Constructive Logic (15-317), Fall2021
Assignment 4: Quantification and Arithmetic

Instructor: Karl Crary
TAs: Avery Cowan, Katherine Cordwell, Matias Scharager, Antian Wang

Due: Submit to Gradescope by Wednesday, Sept 29, 2021, 11:59 pm

Reminder that the first midterm is on Monday, October 4th. The assignments in this course must be submitted electronically through Gradescope. Written homework PDFs and coding SML files will both go to Gradescope. For this homework, you will only be submitting SML files:

- `hw4.sml` (your coding solutions)

1 Quantification

Distribution of Quanitiers is Punishable by Law

In class, we saw that universal quantification distributes over conjuction, that is,

$$(\forall x : \tau.A(x) \wedge B(x)) \equiv (\forall x : \tau.A(x)) \wedge \forall x : \tau.B(x) \textit{ true}.$$

In this section, we will explore various other distributivity properties of quantifiers.

Task 1 (12 points)**.** Dually, existential quantification distributes over disjunction, that is,

$$(\exists x : \tau.A(x) \vee B(x)) \equiv (\exists x : \tau.A(x)) \vee \exists x : \tau.B(x) \textit{ true}.$$

In this task, you will show this equivalence by giving a natural deduction proof of each of the following directions:

a. $(\exists x : \tau.A(x) \vee B(x)) \supset (\exists x : \tau.A(x)) \vee \exists x : \tau.B(x)$ *true*

b. $(\exists x : \tau.A(x)) \vee (\exists x : \tau.B(x)) \supset \exists x : \tau.A(x) \vee B(x)$ *true*

For clarification on how to use quantifiers in SML, please look at `examples/ha_examples.sml`.

Classy Quantifiers

Task 2 (8 points)**.** For each of the following judgments, give a constructive natural deduction proof if it is constructively valid. If it is not constructively valid, state this by returning NONE. *The following judgments are all classically valid.*[1]

a. $(\neg\forall x : \tau.\neg A(x)) \supset \exists x : \tau.A(x)$ *true*

b. $(\exists x : \tau.A(x)) \supset \neg\forall x : \tau.\neg A(x)$ *true*

For All the Exists

Task 3 (8 points)**.** Consider the following proposition

$$(\forall x : t.A(x)) \wedge \top \supset \exists y : t.A(y)$$

When trying to prove this proposition, you're likely to run into a snag. But we can fix it. Replace the $\top$ with another proposition that would make the proof tree work. There're a few rules though. You can't use $\bot$ [2] and you can't use the $A(\_)$ atom.

a. Provide a proposition that would make the above statement provable if it replaced $\top$.

b. Provide a proof of the now provable statement.

[1]This problem first appeared in 15-317 HW's a decade ago. So like these propositions, it is truly a classic

[2]It becomes vacuous so that's too easy

Heyting Arithmetic

Thus far, the propositions we've discussed have been true for generic atoms. With Heyting Arithmetic, we will finally be able to reason about the properties of math. In our case, the naturals and equality.

Rules of Equality

Task 4 (12 points)**.** Prove the following proposition

$$\forall x : \texttt{nat}. \forall y : \texttt{nat}. \forall z : \texttt{nat}.\ x = y \wedge y = z \supset x = z$$

For clarification on how to use Heyting Arithmetic, please look at `support/examples/ha_examples.sml`.