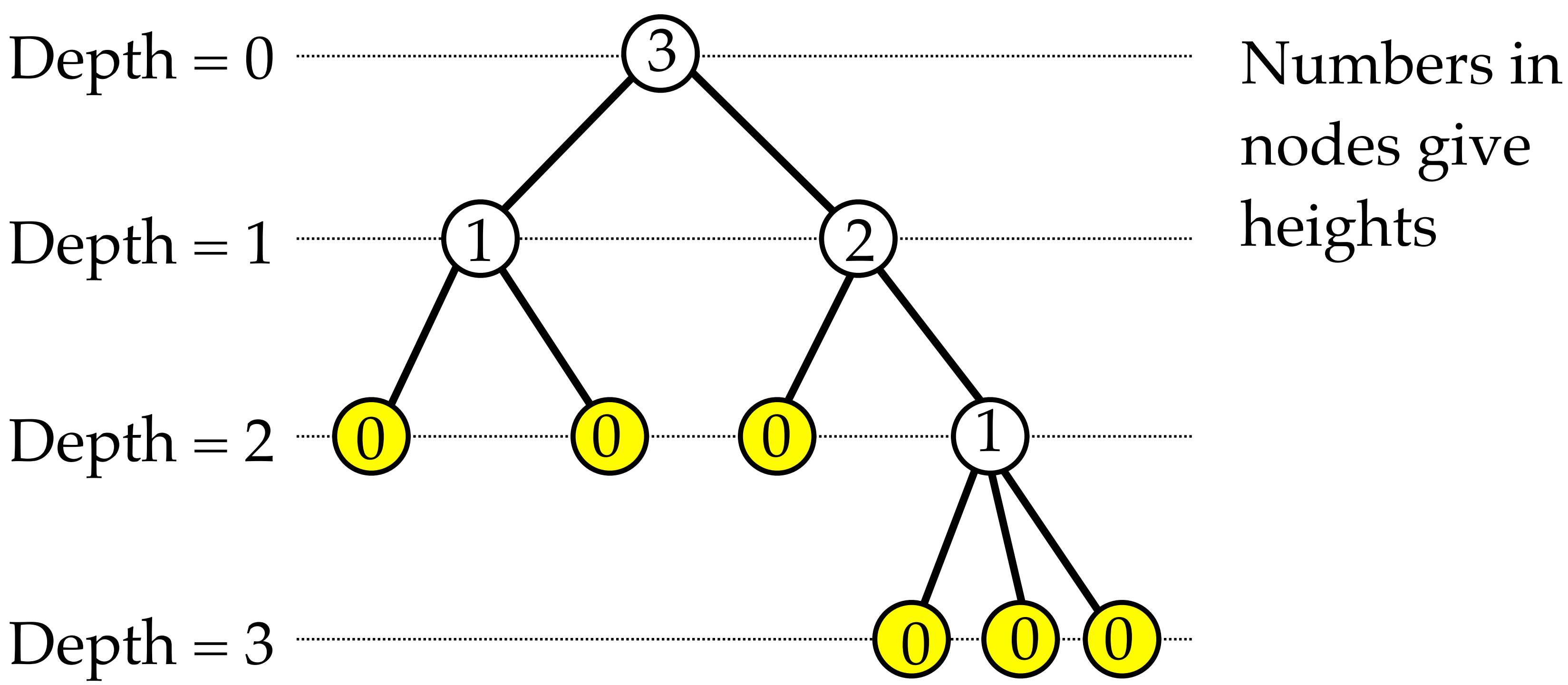
Unfortunately, different authors use different tree terminology

- r is the *parent* of its *children* $r_1, r_2, \ldots, r_k$.
- $r_1, r_2, \ldots, r_k$ are *siblings*.
- *root* = distinguished node, usually drawn at top. Has no parent.
- If all children of a node are **nil**, the node is a *leaf*. Otherwise, the node is a *internal node*.
- A *path* in the tree is a sequence of nodes $u_1, u_2, \ldots, u_m$ such that each of the edges (u, u_{i+1}) exists.
- A node u is an *ancestor* of v if there is a path from u to v.
- A node u is a *descendant* of v if there is a path from v to u.



Height & Depth

- The ***height*** of node u is the length of the longest path from u to a leaf.
- The ***depth*** of node u is the length of the path from the root to u.
- Height of the tree = maximum depth of its nodes.
- A ***level*** is the set of all nodes at the same depth.

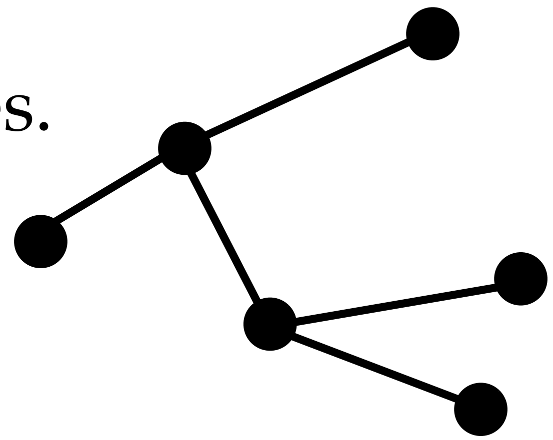


Subtrees, forests, and graphs

- A *subtree* rooted at u is the tree formed from u and all its descendants.

- A *forest* is a (possibly empty) set of trees.
  The set of subtrees rooted at the children of r form a forest.

- As we've defined them, trees are **not** a special case of graphs:
  - Our trees are *oriented* (there is a root which implicitly defines directions on the edges).
  - A *free tree* is a connected graph with no cycles.



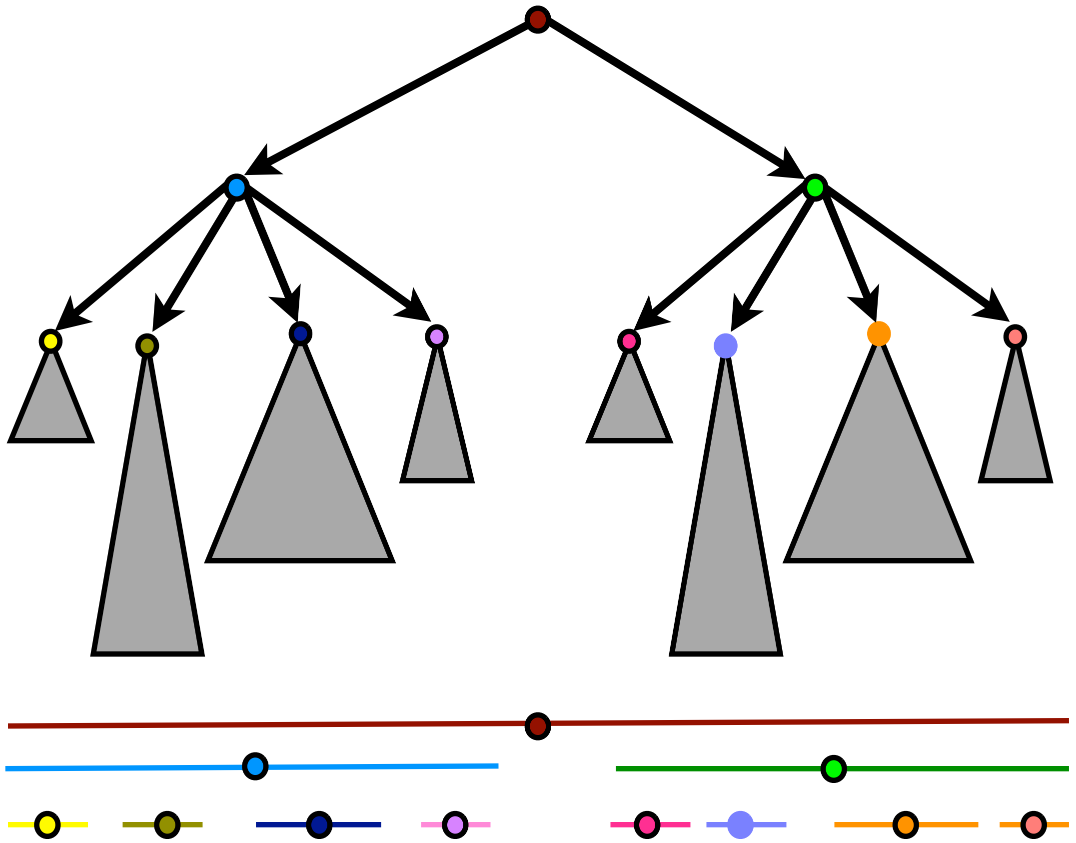
Alternative Definition – Rooted Tree

- A *tree* is a finite set T such that:
  - one element $r \in T$ is designated the *root*.
  - the remaining nodes are partitioned into $k \geq 0$ disjoint sets T_1, T_2, ..., T_k, each of which is a tree.

This definition emphasizes the ***partitioning*** aspect of trees:

As we move down the we're dividing the set of elements into more and more parts.

Each part has a distinguished element (that can represent it).



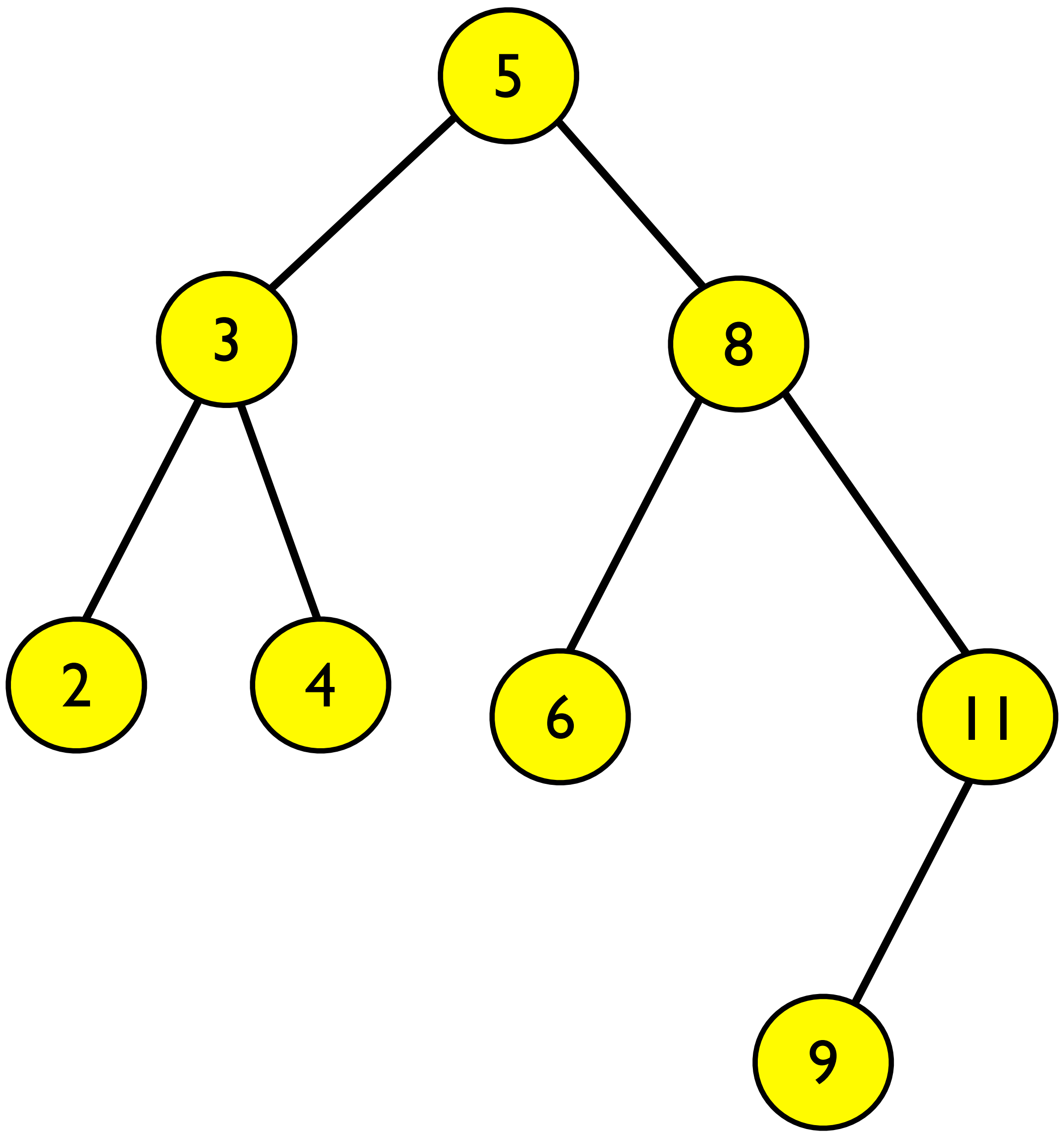
Binary Search Trees

Binary Search Trees (BST)

- **BST Property:** If a node has key *k* then keys in the **left** subtree are $< k$ and keys in the **right** subtree are $> k$.

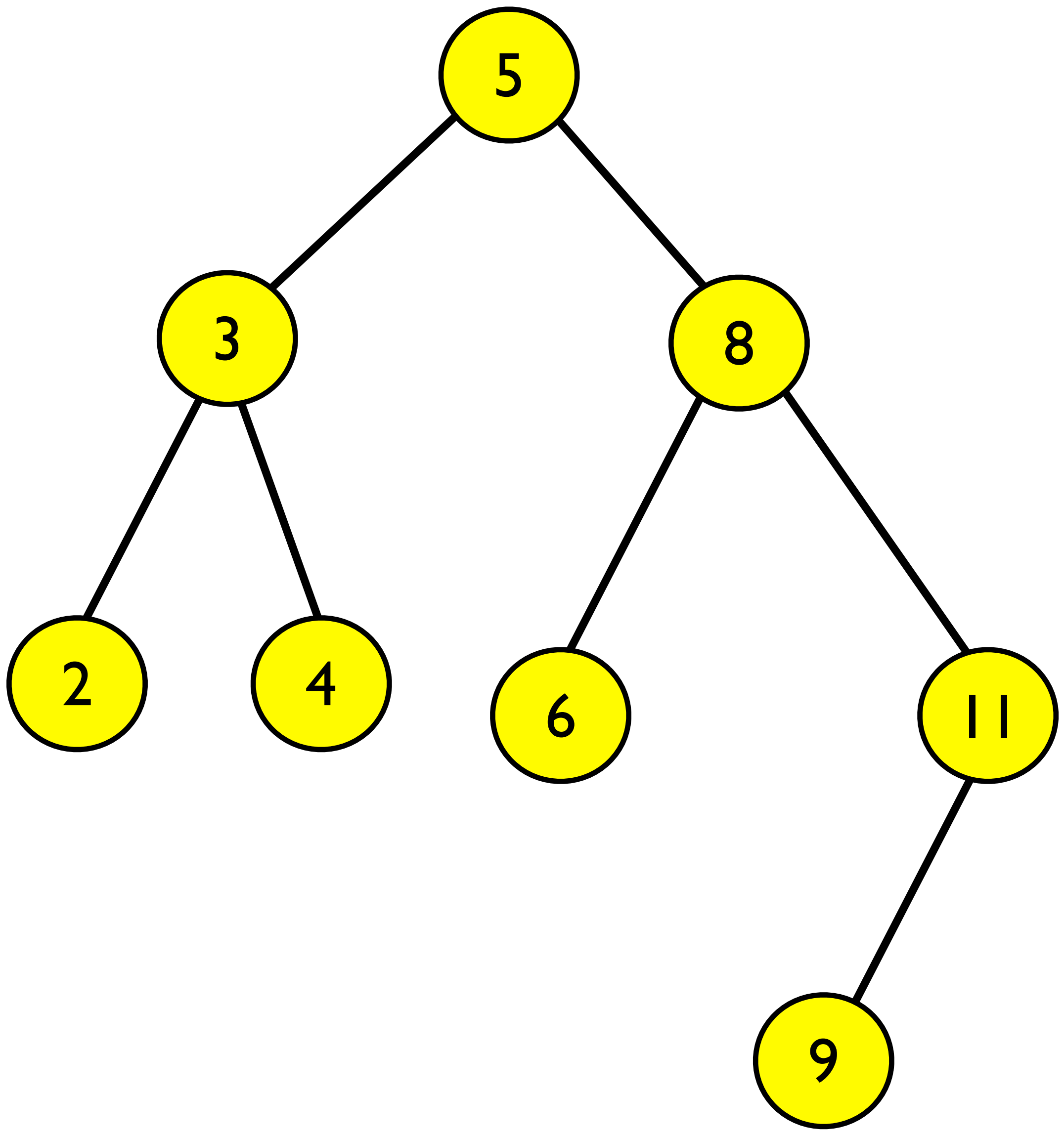
- For convenience, we disallow duplicate keys.

- Good for implementing the *dictionary ADT* we've already seen: insert, delete, find.



BST Find

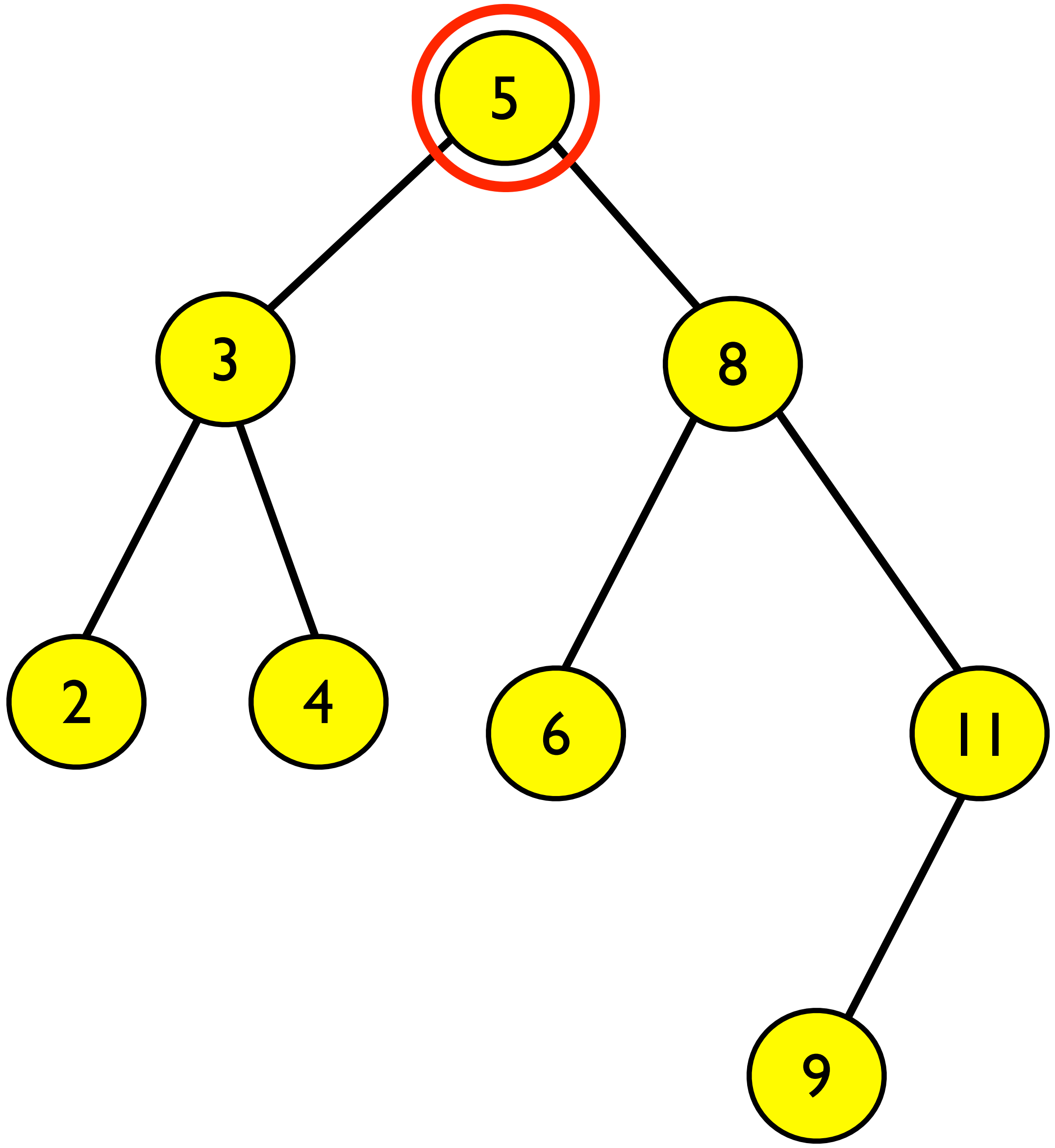
Find $k = 6$:



BST Find

Find $k = 6$:

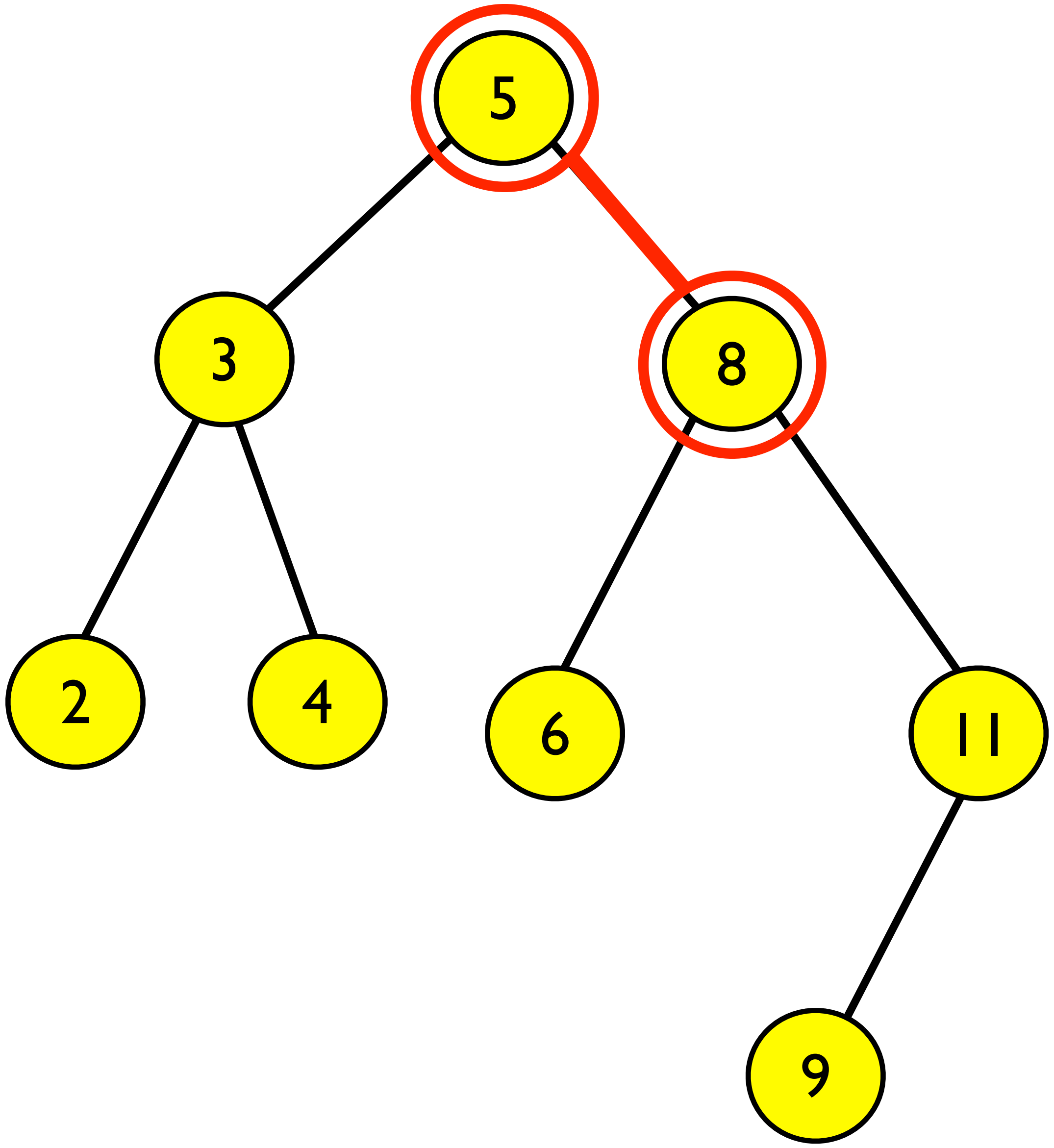
Is $k < 5$?



