Constructive Logic (15-317), Fall2021
Assignment 2: Harmony

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Due: Wednesday, September 15, 2021, 11:59 pm

The assignments in this course must be submitted electronically through Gradescope. Written homework PDFs and coding SML files will both go to Gradescope. For this homework, you will only be submitting SML files:

- `hw2.sml` (your coding solutions)

More Proofs? Deduce that!

Task 1 (12 points)**.** Prove the following theorems using natural deduction logic in SML. Remember that *not A* is syntatic sugar for *A implies False*. You can look at `support/nd_examples.sml` for reference natural deduction proof trees.

a. `prove absurdity:` $A \wedge \neg A \supset B$

b. `prove sCombinator:` $(A \supset B) \supset (A \supset B \supset C) \supset (A \supset C)$

c. `prove deMorgin:` $\neg(A \vee B) \supset \neg A \wedge \neg B$

d. `prove deMorgout:` $\neg A \wedge \neg B \supset \neg(A \vee B)$

You can compile your code by running the following command in your command line. This will check whether the proof is well-formatted, but it will not check that the proof is correct; that is your responsibility.

```
$ smlnj -m sources.cm
```

You can pretty-print your natural deduction proofs by running the following command in your repl. This should make it easier for you to check your work:

```
>> Out.print_nd Homework2.{proof_name_here}
```

Harmony

Task 2 (11 points)**.** Consider a connective $\odot$ with the following elimination rules:

$$\dfrac{A \odot B\ \mathit{true} \qquad \begin{array}{c}\overline{A\ \mathit{true}}^{\,u} \quad \overline{B\ \mathit{true}}^{\,v}\\ \vdots \\ C\ \mathit{true}\end{array}}{C\ \mathit{true}}\ \odot E^{u,v}$$

(Normally we take the verificationist perspective that introduction rules come first to define a connective, but this time we'll go in the opposite direction.)

a. Come up with a set of zero or more introduction rules for this connective. For clarification on how to create new rules, please look at `examples/rc_examples.sml`.

b. Show that the connective is locally sound for your choice of introduction rules. For clarification on how to write a reduction, please look at `examples/h_examples.sml`

c. Show that the connective is locally complete for your choice of introduction rules. For clarification on how to write an expansion, please look at `examples/h_examples.sml`

d. Is it possible to come up with a notational definition $A \odot B \triangleq$ ______________ so that both your defined introduction rule(s) as well as the elimination rule given above are merely derived rules? If yes, provide SOME(p) where p is the appropriate proposition. Otherwise, return NONE.

Task 3 (10 points)**.** Consider a connective $\ltimes$ defined by the following rules:

$$\dfrac{A\ \mathit{true} \qquad \begin{array}{c}\overline{A\ \mathit{true}}^{\,u}\\ \vdots \\ B\ \mathit{true}\end{array}}{A \ltimes B\ \mathit{true}}\ \ltimes I^{u} \qquad\qquad \dfrac{A \ltimes B\ \mathit{true}}{B\ \mathit{true}}\ \ltimes E$$

a. Is this connective locally sound? If so, provide the local reduction; if not, give (without proving) a replacement[1] for the $\ltimes E$ rule to make the connective harmonious.

b. Is this connective locally complete? If so, provide the local expansion; if not, give (without proving) a replacement[2] for the $\ltimes E$ rule to make the connective harmonious.

[1] Replacement: a person or thing that takes the place of another

[2] If you didn't know this, now you do

Hype for *hyp*s

Consider the following notations for a proofs of $A \supset B \supset A \wedge B$.

Floating Hypothesis Notation

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A\ \mathit{true}}\,x \qquad \dfrac{}{B\ \mathit{true}}\,y}{A \wedge B\ \mathit{true}}\wedge I}{B \supset (A \wedge B)\ \mathit{true}}\supset I_y}{A \supset (B \supset (A \wedge B))\ \mathit{true}}\supset I_x$$

Context Notation

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A\ \mathit{true}, B\ \mathit{true} \vdash A\ \mathit{true}}\,hyp \qquad \dfrac{}{A\ \mathit{true}, B\ \mathit{true} \vdash B\ \mathit{true}}\,hyp}{A\ \mathit{true}, B\ \mathit{true} \vdash A \wedge B\ \mathit{true}}\wedge I}{A\ \mathit{true} \vdash B \supset (A \wedge B)\ \mathit{true}}\supset I}{\vdash A \supset (B \supset (A \wedge B))\ \mathit{true}}\supset I$$

The Γ notation of hypotheses can be created by slightly modifying the rules of Natural Deduction to carry a context. (A context is a set of hypotheses.)

$$\dfrac{\Gamma \vdash A\ \mathit{true} \quad \Gamma \vdash B\ \mathit{true}}{\Gamma \vdash A \wedge B\ \mathit{true}}\wedge I \qquad \dfrac{\Gamma \vdash A \wedge B\ \mathit{true}}{\Gamma \vdash A\ \mathit{true}}\wedge E_1 \qquad \dfrac{\Gamma \vdash A \wedge B\ \mathit{true}}{\Gamma \vdash B\ \mathit{true}}\wedge E_2$$

$$\dfrac{\Gamma \vdash A\ \mathit{true}}{\Gamma \vdash A \vee B\ \mathit{true}}\vee I_1 \qquad \dfrac{\Gamma \vdash B\ \mathit{true}}{\Gamma \vdash A \vee B\ \mathit{true}}\vee I_2 \qquad \dfrac{\Gamma \vdash A \vee B\ \mathit{true} \quad \Gamma, A\ \mathit{true} \vdash C\ \mathit{true} \quad \Gamma, B\ \mathit{true} \vdash C\ \mathit{true}}{\Gamma \vdash C\ \mathit{true}}\vee E$$

$$\dfrac{\Gamma, A\ \mathit{true} \vdash B\ \mathit{true}}{\Gamma \vdash A \supset B\ \mathit{true}}\supset I \qquad \dfrac{\Gamma \vdash A \supset B\ \mathit{true} \quad \Gamma \vdash A\ \mathit{true}}{\Gamma \vdash B\ \mathit{true}}\supset E$$

$$\dfrac{}{\Gamma \vdash \top\ \mathit{true}}\top I \qquad \dfrac{\Gamma \vdash \bot\ \mathit{true}}{\Gamma \vdash C\ \mathit{true}}\bot E$$

Finally, the hypothesis rule, which allows you to conclude a hypothesis. We view contexts as unordered sets, so the J in this rule need not be written last.

$$\dfrac{}{\Gamma, J \vdash J}\,hyp$$

Task 4 (4 points)**.** Consider the following proof and write the exact corresponding proof using context notation. Don't shorten the proof. For clarification on how to write proofs in context natural deduction, please look at `examples/cnd_examples.sml`. Since this problem focuses on notation, we will not be accepting partial credit regrades for this task. As such, we recommend carefully reading your proof before submission.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(A \supset A) \wedge (A \supset A)\ \mathit{true}}\,u}{A \supset A\ \mathit{true}}\wedge E_2 \qquad \dfrac{\dfrac{\dfrac{}{(A \supset A) \wedge (A \supset A)\ \mathit{true}}\,u}{A \supset A\ \mathit{true}}\wedge E_1 \quad \dfrac{}{A\ \mathit{true}}\,w}{A\ \mathit{true}}\supset E}{A\ \mathit{true}}\supset E}{(A \supset A)\ \mathit{true}}\supset I_w}{(A \supset A) \wedge (A \supset A) \supset (A \supset A)\ \mathit{true}}\supset I_u$$