Constructive Logic (15-317), Spring 2021
Assignment 11: Linear and Modal Logic

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Due: Wednesday, December 1, 11:59 pm

The assignments in this course must be submitted electronically through Gradescope. For this homework, you will be submitting both written pdf files and Dcheck coding files:

- `hw.deriv` (your coding solutions)
- `hw.pdf` (your written solutions)

The coding portion will use the experimental Dcheck derivation checker. You can find documentation and examples on the Software page at the course web site (`cs.cmu.edu/~crary/317-f21/software.html`). That document has been updated with information on preparing derivations using linear and modal logic.

In order for us to grade this assignment in time for you to prepare for the final exam, it must be handed in by December 1. **No late days can be used on this assignment.**

1 Practicing Linear Logic

Task 1 (16 pts)**.** Provide derivations for the following Linear Logic judgements using Dcheck syntax. Derivation names are given in the starter code.

1. $\Vdash (A \multimap B \multimap C) \multimap (A \otimes B \multimap C)$ *true*
2. $\Vdash (((A \otimes \top) \mathbin{\&} (B \otimes \top)) \multimap C) \multimap (A \multimap B \multimap C)$ *true*
3. $\Vdash ((A \multimap C) \oplus (B \multimap C)) \multimap (A \mathbin{\&} B) \multimap C$ *true*
4. $\Vdash ((A \multimap 0) \oplus (B \multimap 0)) \multimap (A \otimes B) \multimap 0$ *true*

2 Linear Harmony

Just like we did in the beginning of the course, we can check a local correctness condition for the rules of linear natural deduction: proof-theoretic harmony.[1] Hint: exhibiting local reductions and expansions in linear logic is subtle: you must be sure to not constrain the contexts Δ in your local reductions and expansions any more than is warranted by the rules of linear logic.

Remark 1 (Linear substitution principle)**.** When exhibiting local reductions and expansions, you will need to use substitutions $[\mathcal{D}/u]\mathcal{E}$. These are governed by the *linear substitution principle*, which states:

$$\text{If } \Gamma \Vdash \overset{\mathcal{D}}{A}\ \textit{true} \text{ and } \Gamma', u : A\ \textit{true} \Vdash \overset{\mathcal{E}}{B}\ \textit{true}, \text{ then } \Gamma, \Gamma' \Vdash \overset{[\mathcal{D}/u]\mathcal{E}}{B}\ \textit{true}.$$

You *must* ensure that your resulting derivations have correct contexts.

Task 2 (10 pts)**.** Verify that tensor $\otimes$ satisfies local soundness and completeness.

Task 3 (10 pts)**.** Verify that that $\mathbf{1}$ satisfies local soundness and completeness.

Task 4 (10 pts)**.** Verify that $\top$ satisfies local soundness and completeness.

The relevant rules are:

$$\frac{\Delta \Vdash A\ \textit{true} \quad \Delta' \Vdash B\ \textit{true}}{\Delta, \Delta' \Vdash A \otimes B\ \textit{true}}\ \otimes\mathsf{I} \qquad \frac{\Delta \Vdash A \otimes B\ \textit{true} \quad \Delta', u : A\ \textit{true}, v : B\ \textit{true} \Vdash C\ \textit{true}}{\Delta, \Delta' \Vdash C\ \textit{true}}\ \otimes\mathsf{E}^{u,v}$$

$$\frac{}{\cdot \Vdash \mathbf{1}\ \textit{true}}\ \mathbf{1}\mathsf{I} \qquad \frac{\Delta \Vdash \mathbf{1}\ \textit{true} \quad \Delta' \Vdash C\ \textit{true}}{\Delta, \Delta' \Vdash C\ \textit{true}}\ \mathbf{1}\mathsf{E} \qquad \frac{}{\Delta \Vdash \top\ \textit{true}}\ \top\mathsf{I}$$

[1] Harmony is a necessary condition for the correctness of rules, but not a sufficient condition.

3 Applications

In class, we looked at *Blocks World*, an example of encoding *state* in linear logic. Blocks World is a class of scenarios in which there is a table, some number of blocks which can be stacked on top of each other, and a robotic arm which can pick up and move blocks. The following atomic predicates are used:

1. empty means that the robotic arm's hand is empty.
2. $\mathsf{holds}(()x)$ means that the robotic arm's hand is holding x.
3. $\mathsf{clear}(x)$ means that the block x does not have anything on top of it.
4. $\mathsf{on}(x, y)$ means that the block x is directly on top of the block y.
5. $\mathsf{on\_table}(x)$ means that the block x is sitting directly on the table.

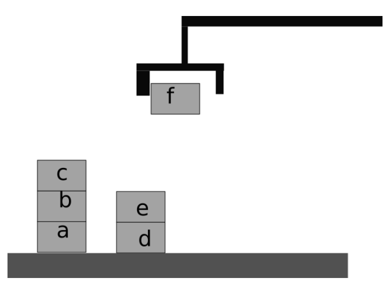
The possible state transitions for Blocks World are given by the following axioms:

1. $\forall x, y.\ \mathsf{empty} \otimes \mathsf{clear}(x) \otimes \mathsf{on}(x, y) \multimap \mathsf{holds}(x) \otimes \mathsf{clear}(y)$
2. $\forall x.\mathsf{empty} \otimes \mathsf{clear}(x) \otimes \mathsf{on\_table}(x) \multimap \mathsf{holds}(x)$
3. $\forall x, y.\ \mathsf{holds}(x) \otimes \mathsf{clear}(y) \multimap \mathsf{empty} \otimes \mathsf{on}(x, y) \otimes \mathsf{clear}(x)$
4. $\forall x.\ \mathsf{holds}(x) \multimap \mathsf{empty} \otimes \mathsf{on\_table}(x) \otimes \mathsf{clear}(x)$

Task 5 (10 pts)**.** So far we have assumed that the table is infinitely broad and can therefore accomodate any number of blocks. **Now consider the case that the table in fact only has a finite number of spaces for blocks on it, and modify the axioms of Blocks World above in order to preserve this invariant.**

You should use an atomic predicate space, which represents an open space on the table. A correct solution to this task will not depend on the size of the table.

Task 6 (10 pts)**.** Consider the following Blocks World scenario:



Write a formula in linear logic which expresses this configuration, assuming that **four** blocks can fit directly on the table.

4 Fun with Modal Logic

Task 7 (20 pts)**.** Provide derivations for the following Modal Logic judgements using Dcheck syntax. Derivation names are given in the starter code.

1. $\Box T$ *true*
2. $\Box A \supset \Diamond\neg A \supset \Diamond F$ *true*
3. $(\Box A \wedge \Box B) \supset \Box(A \wedge B)$ *true*
4. $\Diamond(\Box A \wedge \Diamond(A \supset B)) \supset \Diamond B$ *true*
5. $\Diamond\Box\Diamond A \supset \Diamond A$ *true*