BST Find

Find $k = 6$:

Is $k < 5$? No, go right

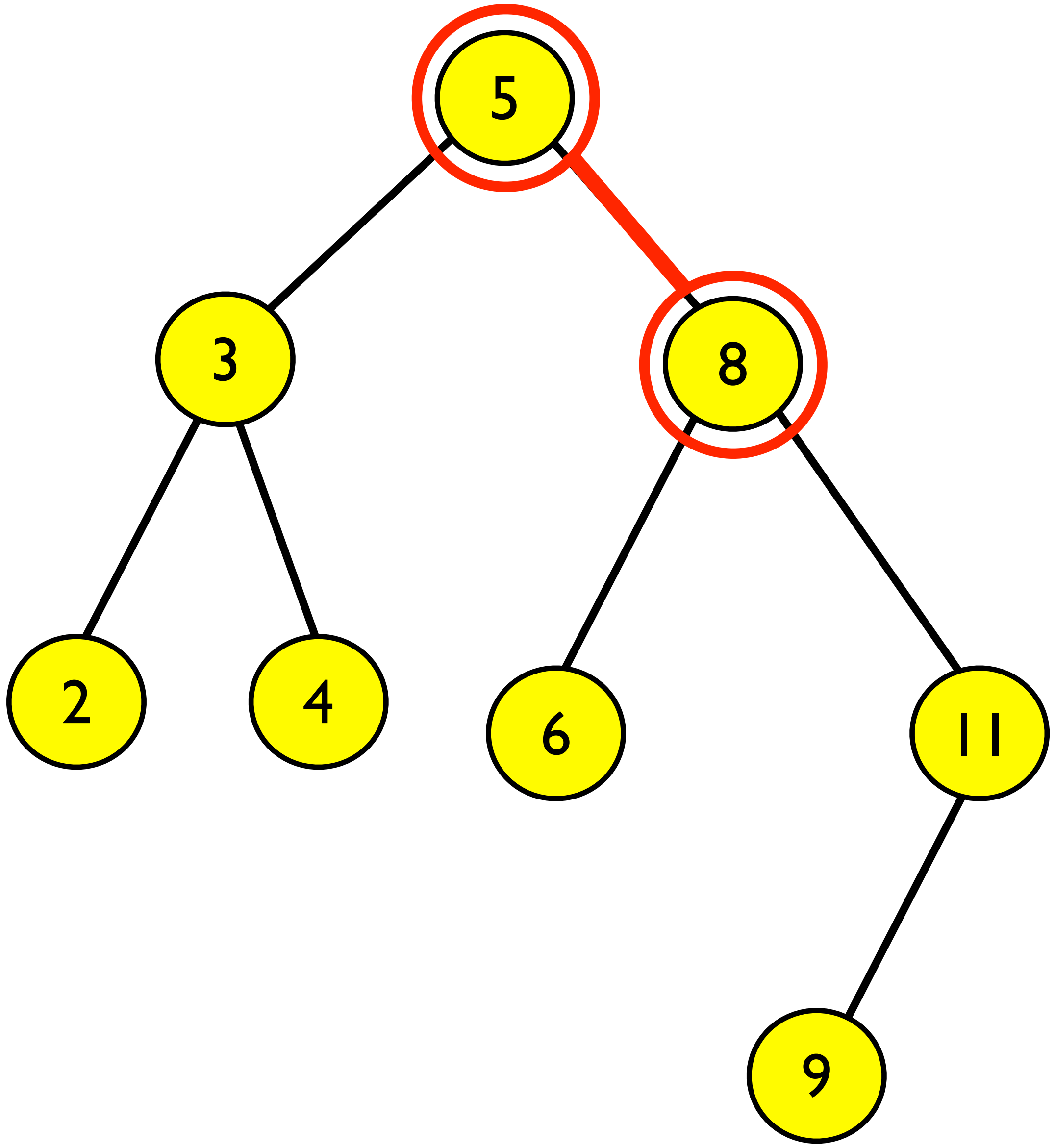


BST Find

Find $k = 6$:

Is $k < 5$? No, go right

Is $k < 8$?

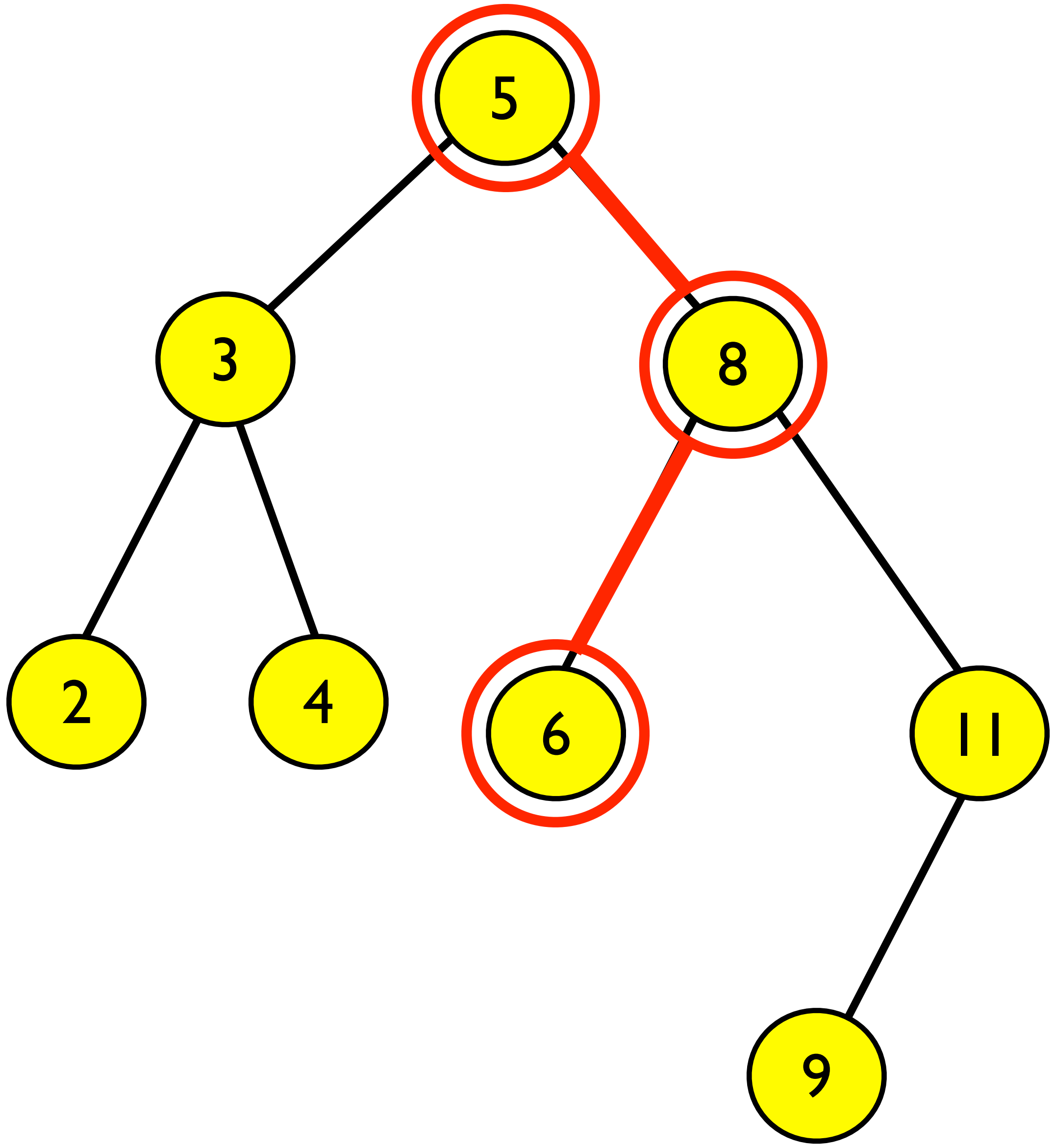


BST Find

Find $k = 6$:

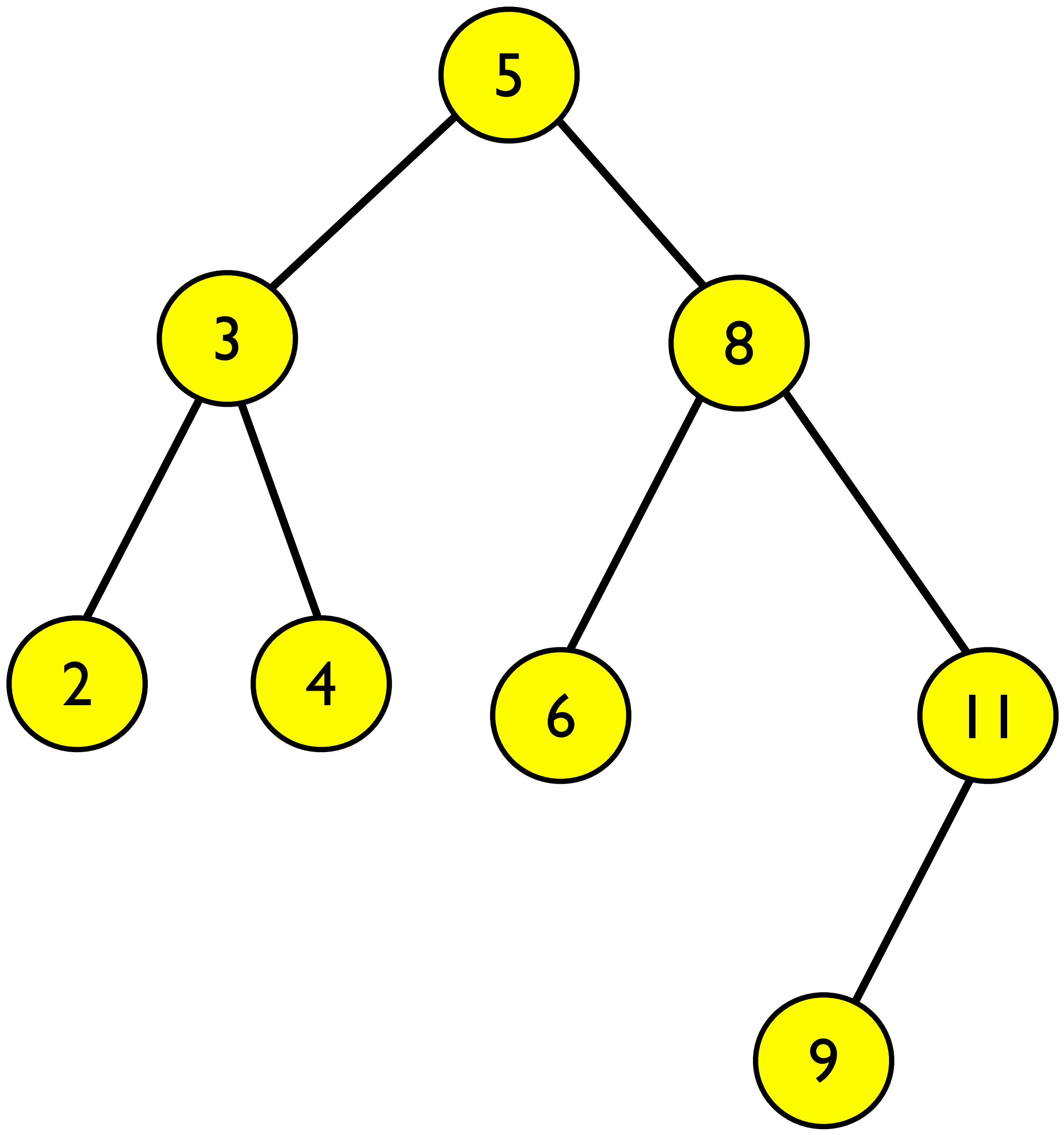
Is $k < 5$? No, go right

Is $k < 8$? Yes, go left



BST Find

Find $k = 9$:



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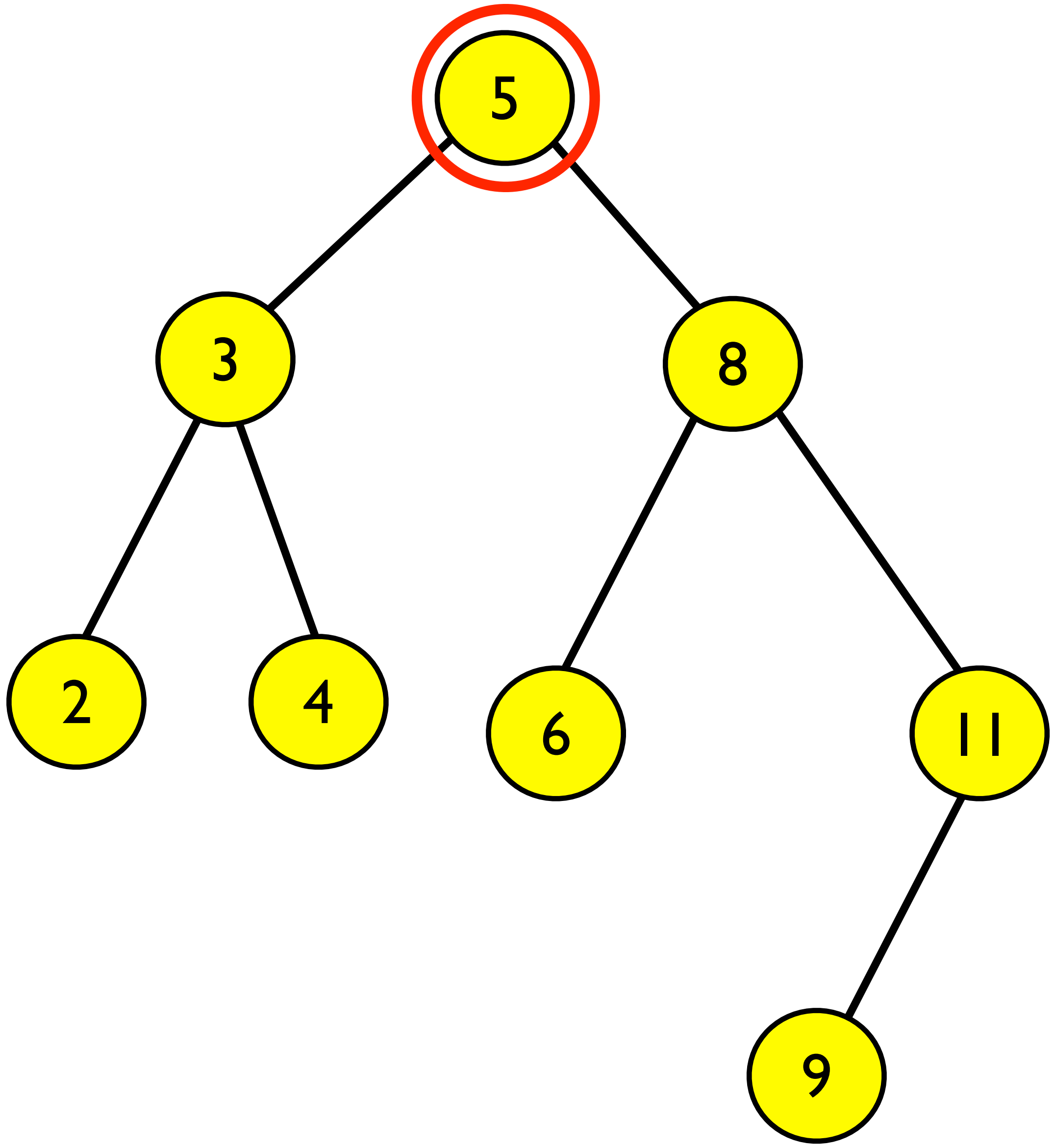

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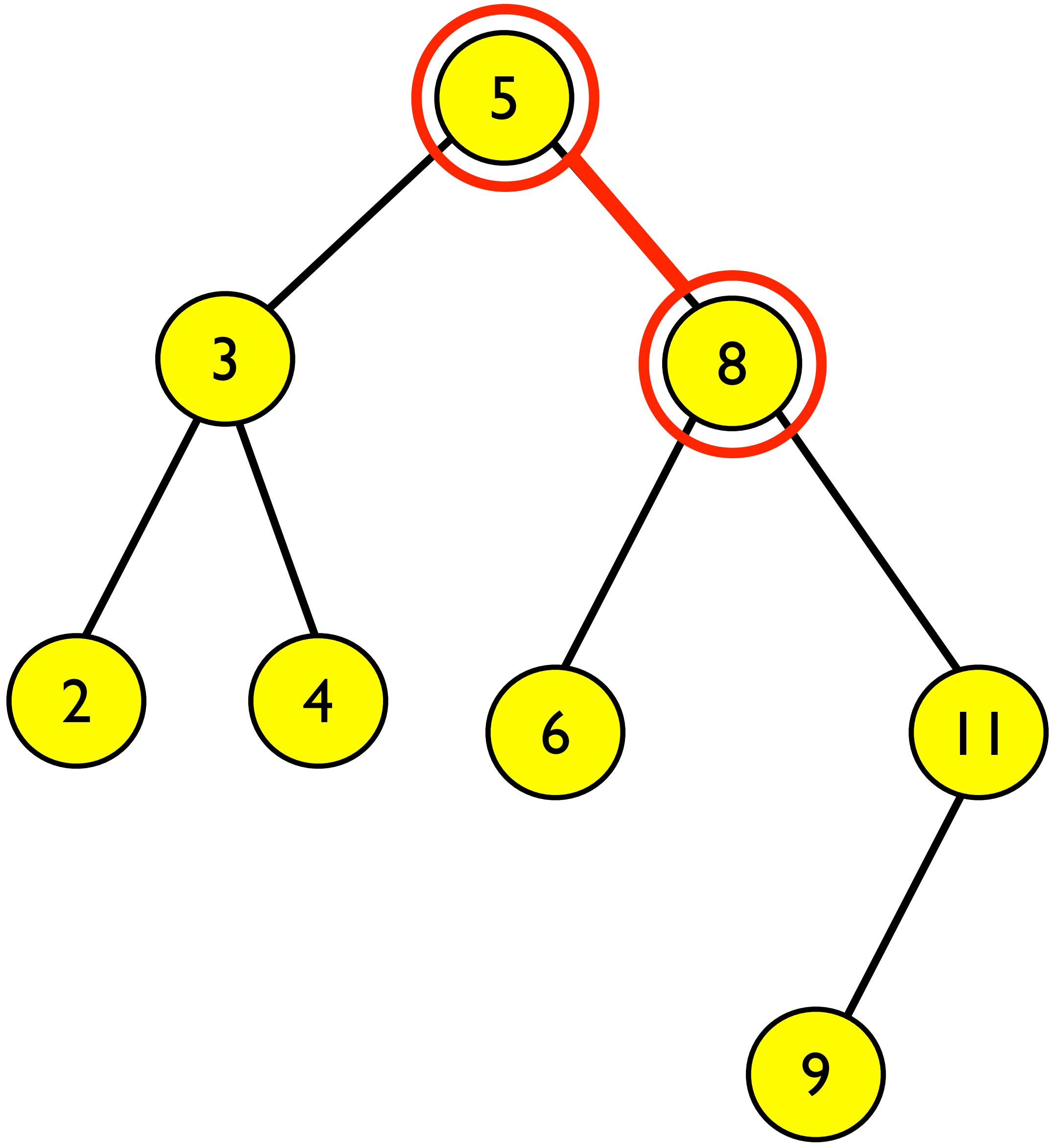

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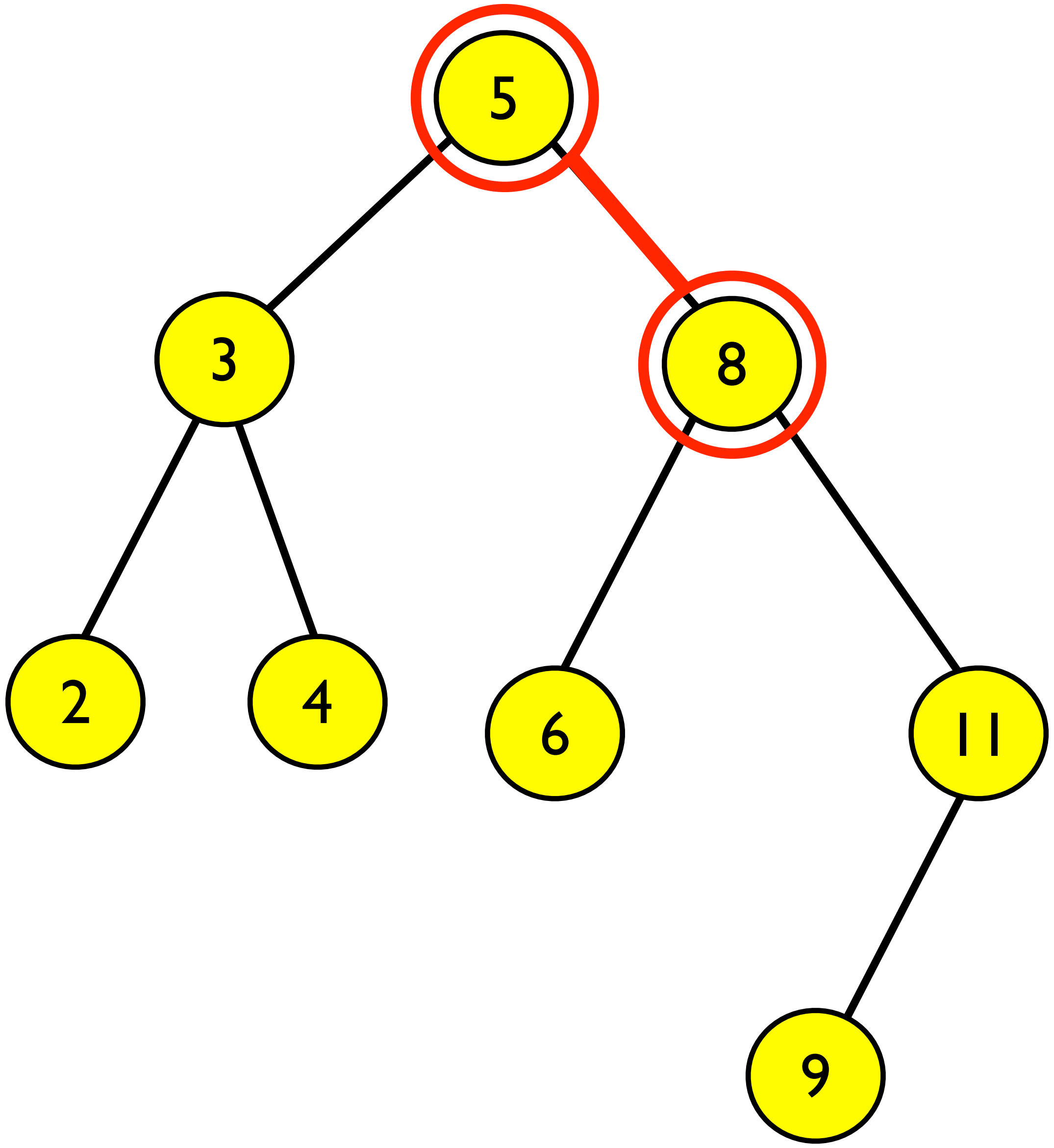

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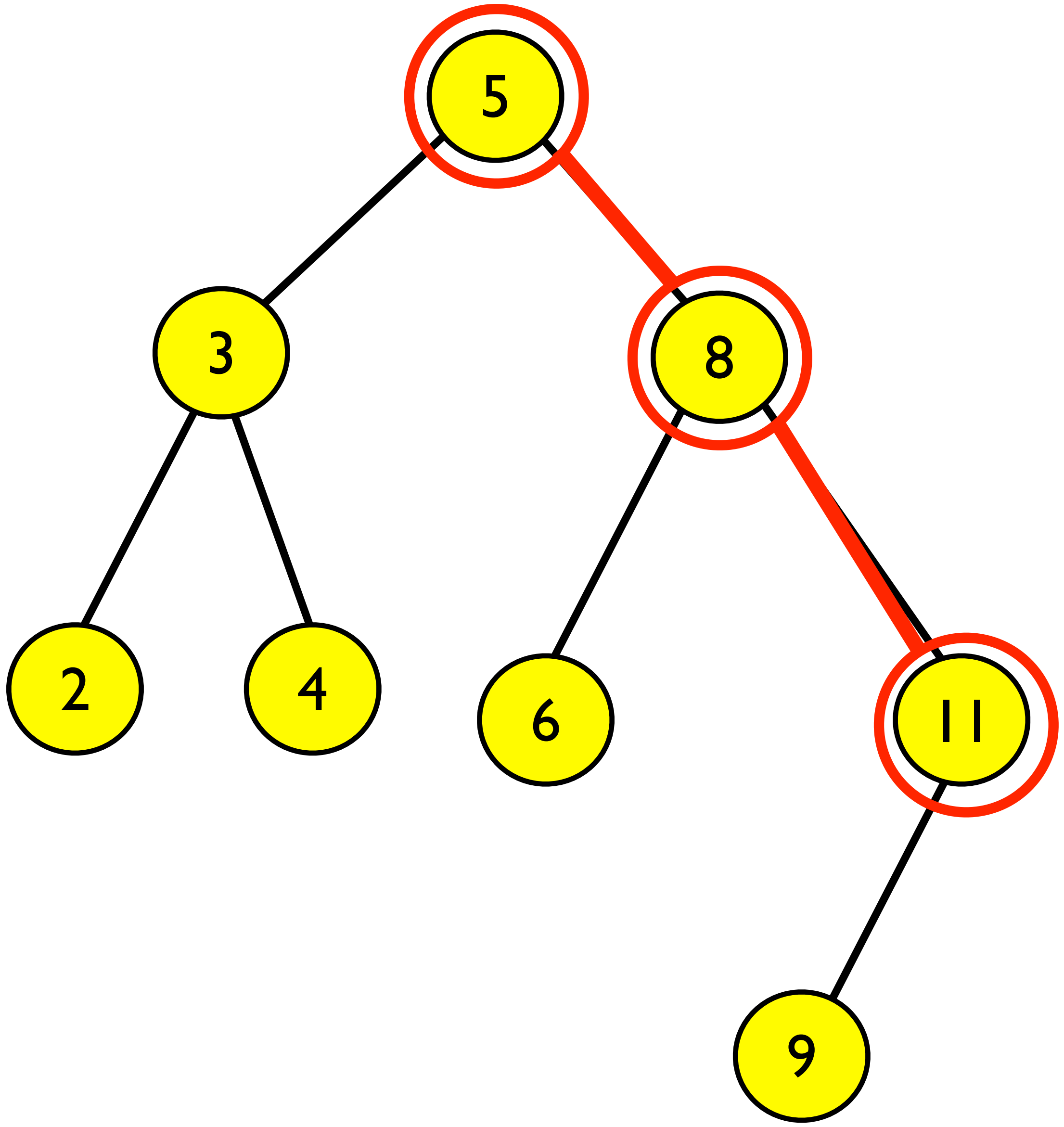


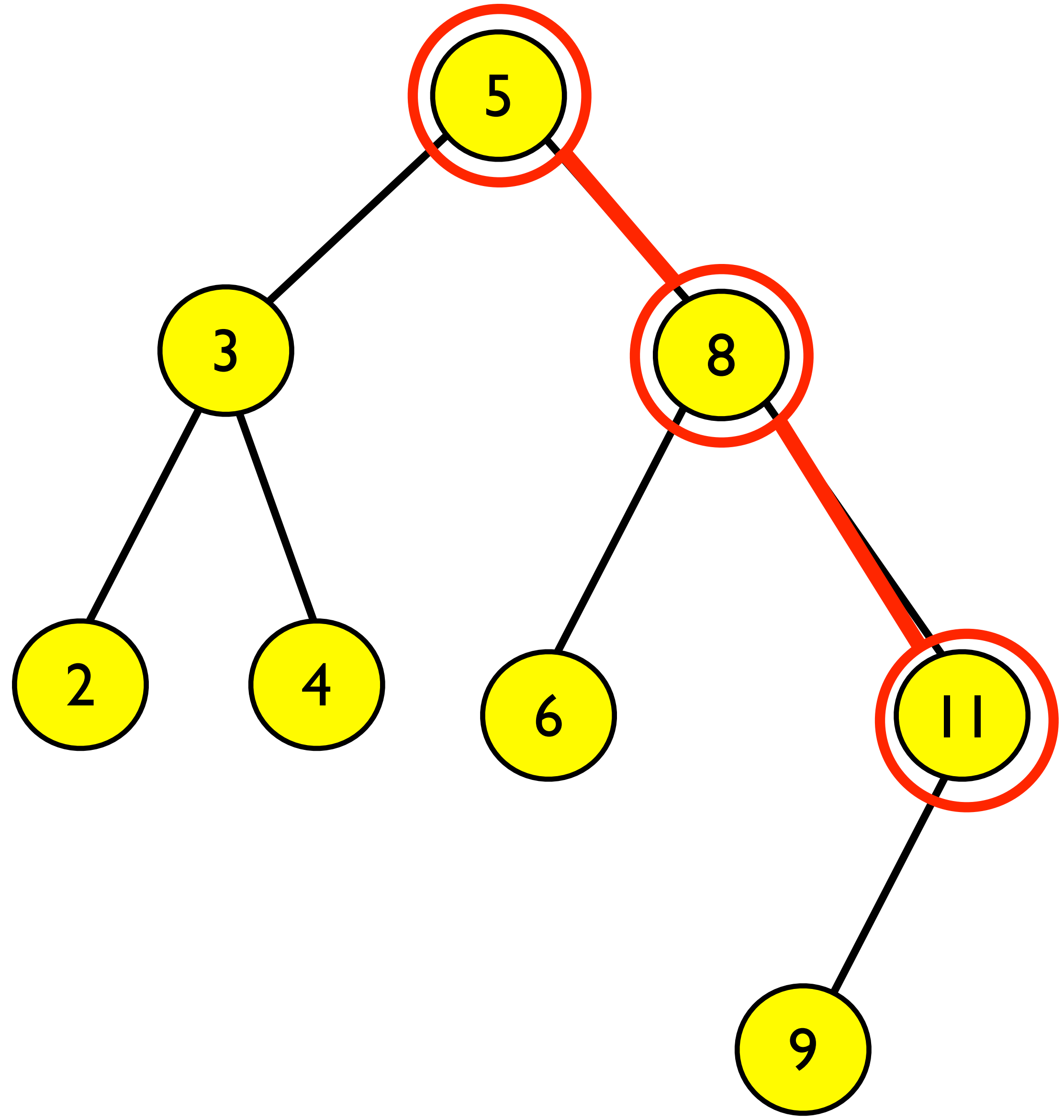
BST Find

Find $k = 9$:

Is $k < 5$? No, go right

Is $k < 8$? No, go right

Is $k < 11$?



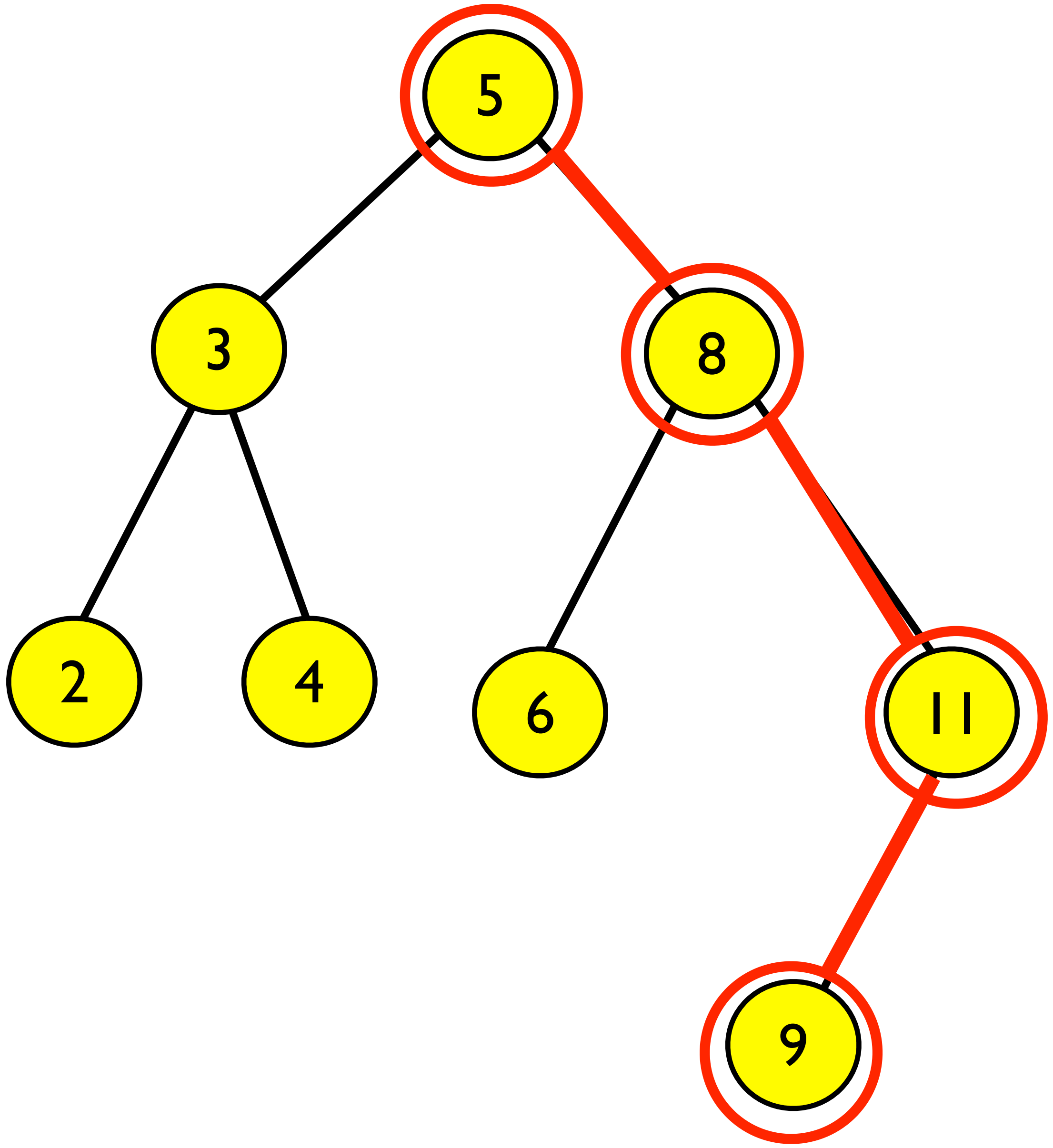
BST Find

Find $k = 9$:

Is $k < 5$? No, go right

Is $k < 8$? No, go right

Is $k < 11$? Yes, go left



BST Find

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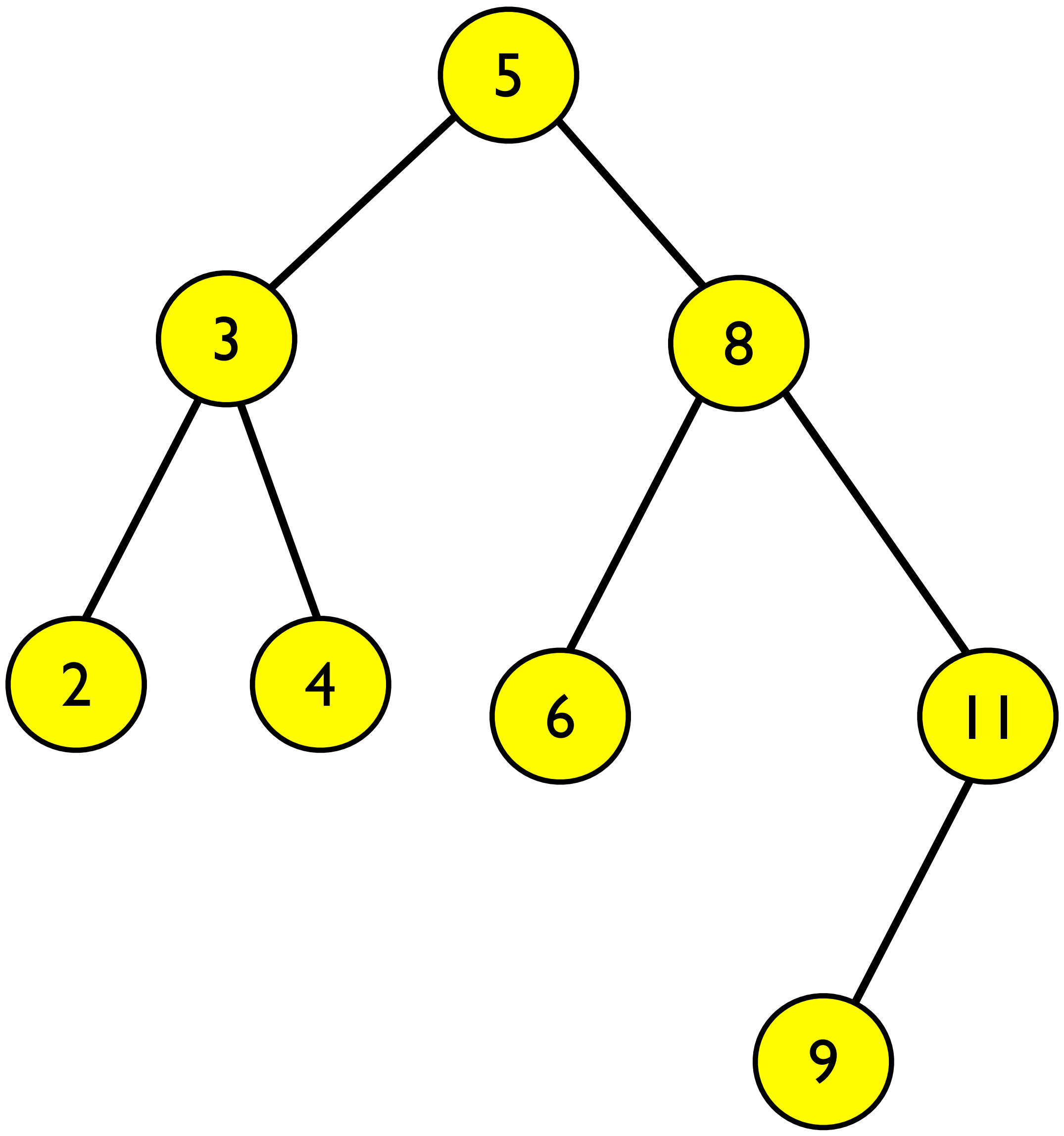
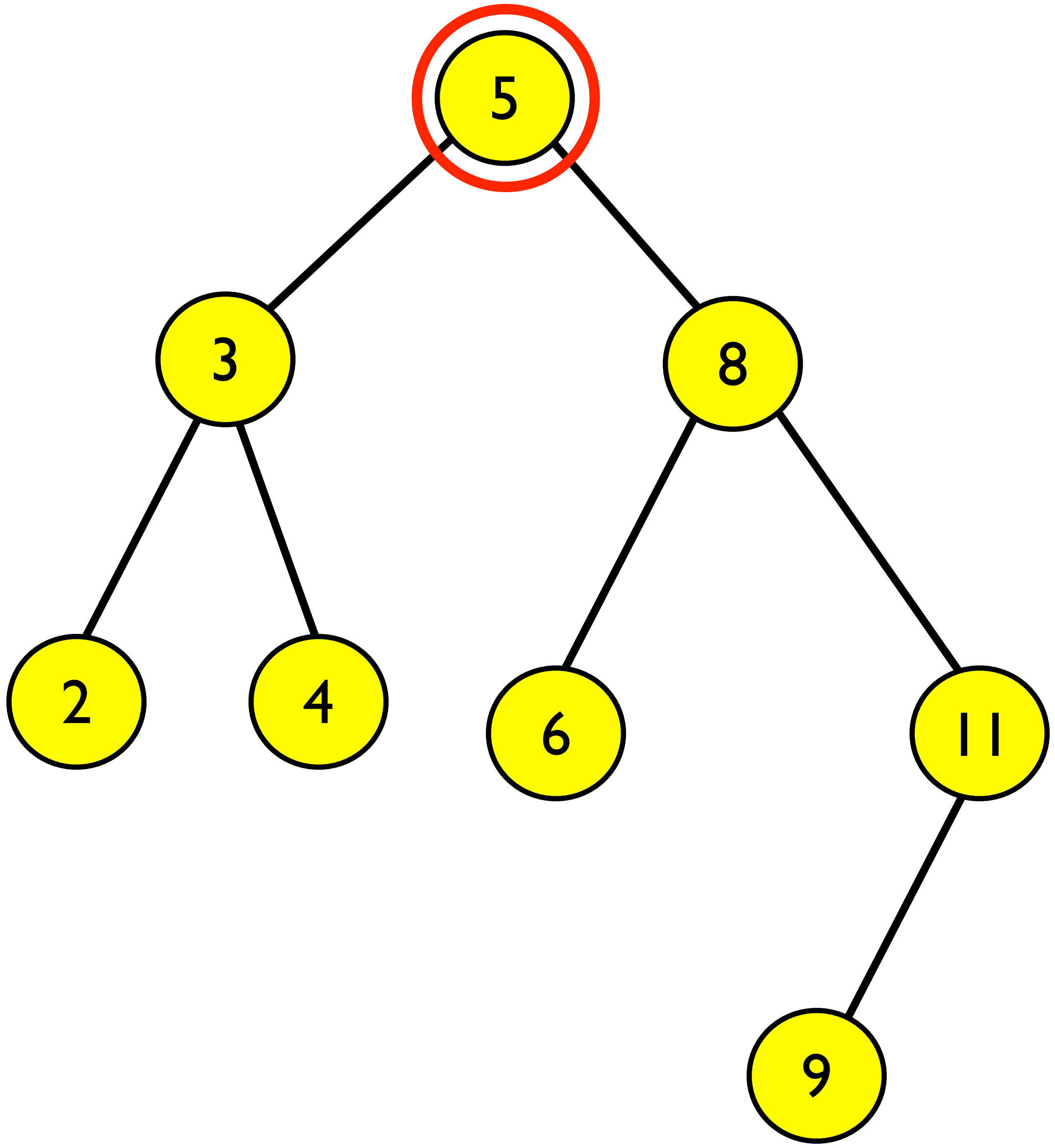


BST Find

Find $k = 13$:

Is $k < 5$?



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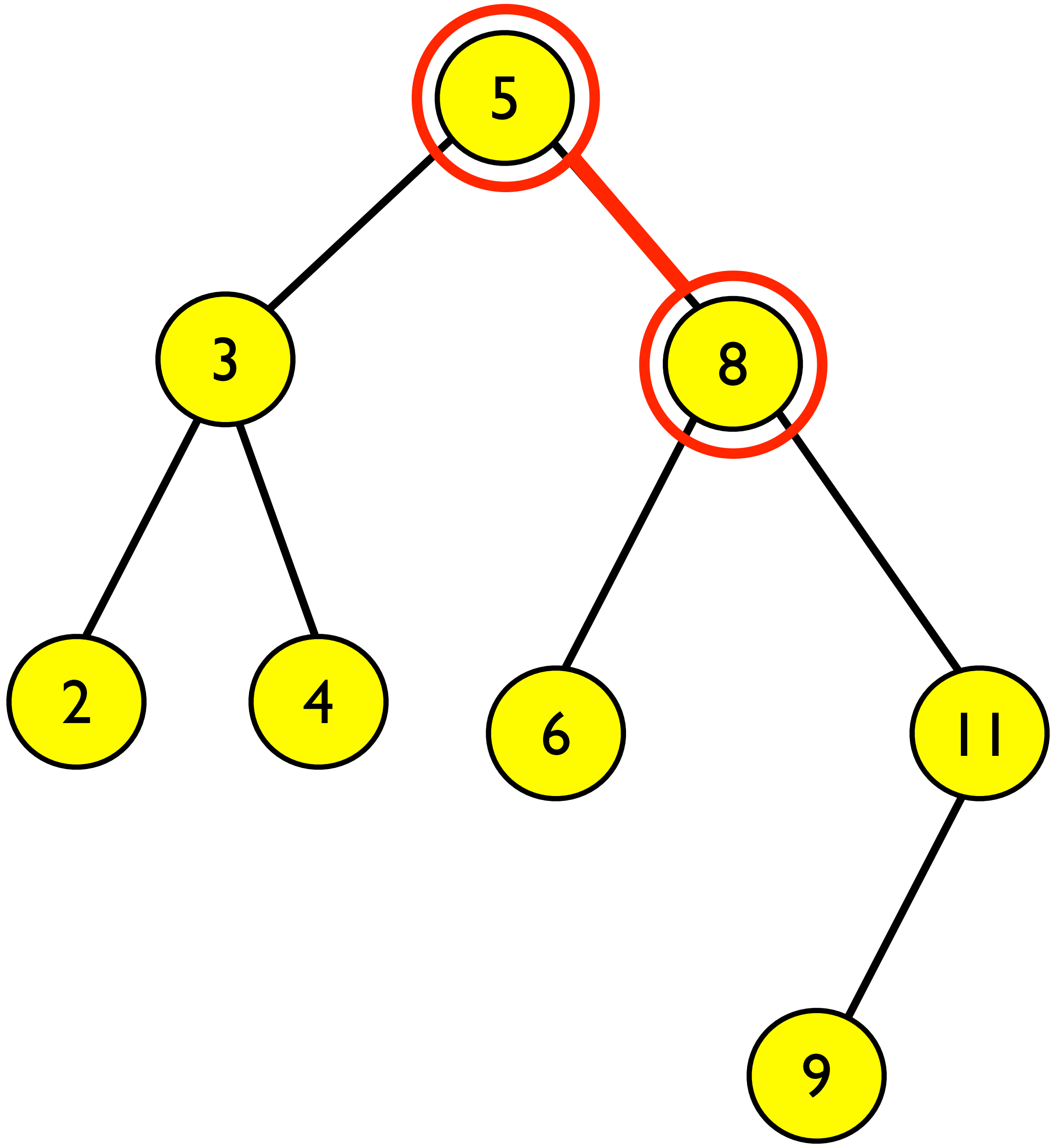

BST Find

Find $k = 13$:

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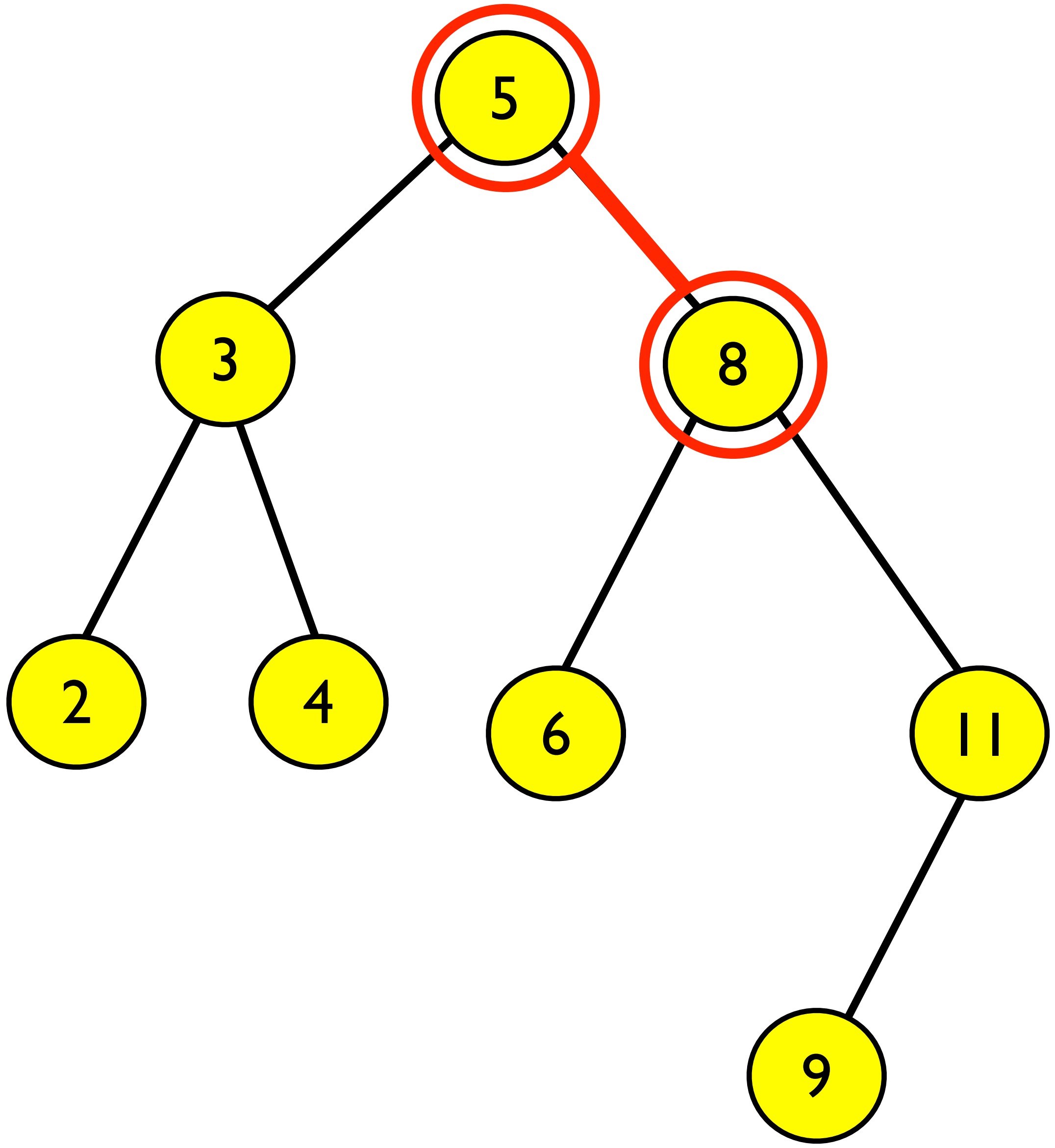

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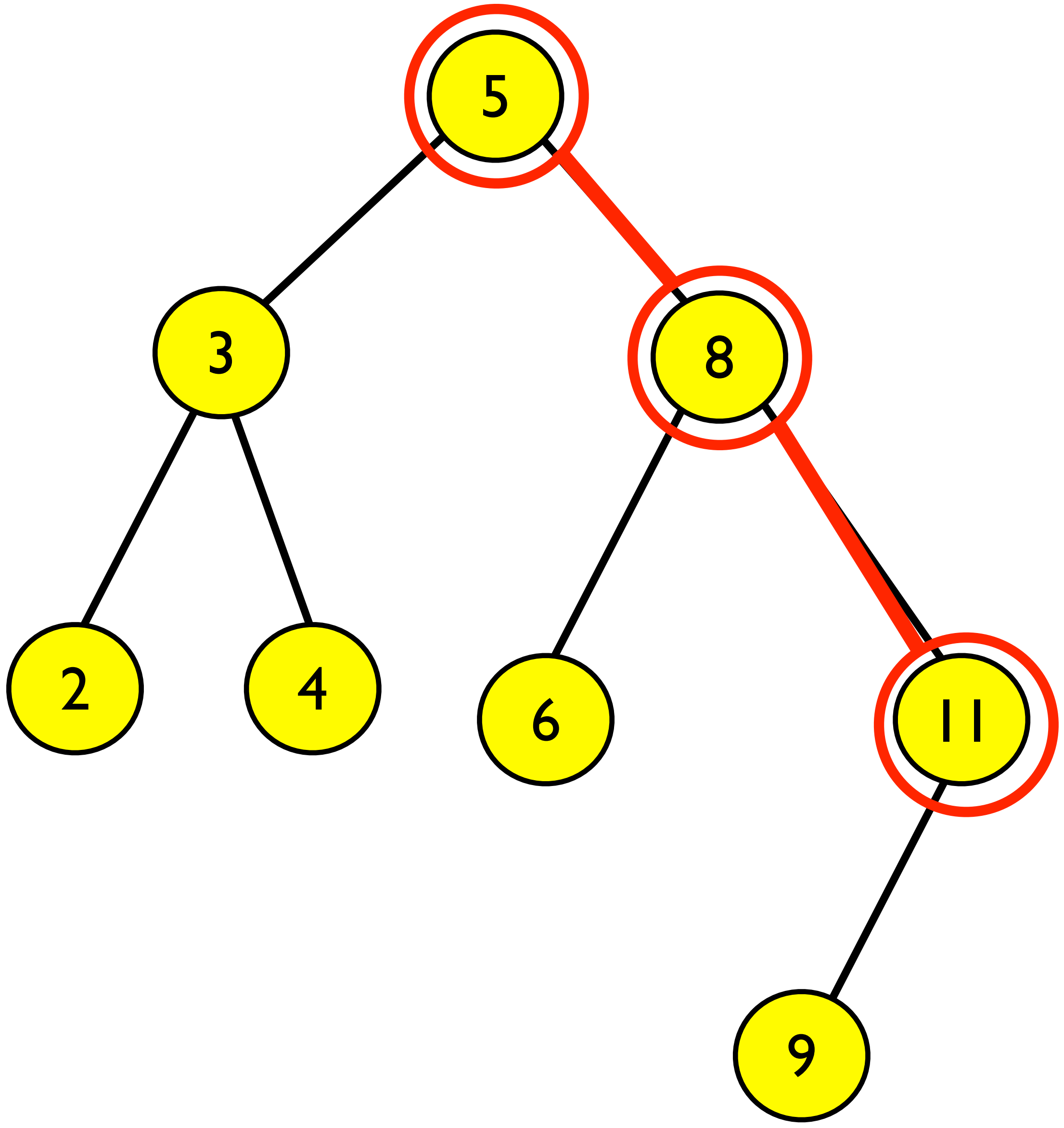

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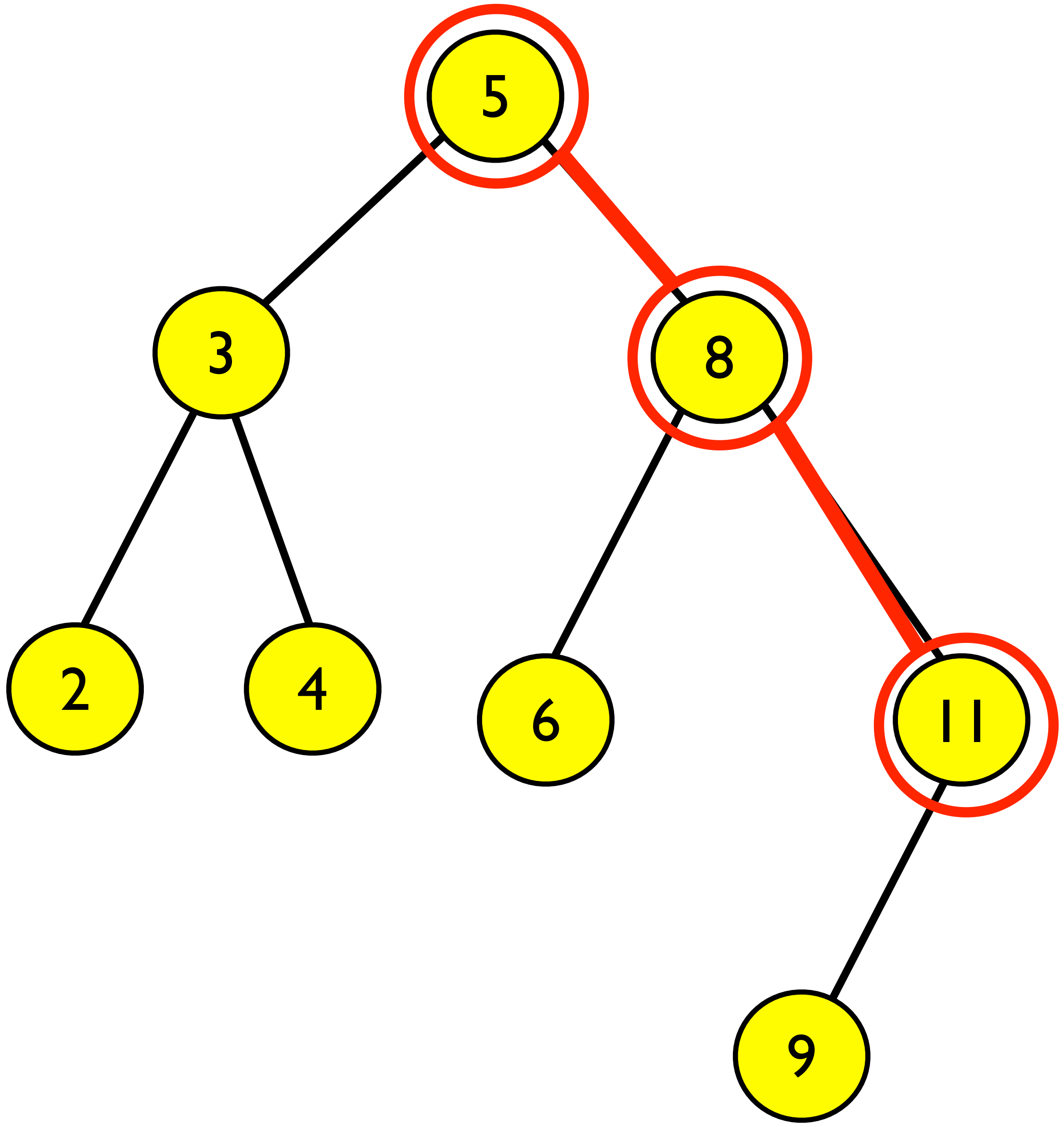

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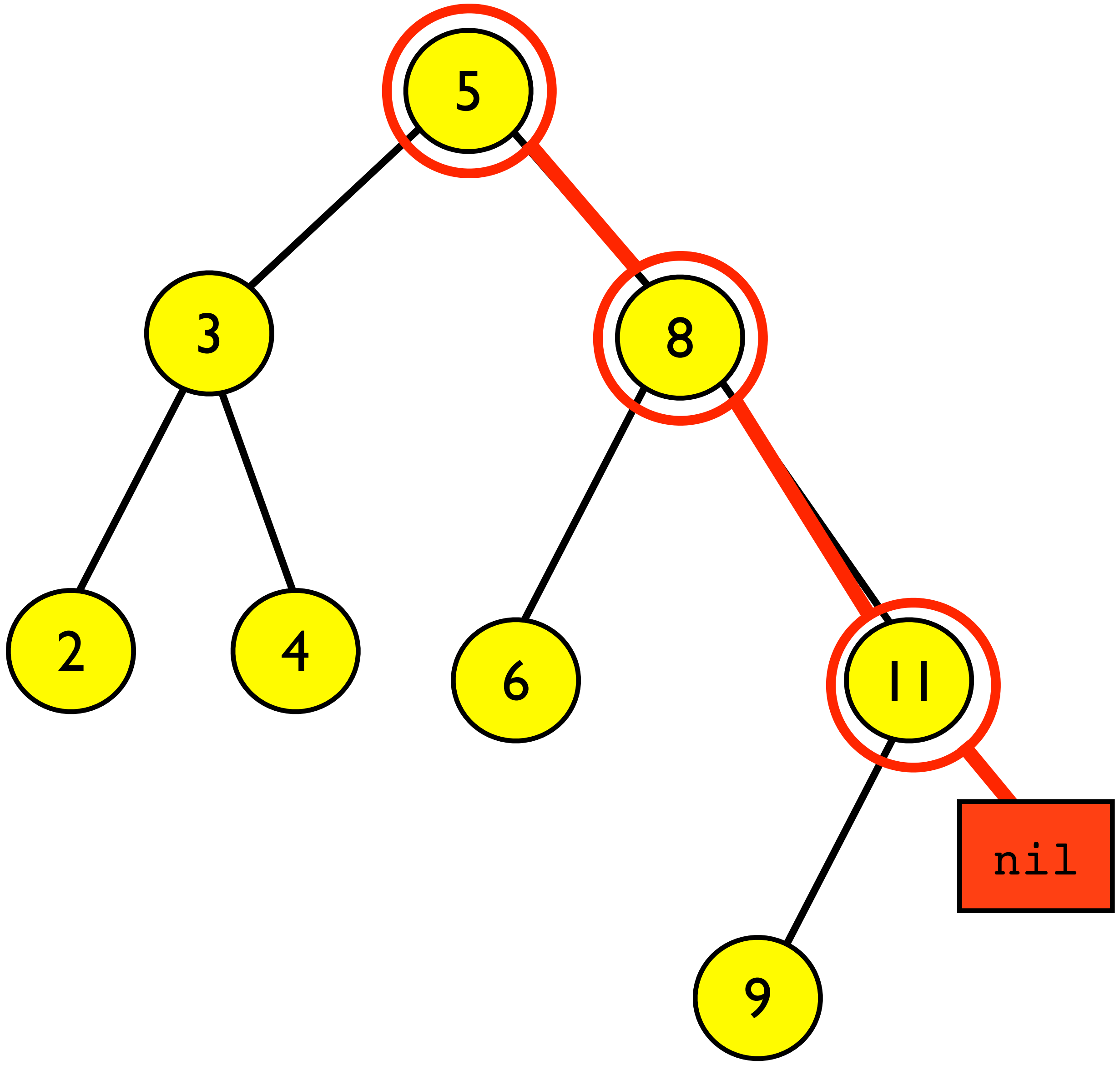


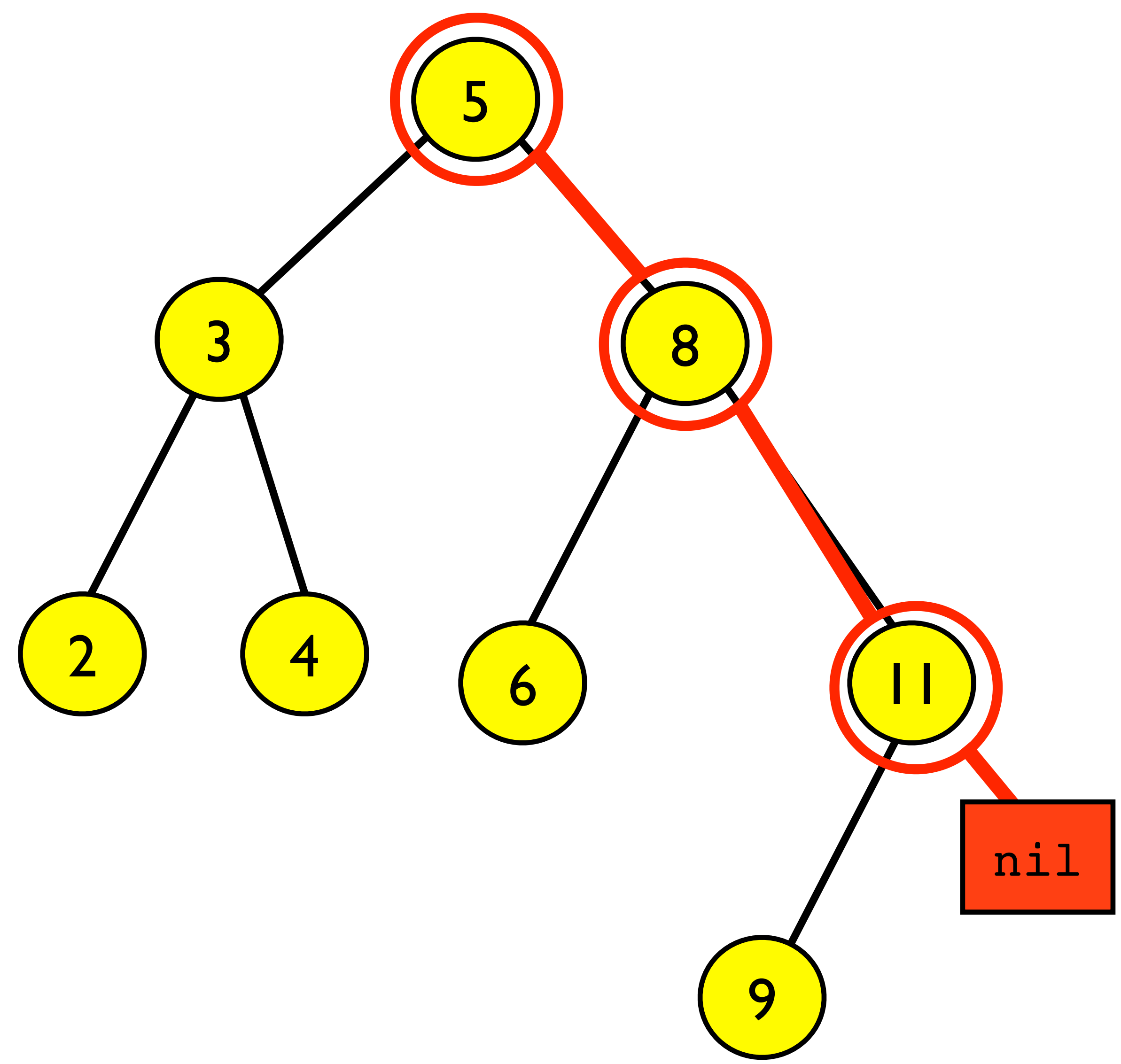
BST Insert

```
insert(T, K):
   q = NULL
   p = T
   while p != nil and p.key != K:
      q = p
      if p.key < K:
        p = p.right
      else if p.key > K:
        p = p.left

   if p != nil: error DUPLICATE

   N = new Node(K)
   if q.key > K:
     q.left = N
   else:
     q.right = N
```

Same idea as BST Find



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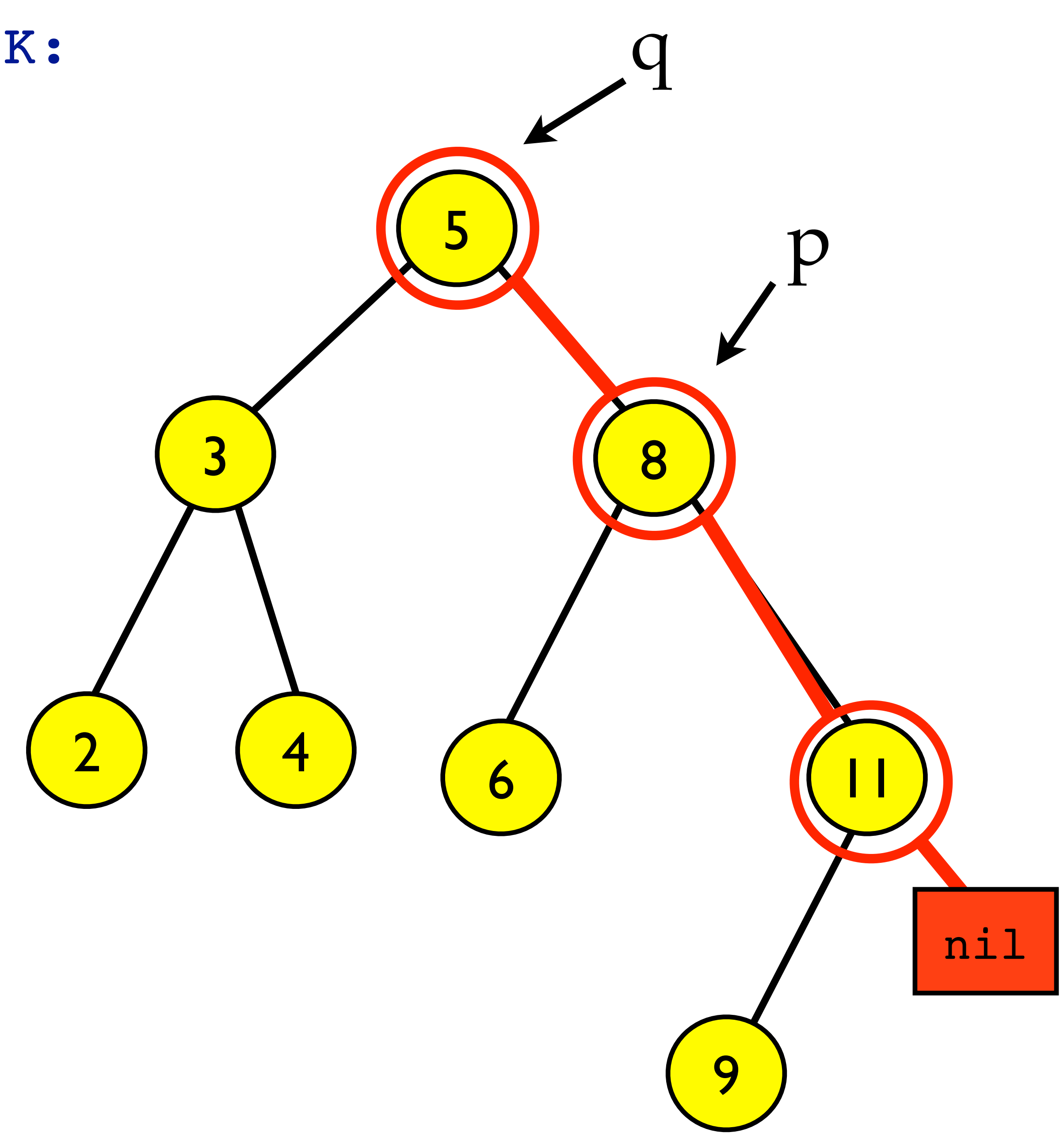

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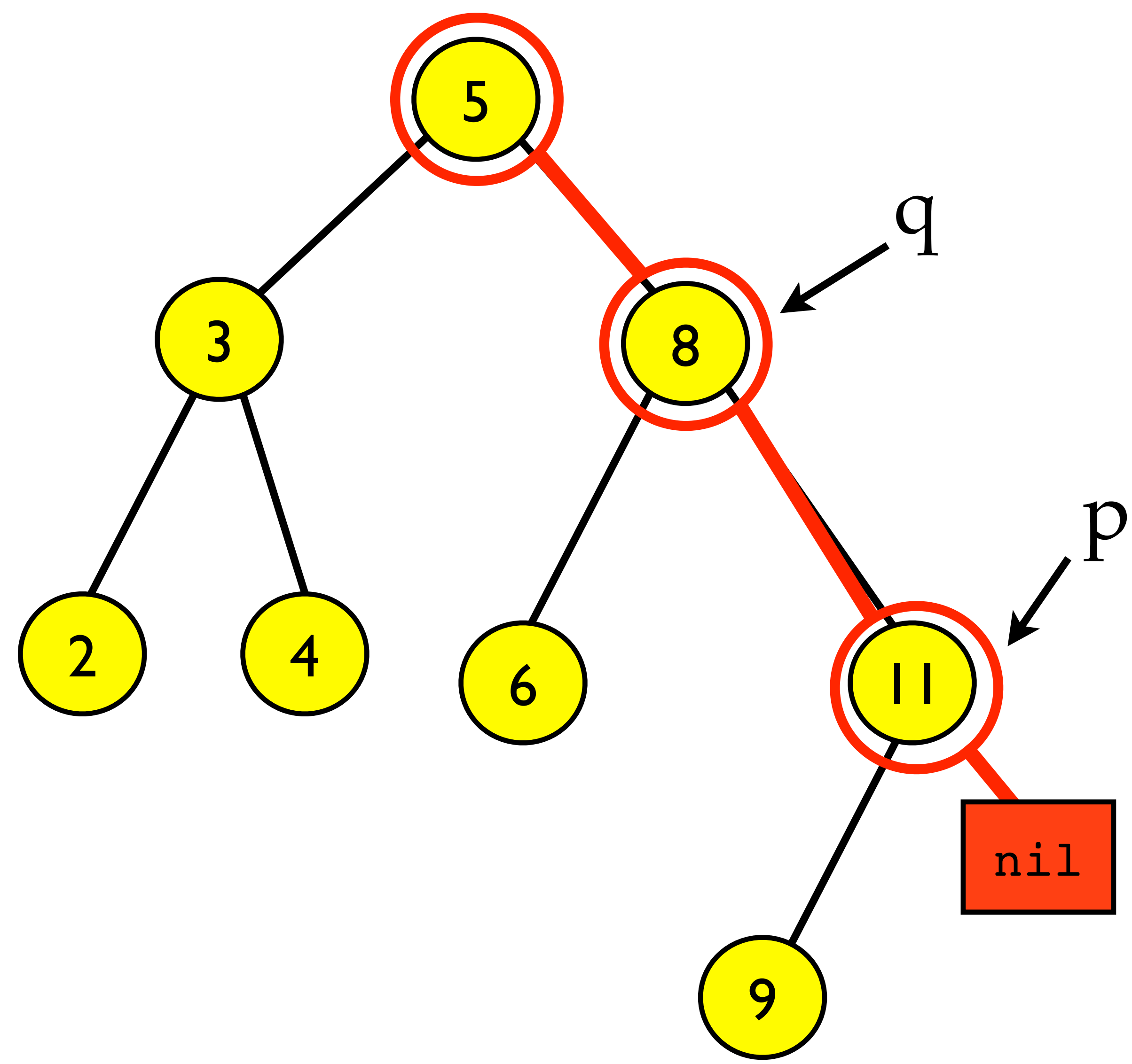

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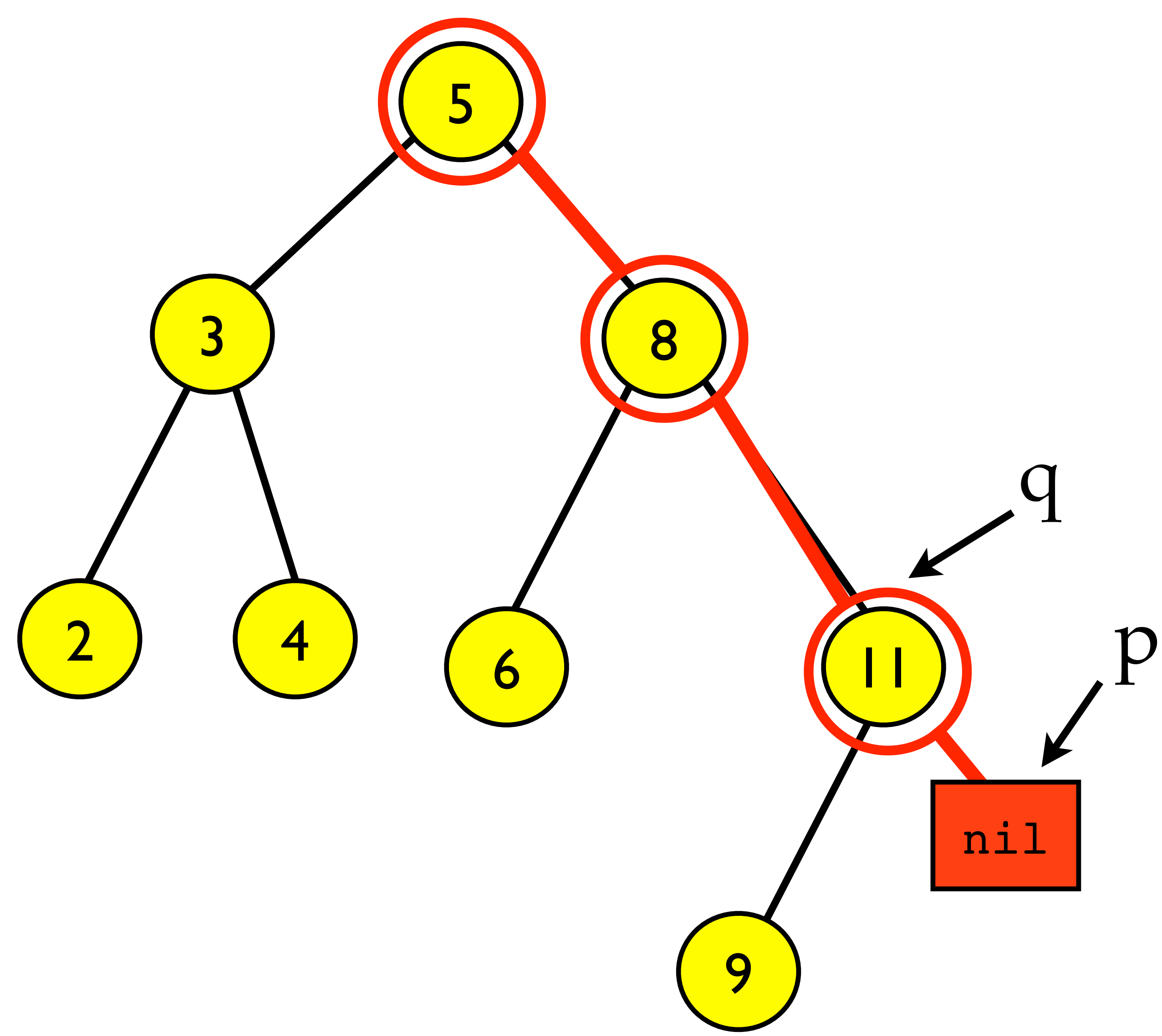
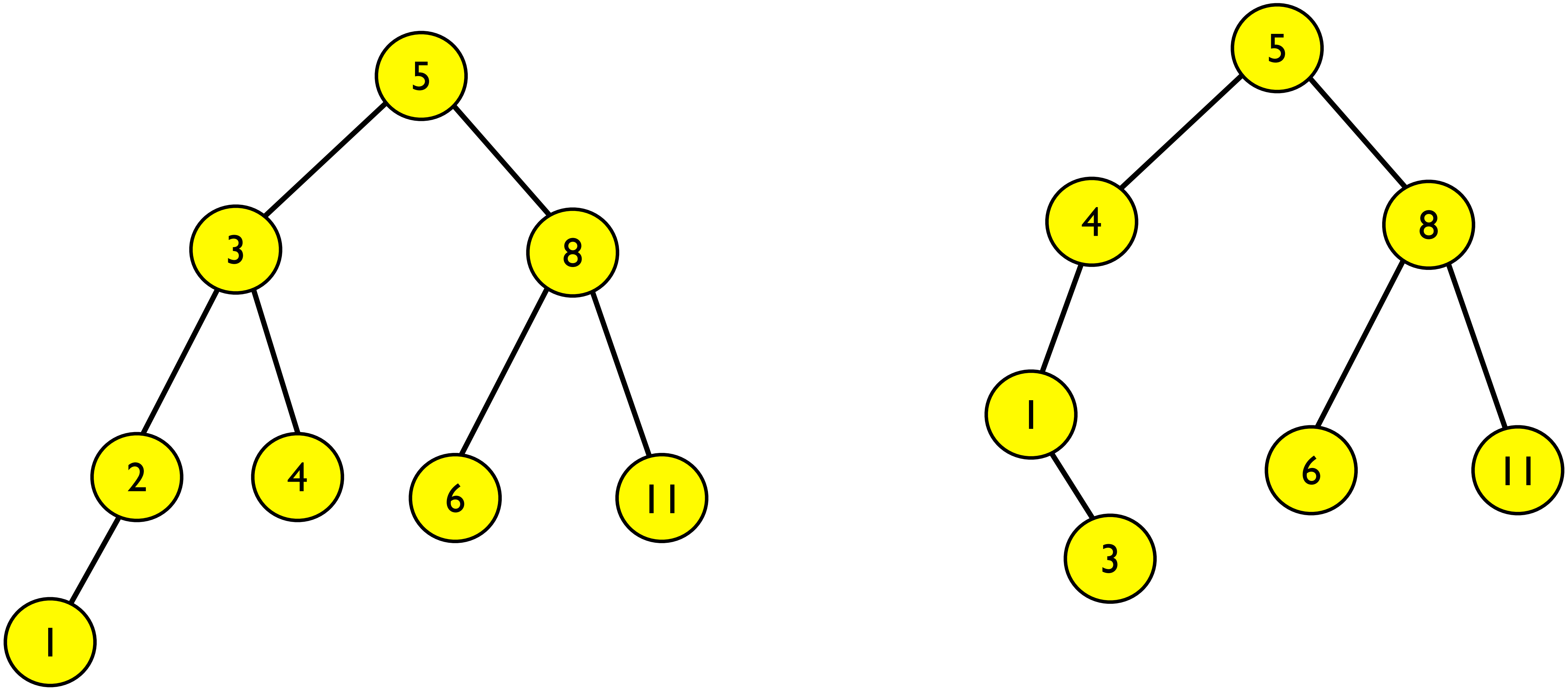


BST FindMin



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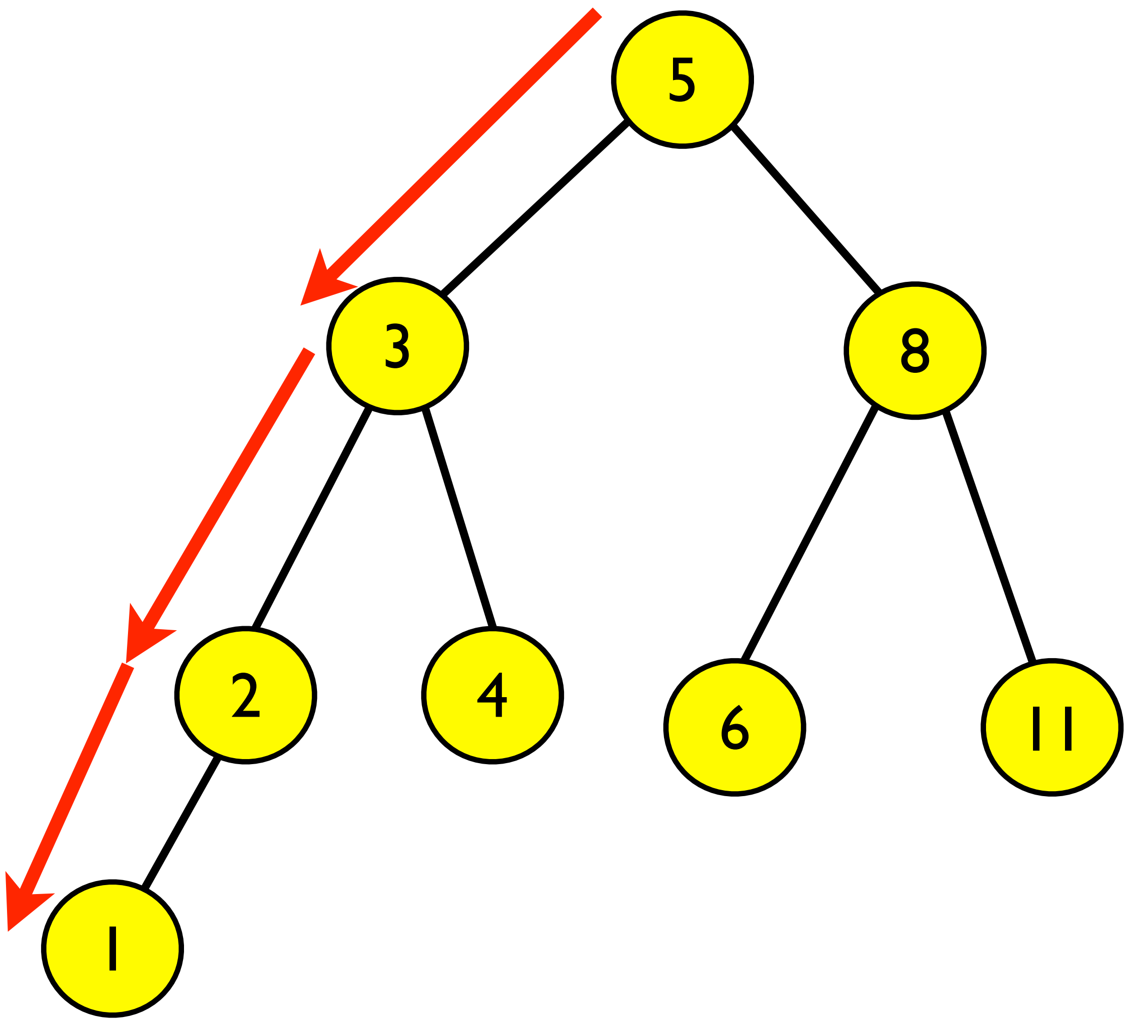
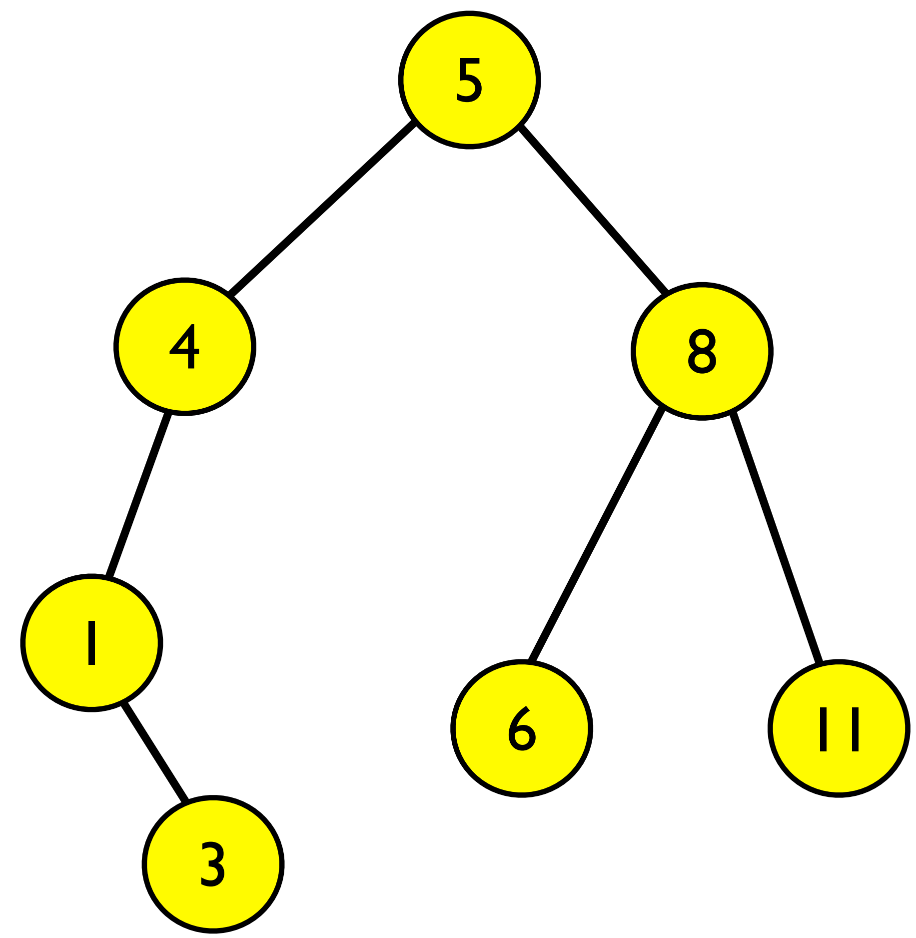
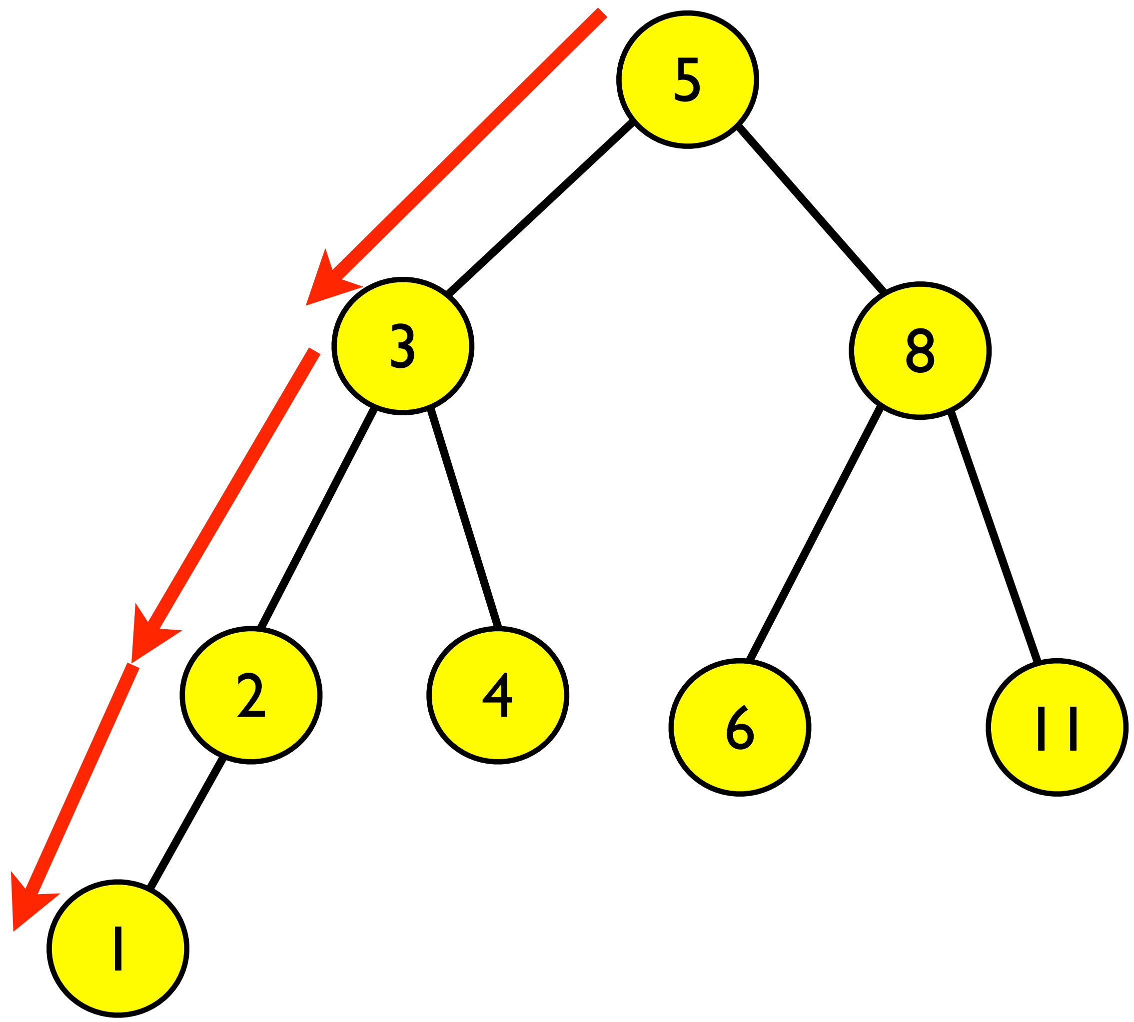
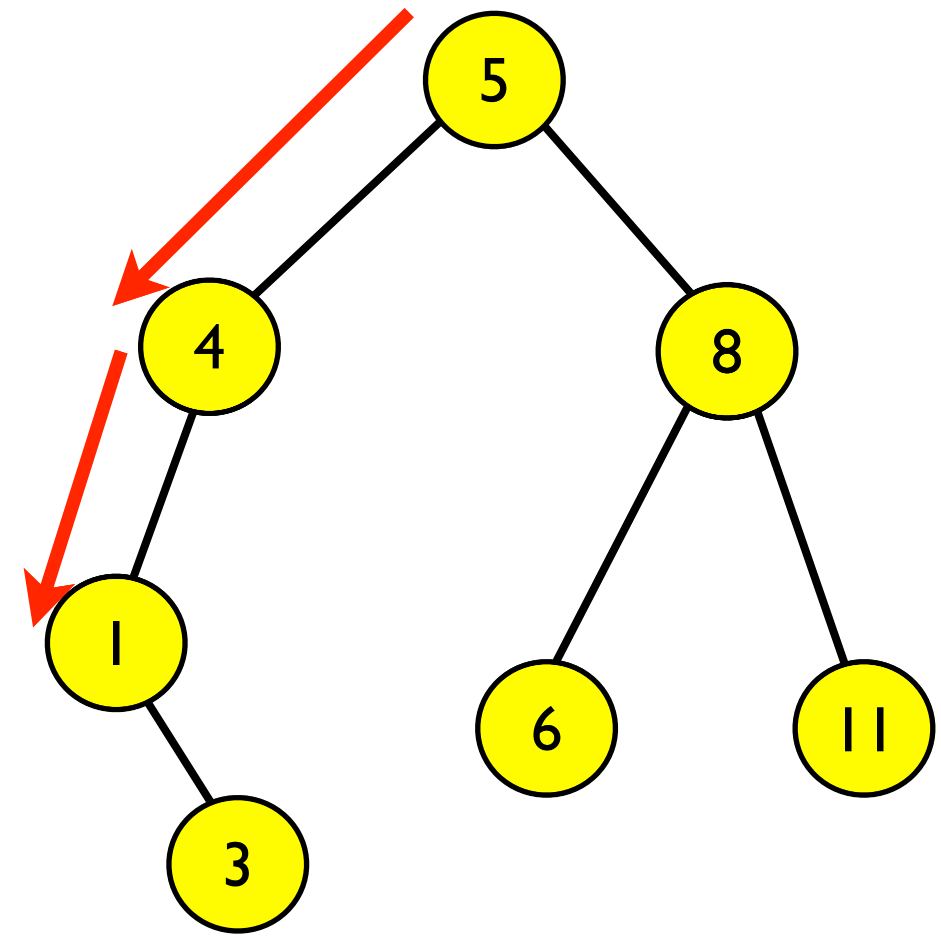




Walk left until you can't go left any more

BST FindMin

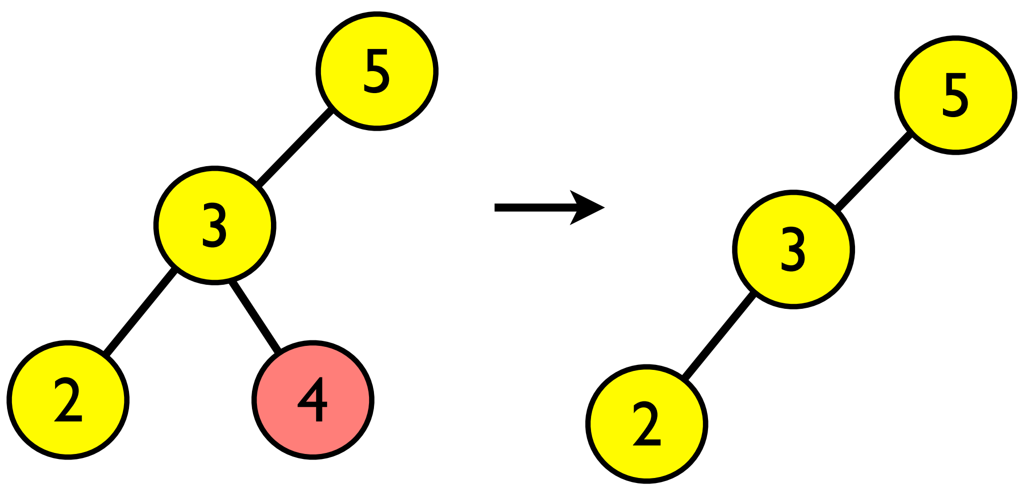




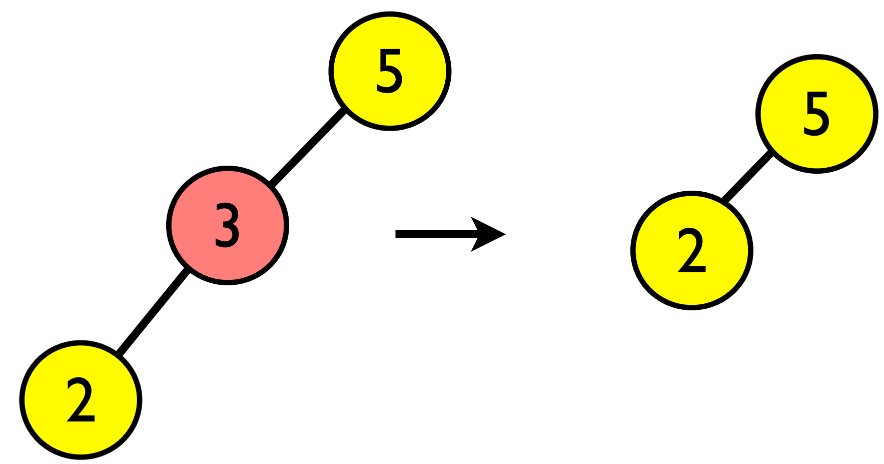
Walk left until you can't go left any more

BST Delete

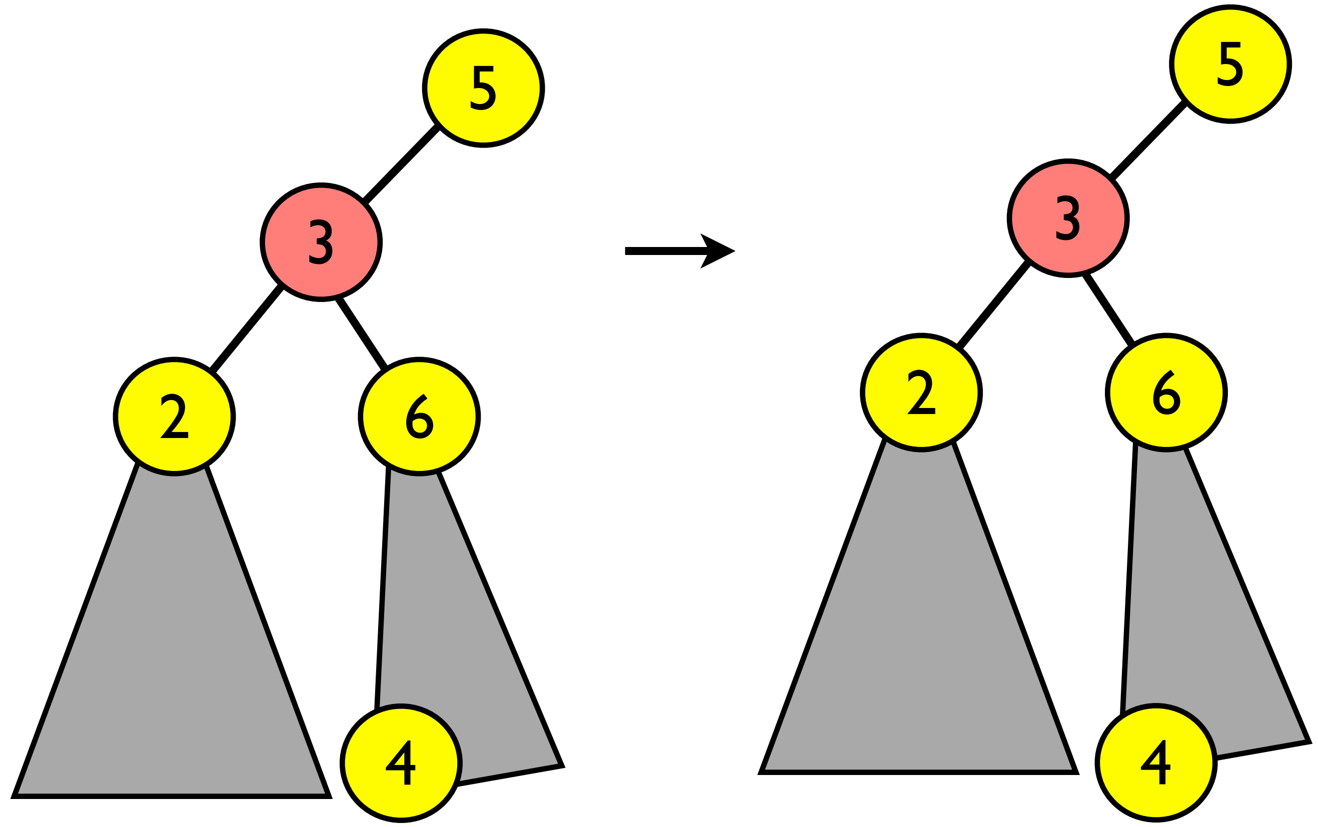
Node is a leaf:



Node has 1 child:



Node has 2 children:



BST Delete

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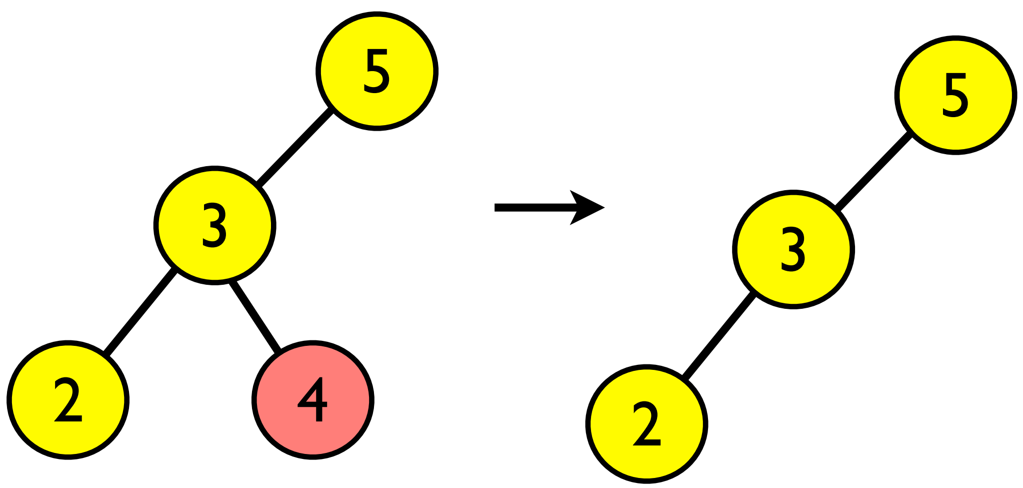

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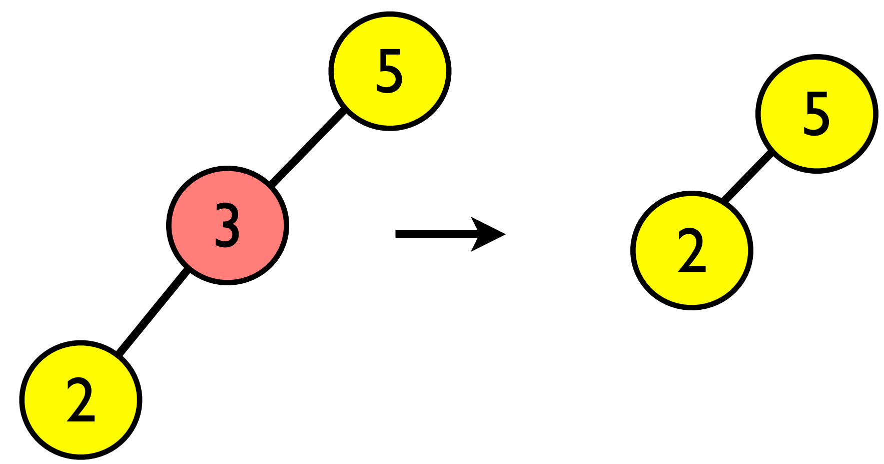
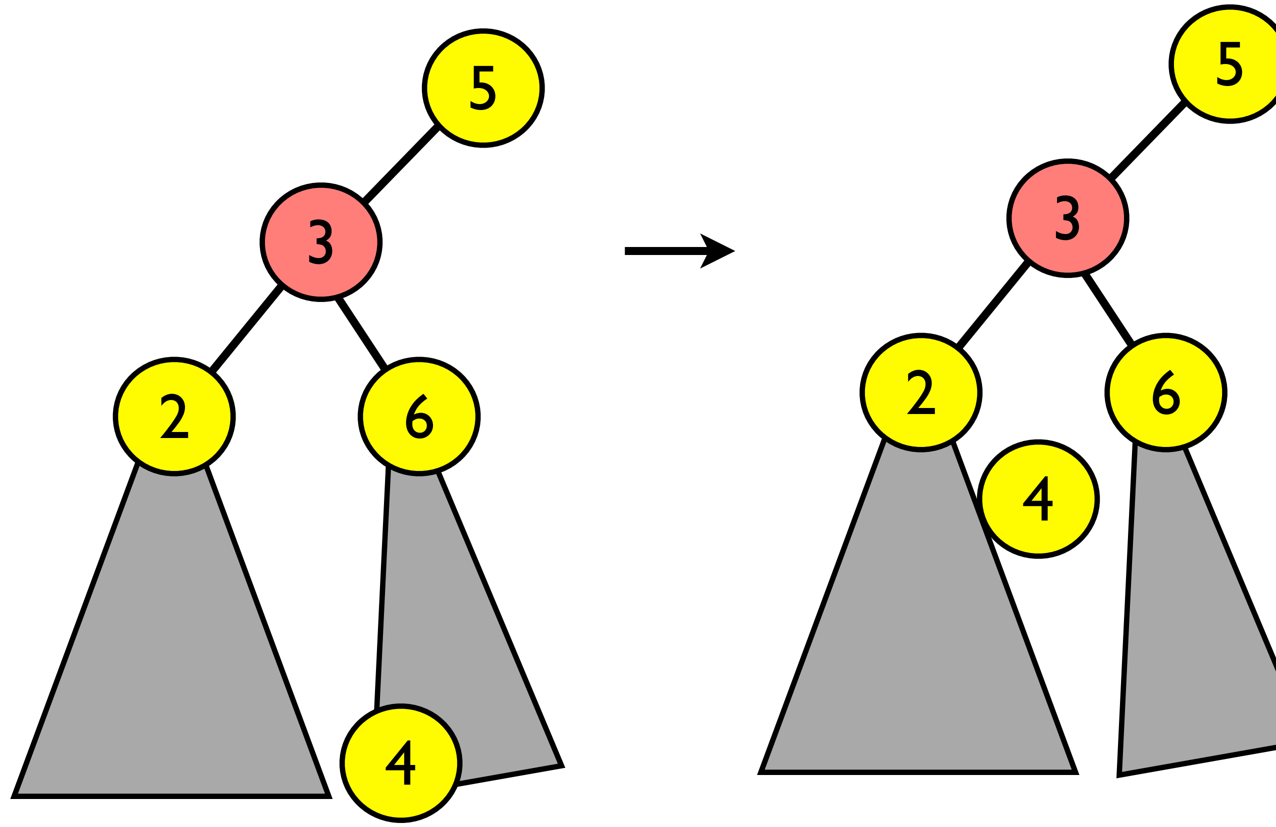
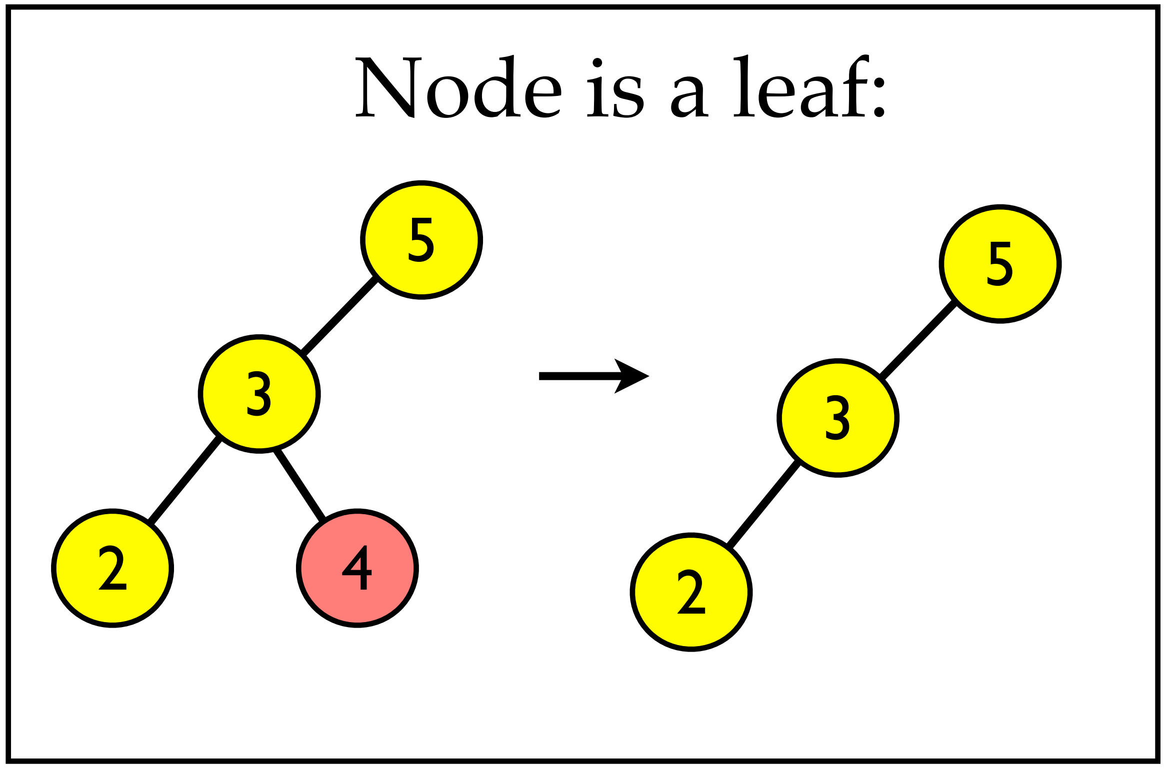
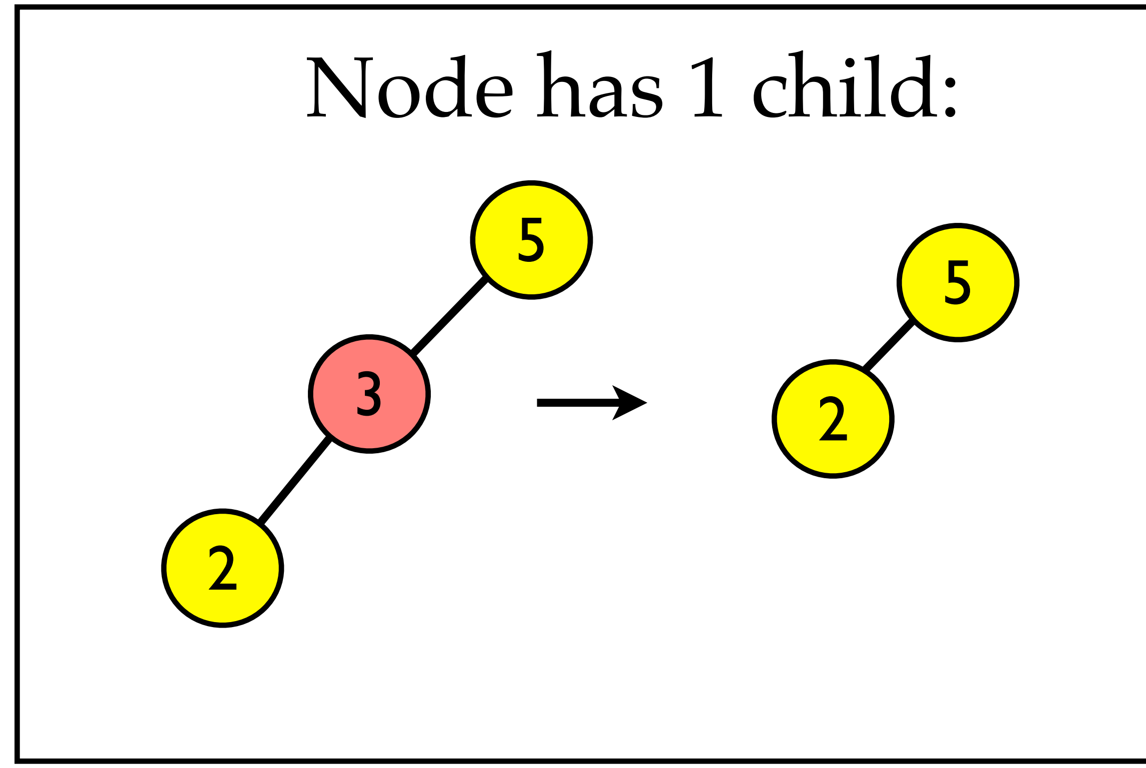


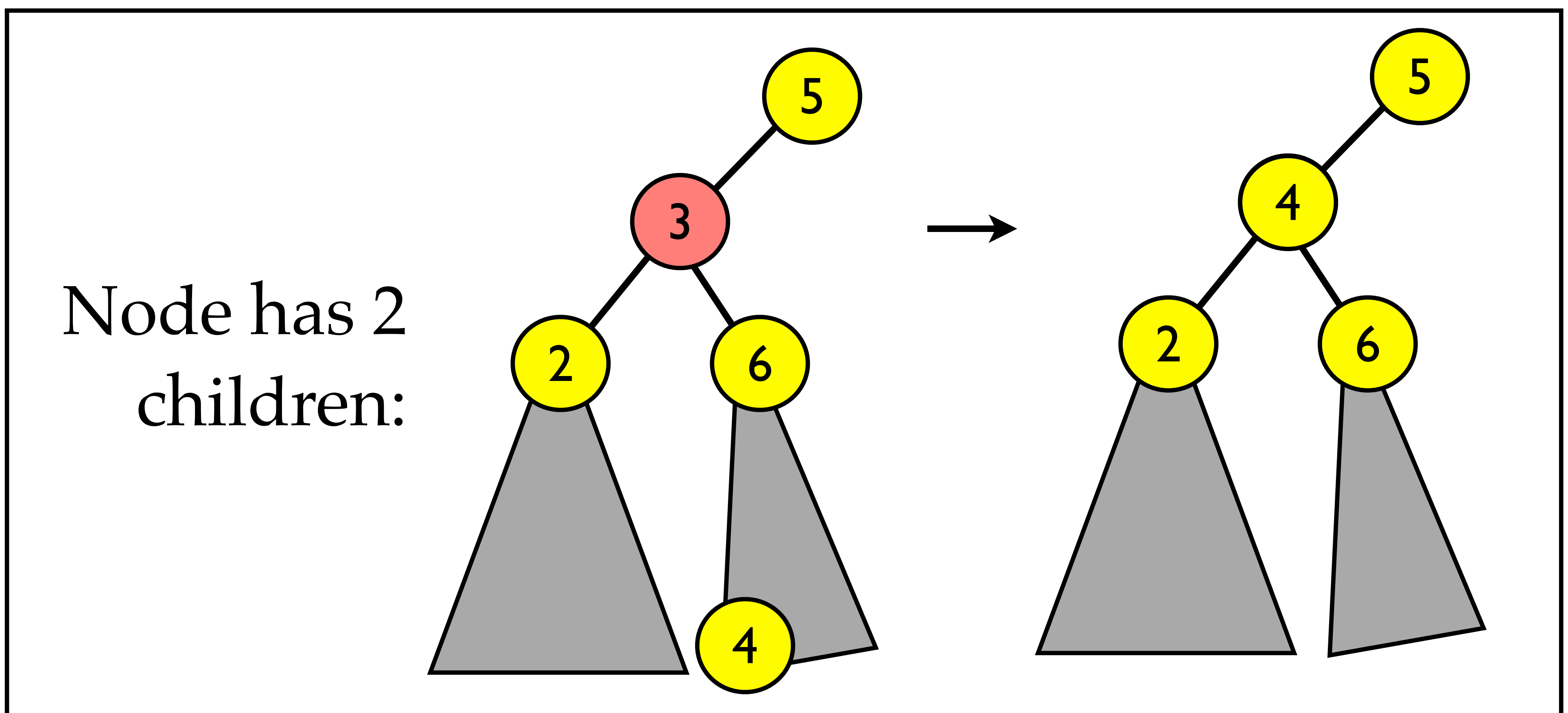
Node has 2 children:



BST Delete


Node is a leaf:
5
3
2
4
5
3
2


Node has 1 child:
5
3
2
5
2


Node has 2 children:
5
3
2
6
4
5
4
2
6

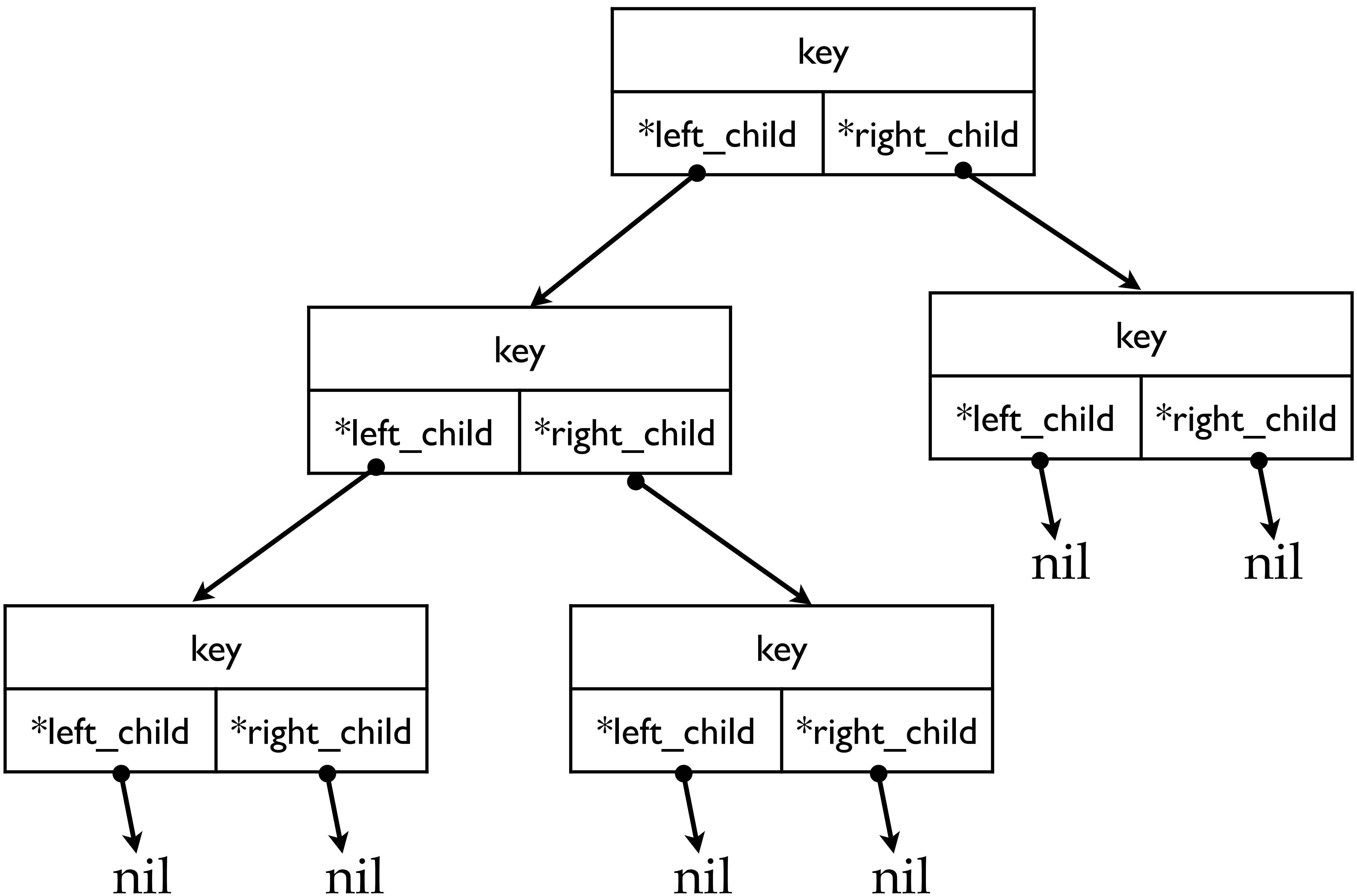
BST Operations Summary

- Find: walk left or right according to the key comparison.
- Insert: Put the new node where a Find for it would have fallen off the tree.
- Delete:
  - If deleting a leaf, just remove it.
  - If deleting a node u with 1 child, move that child up to be a child of u's parent.
  - If deleting a node u with 2 children: find the smallest key in the subtree rooted at u, delete it, and replace u with that key.

- What’s the worst possible insertion order?

- What’s the best possible insertion order?

Binary Tree Representation



BST Find Code

```
type BSTNode struct {
   key int
   left, right *BSTNode
}
```

A node contains the data (here key) plus pointers to the left and right children.

Recursive implementation:

```
func BSTFind(root *BSTNode, k int) *BSTNode {
   if root != nil {
      if k == root.key { return root }
      if k < root.key { return BSTFind(root.left, k) }
      if k > root.key { return BSTFind(root.right, k) }
   }
   return nil
}
```

How much memory is used?

BST Find: Non-recursive

We update “root” so that it points to the current node:

Also extended so that this returns both the node and it’s parent

```
func BSTFind(root *BSTNode, k int) (*BSTNode, *BSTNode) {
   var parent *BSTNode = nil
   for root != nil {
      if k == root.key { return parent, root }
      parent = root
      if k < root.key {
         root = root.left
      } else if k > root.key {
         root = root.right
      }
   }
   return parent, root
}
```

BST Insert Code

```
func BSTInsert(root *BSTNode, k int) (*BSTNode, bool) {
   newNode := CreateBSTNode(k)
   if root == nil { return newNode, true }

   parent, current := BSTFind(root, k)
   // if key is already in the tree, report error
   if current != nil { return root, false}

   if newNode.key < parent.key {
      parent.left = newNode
   } else {
      parent.right = newNode
   }

   return root, true
}
```

Decide if new node should be a left or right child of where we fell off the tree

BST FindMin Code

```
func BSTFindMin(root *BSTNode) *BSTNode {
    if root == nil { return nil }
    for root.left != nil {
        root = root.left
    }
    return root
}
```

Look ahead: will going left make us fall off the tree?

BST Delete Code

```
func BSTDelete(root *BSTNode, k int) (*BSTNode, bool) {
   if root == nil { return nil, false }
   parent, current := BSTFind(root, k)
   if current == nil { return root, false } // didn't find

   var pPointer **BSTNode    // !!!
   if parent != nil {
      if current.key < parent.key {
         pPointer = &parent.left
      } else {
         pPointer = &parent.right
      }
   }

   switch {
   case current.left != nil && current.right != nil:
      min := BSTFindMin(current)
      BSTDelete(current, min.key)
      current.key = min.key
   case current.left == nil && current.right == nil:
      *pPointer = nil
   case current.left != nil:
      *pPointer = current.left
   case current.right != nil:
      *pPointer = current.right
   }

   return root, true
}
```

Find the node to delete and its parent

Coding jujutsu: pPointer is a pointer to the pointer in the parent that we have to change during the delete

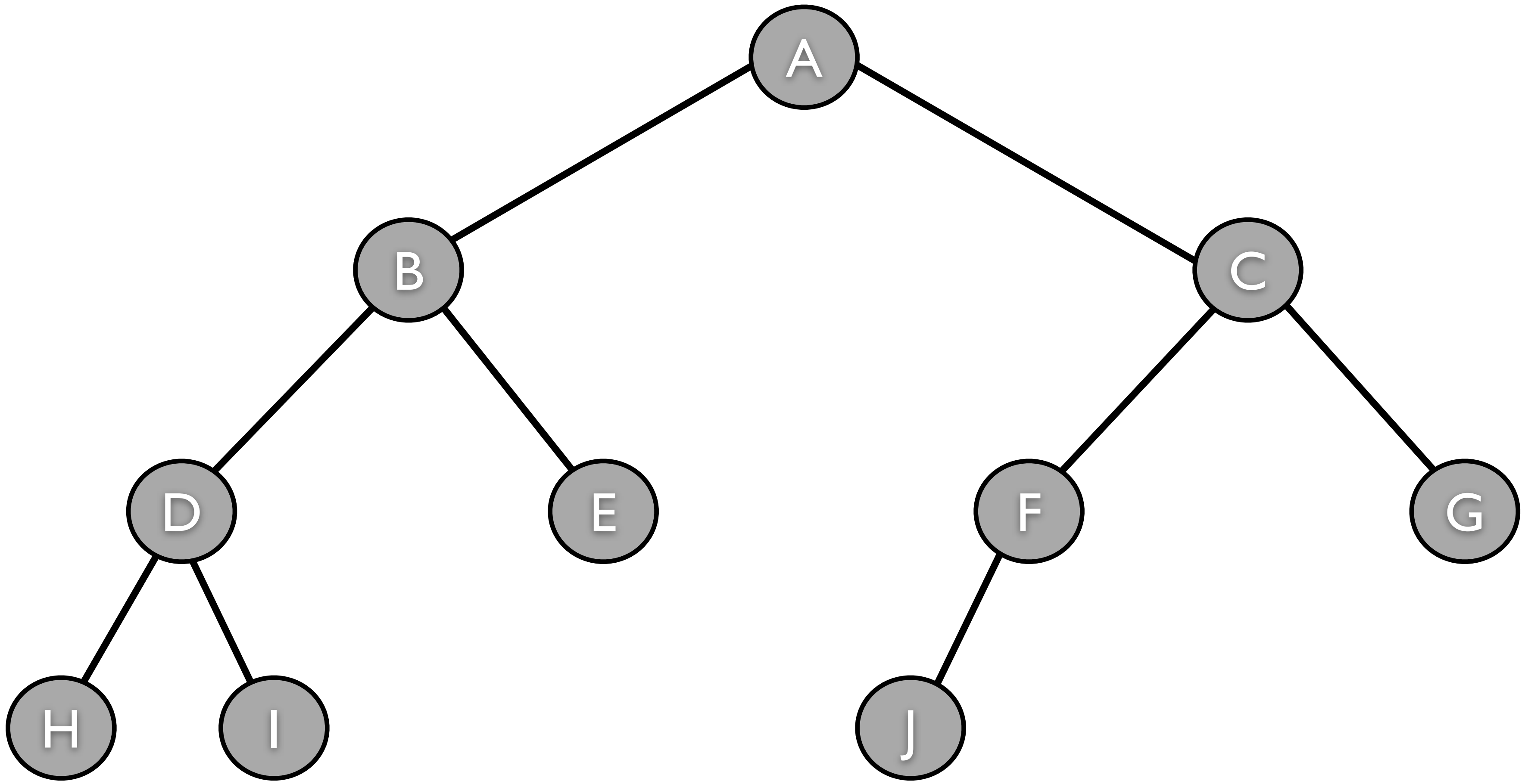
The delete cases depend on which children exist in the node we are deleting

Summary

- Binary search trees are a fundamental data structure supporting the “dictionary” (aka map, associative array) operations.
- The requirement that the keys be unique is not crucial: it just adds a few more special cases to the code.
- The running time of all the operations is proportional to the height of the tree.
- Standard BSTs don’t do anything to keep the height small.

More about trees

Binary Tree Traversals



inorder: HDIBEAJFCG
preorder: ABEHIECFJG
postorder: HIDEBJFGCA

```
func traverse(T *Node) {
  if(T != nil) {
      PREORDER(T);
      traverse(T.left);
      INORDER(T);
      traverse(T.right);
      POSTORDER(T);
   }
}
```

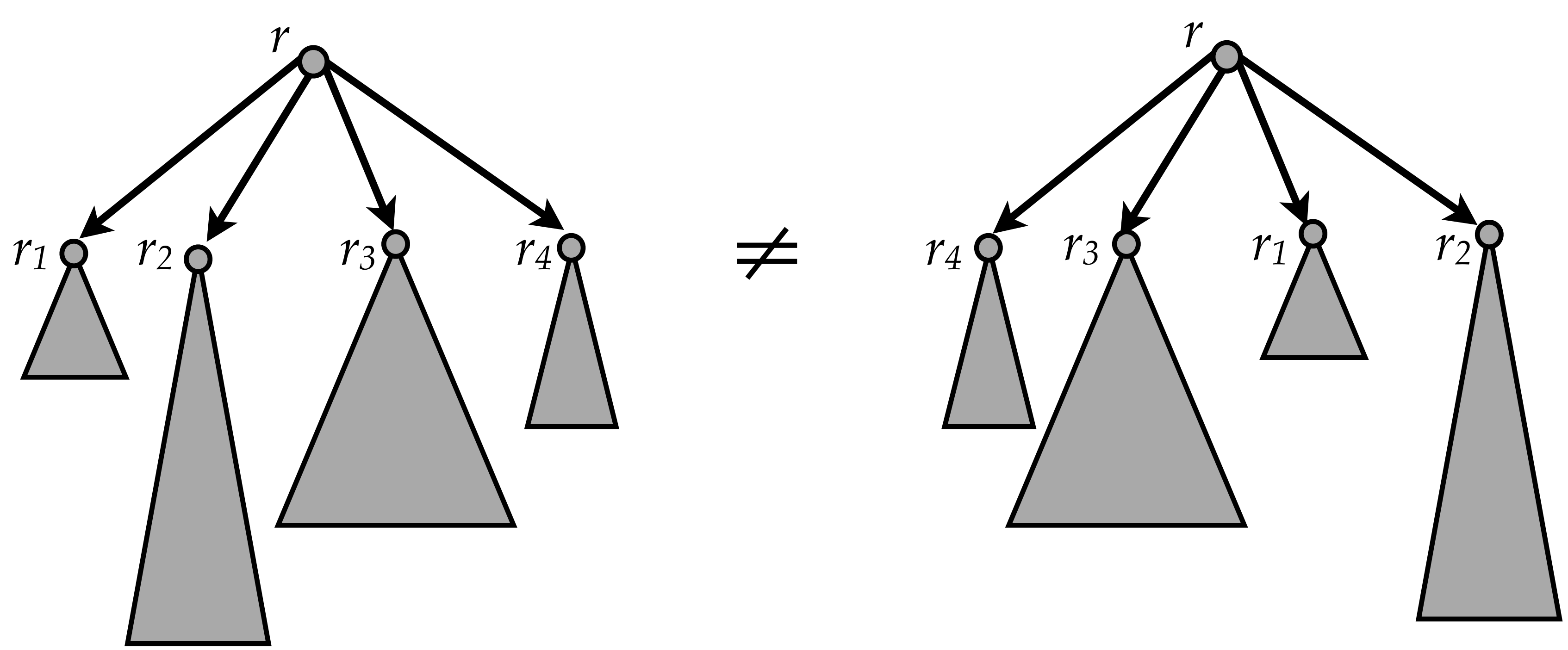
How much space is used?

Basic Properties

- Every node except the root has exactly one parent.
- A tree with n nodes has n-1 edges
(every node except the root has an edge to its parent).
- There is exactly one path from the root to each node.
(Suppose there were 2 paths, then some node along the 2 paths would have 2 parents.)

Binary Trees – Definition

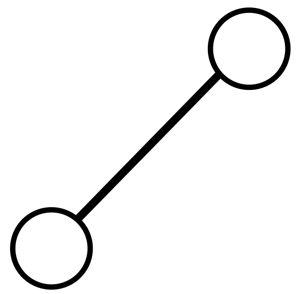
- An *ordered* tree is a tree for which the order of the children of each node is considered important.



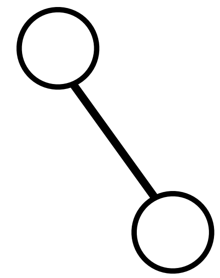
- A *binary tree* is an ordered tree such that each node has ≤ 2 children.
- Call these two children the *left* and *right* children.

Example Binary Trees

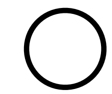
The edge cases:



Only left child



Only right child

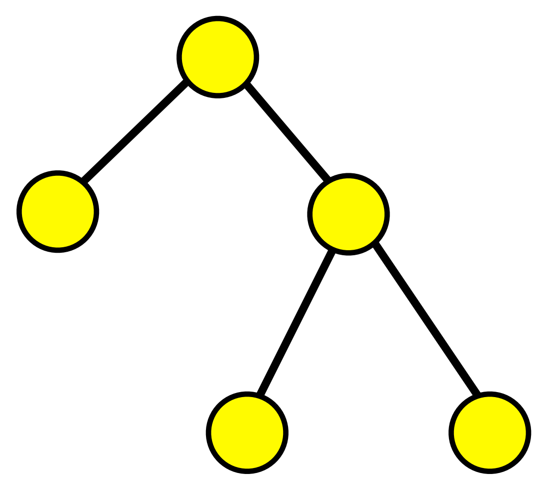


Single node

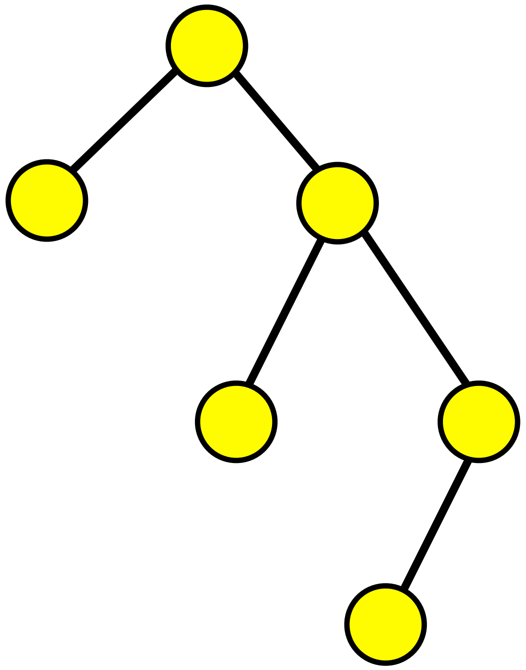
Λ

Empty Binary Tree

Small binary tree:

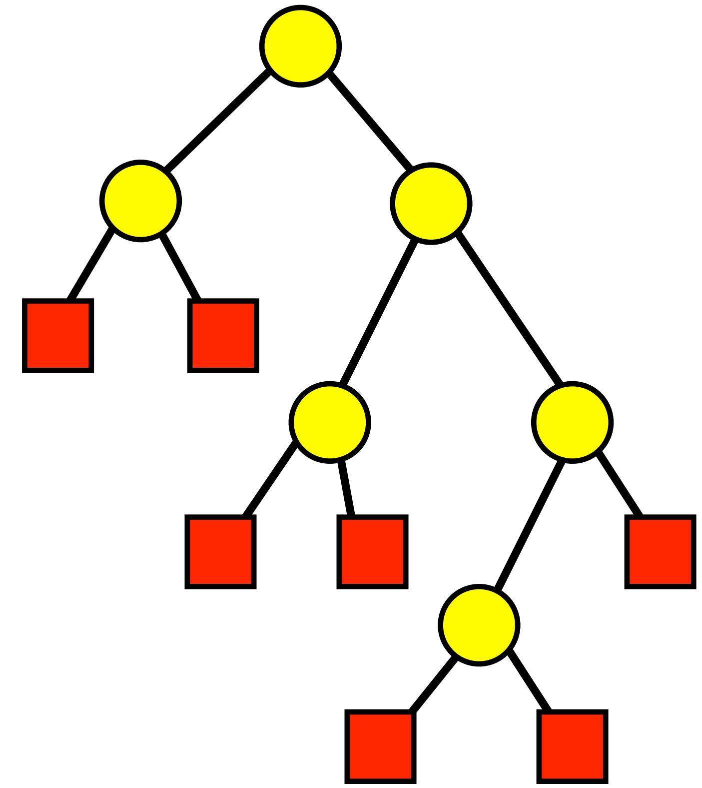


Extended Binary Trees



Replace each missing child with *external node*

Do you need a special flag to tell which nodes are external?



Binary tree

Extended binary tree

Every internal node has exactly 2 children.

Every leaf (external node) has exactly 0 children.

Each external node corresponds to one Λ in the original tree – let's us distinguish different instances of Λ.

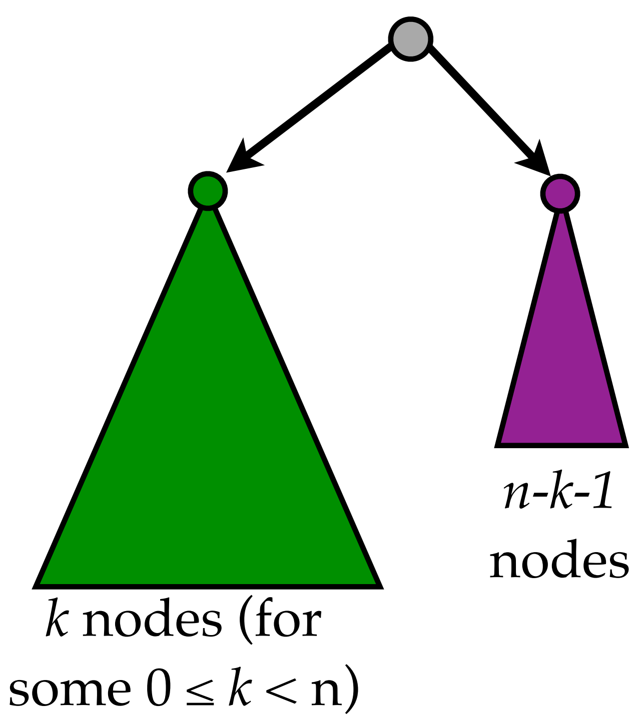
of External Nodes in Extended Binary Trees

Thm. *An extended binary tree with* n *internal nodes has* n+1 *external nodes.*

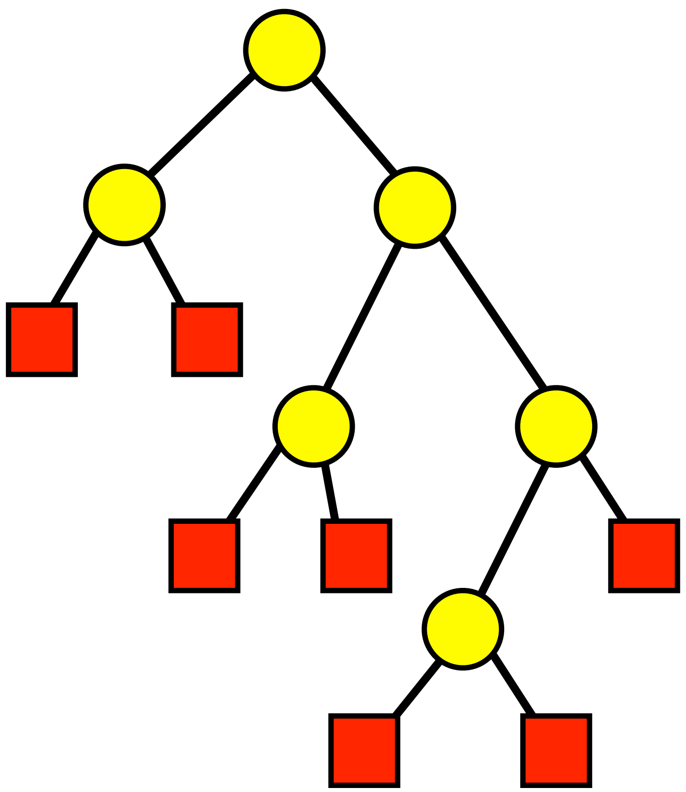
Proof. By induction on n.
X(n) := number of external nodes in binary tree with n internal nodes.

Base case: $X(0) = 1 = n + 1$.

Induction step: Suppose theorem is true for all $i < n$.
Because $n \geq 1$, we have:



$$\begin{aligned} X(n) &= X(k) + X(n-k-1) \\ &= k+1 + n-k-1 + 1 \\ &= n + 1 \quad \square \end{aligned}$$



Extended binary tree

Alternative Proof

Thm. *An extended binary tree with* n *internal nodes has* n+1 *external nodes.*

Proof. Every node has 2 children pointers, for a total of $2n$ pointers.

Every node except the root has a parent, for a total of n - 1 nodes with parents.

These n - 1 parented nodes are all children, and each takes up 1 child pointer.

(pointers) - (used child pointers) = (unused child pointers)

$$2n - (n-1) = n + 1$$

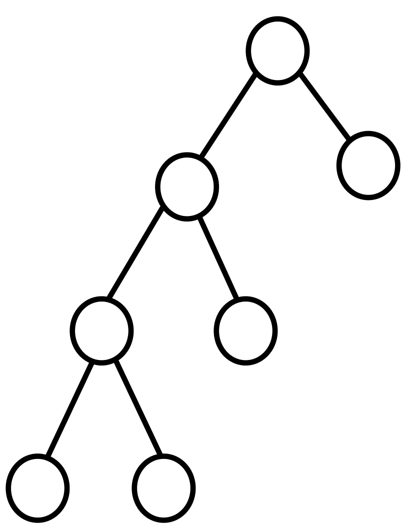
Thus, there are $n + 1$ null pointers.

Every null pointer corresponds to one external node by construction. □

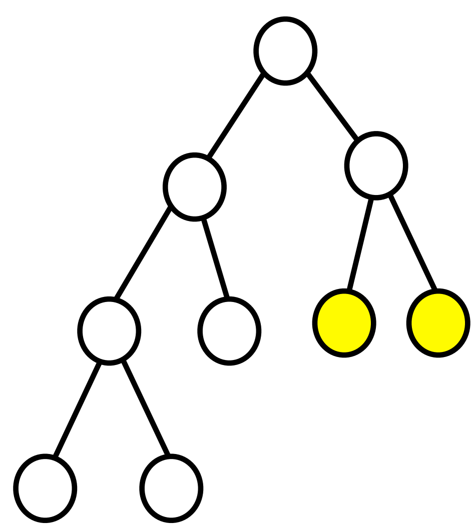
Full and Complete Binary Trees

Unfortunately, different authors use different tree terminology

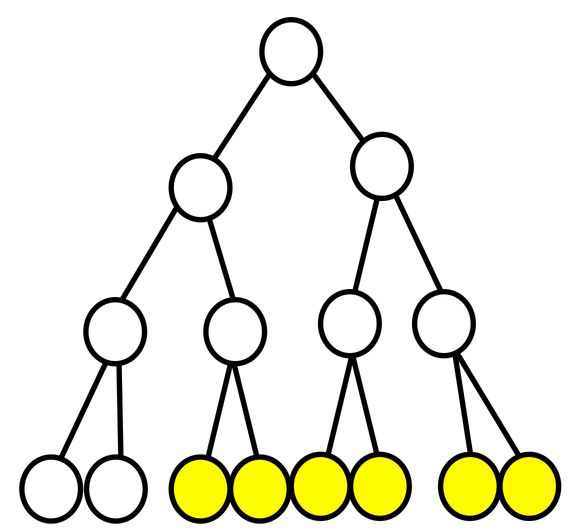
- If every node has either 0 or 2 children, a binary tree is called *full*.
- If the lowest *d-1* levels of a binary tree of height *d* are filled and level *d* is partially filled from left to right, the tree is called *complete*.
- If all *d* levels of a height-*d* binary tree are filled, the tree is called *perfect*.



full



complete



perfect

Nodes in a Perfect Tree of Height h

Thm. *A perfect tree of height h has* $2^{h+1} - 1$ *nodes.*

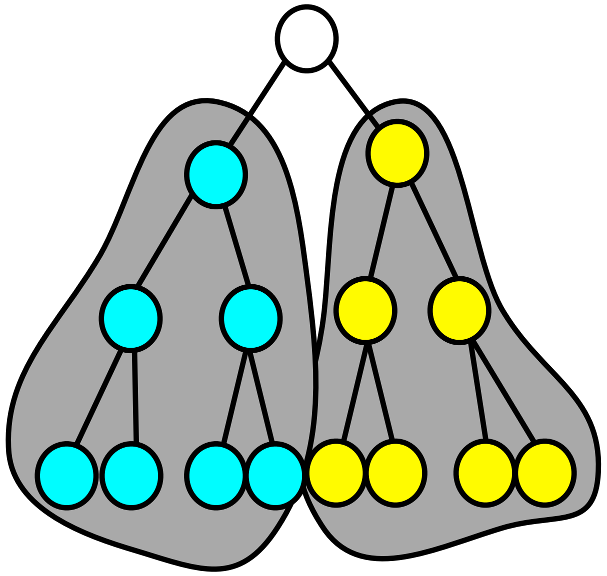
Proof. By induction on h.

Let $N(h)$ be number of nodes in a perfect tree of height h.

Base case: when $h = 0$, tree is a single node. $N(0) = 1 = 2^{0+1} - 1$.

Induction step: Assume $N(i) = 2^{i+1} - 1$ for $0 \le i < h$.

A perfect binary tree of height h consists of 2 perfect binary trees of height h-1 plus the root:



$$
\begin{aligned}
N(h) &= 2 \times N(h - 1) + 1 \\
&= 2 \times (2^{h-1+1} - 1) + 1 \\
&= 2 \times 2^{h} - 2 + 1 \\
&= 2^{h+1} - 1 \quad \square
\end{aligned}
$$

2^h are leaves

2^h - 1 are internal nodes

Full Binary Tree Theorem

Thm. *In a non-empty, full binary tree, the number of internal nodes is always 1 less than the number of leaves.*

Proof. By induction on n.
$L(n)$:= number of leaves in a non-empty, full tree of n internal nodes.

<u>Base case:</u> $L(0) = 1 = n + 1$.

<u>Induction step:</u> Assume $L(i) = i + 1$ for $i < n$.

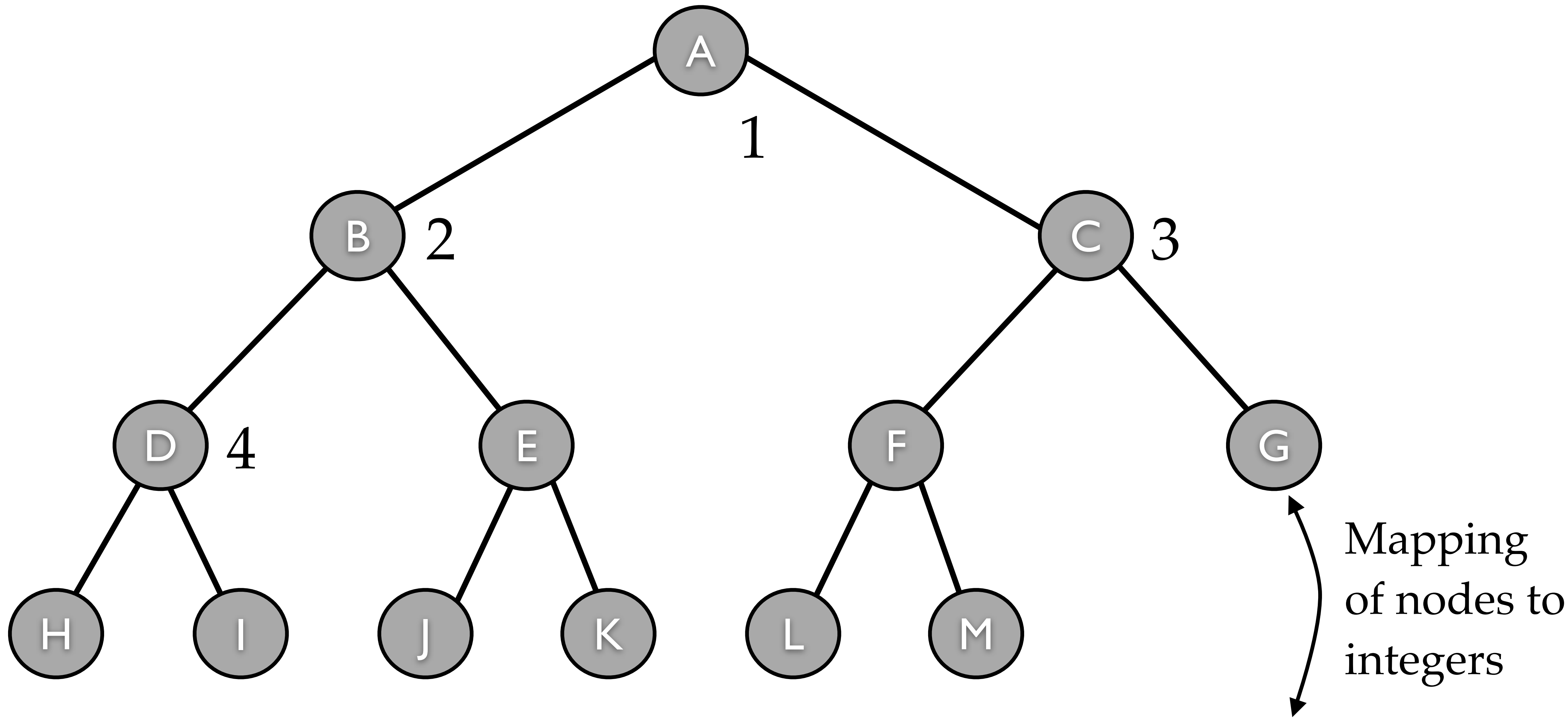
Given T with n internal nodes, remove two sibling leaves.

T′ has n-1 internal nodes, and by induction hypothesis, $L(n\text{-}1) = n$ leaves.

Replace removed leaves to return to tree T.
Turns a leaf into an internal node, adds two new leaves.

Thus: $L(n) = n + 2 - 1 = n + 1$.

Array Implementation for Complete Binary Trees



A	B	C	D	E	F	G	H	I	J	K	L	M		
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

left(i): $2i$ **if** $2i \leq n$ **otherwise** 0

right(i): $(2i + 1)$ **if** $2i + 1 \leq n$ **otherwise** 0

parent(i): $\lfloor i/2 \rfloor$ **if** i ≥ 2 **otherwise** 0

Summary

- Trees are an incredibly common way to organize data:
  - folders on your hard drives
  - URLs: http://www.cs.cmu.edu/~ckingsf/software/sailfish
  - BST, Splay trees, AVL trees, B-trees, Quad-trees, kd-trees, red-black trees, M-trees, … probably thousands of variants that are good for different data and different queries.
- Binary trees in particular are nice because each node partitions the data into 2 subsets and because there are nice relationships between # of nodes and # of leaves, etc.
- Typically, trees are represented using nodes & pointers, though this does not have to be the case.