Constructive Logic (15-317), Spring 2021
Assignment 5: Sequent Calculus, Cut, etc.

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Due: Submit to Gradescope by Wednesday, October 13, 2021, 11:59 pm

The assignments in this course must be submitted electronically through Gradescope. Written homework PDFs and Dcheck code files will both go to Gradescope. The written is required to be typeset. We recommend using LaTeX but other suitable software are also acceptable.

- `hw5.deriv` (your coding solutions)
- `hw5.pdf` (your written solutions)

The coding portion will use the experimental Dcheck derivation checker. You can find documentation and examples on the Software page at the course web site (`cs.cmu.edu/~crary/317-f21/software.html`).

1 Sequent Calculus

Provide derivations of the following Sequent Calculus judgements using Dcheck syntax.

Task 1 (4 points)**.** Define a derivation named `task1` that derives:

$$\Longrightarrow ((P \supset F) \vee Q) \supset (P \supset Q)$$

Task 2 (4 points)**.** Define a derivation named `task2` that derives:

$$\Longrightarrow ((P \supset R) \wedge (Q \supset R)) \supset ((P \vee Q) \supset R)$$

Task 3 (4 points)**.** Define a derivation named `task3` that derives:

$$\Longrightarrow (P \supset (Q \supset R)) \supset ((P \supset Q) \supset (P \supset R))$$

Task 4 (4 points)**.** Define a derivation named `task4` that derives:

$$R \supset P \vee Q, (P \wedge Q) \supset R \Longrightarrow P \supset (Q \supset R)$$

Note that Dcheck takes P, Q, and R to be atomic propositions.

2 Cut for a New Connective

Recall the $\clubsuit$ connective from HW3, replicated below for convenience.

$$\dfrac{\begin{matrix}\overline{A\ \mathit{true}}^{\,u} & \overline{A\ \mathit{true}}^{\,v}\\ \vdots & \vdots \\ B\ \mathit{true} & C\ \mathit{true}\end{matrix}}{\clubsuit(A,B,C)\ \mathit{true}}\clubsuit I^{u,v} \qquad \dfrac{\begin{matrix} & & \overline{B\ \mathit{true}}^{\,u}\\ & & \vdots \\ \clubsuit(A,B,C)\ \mathit{true} & A\ \mathit{true} & D\ \mathit{true}\end{matrix}}{D\ \mathit{true}}\clubsuit E_1^u \qquad \dfrac{\begin{matrix} & & \overline{C\ \mathit{true}}^{\,u}\\ & & \vdots \\ \clubsuit(A,B,C)\ \mathit{true} & A\ \mathit{true} & D\ \mathit{true}\end{matrix}}{D\ \mathit{true}}\clubsuit E_2^u$$

If we converted this connective to sequent calculus we would get the following rules:

$$\dfrac{\Gamma, A \Rightarrow B \quad \Gamma, A \Rightarrow C}{\Gamma \Rightarrow \clubsuit(A,B,C)}\clubsuit R \qquad \dfrac{\Gamma, \clubsuit(A,B,C) \Rightarrow A \quad \Gamma, \clubsuit(A,B,C), B \Rightarrow D}{\Gamma, \clubsuit(A,B,C) \Rightarrow D}\clubsuit L_1 \qquad \dfrac{\Gamma, \clubsuit(A,B,C) \Rightarrow A \quad \Gamma, \clubsuit(A,B,C), C \Rightarrow D}{\Gamma, \clubsuit(A,B,C) \Rightarrow D}\clubsuit L_2$$

Task 5 (8 points)**.** If we wanted to add this connective (and the sequent calculus rules describing it above) to our logic, what would be the additional cases we have to prove for cut to remain admissible? Please list the additional cut cases that would need to be added to the cut proof from class. For each of these cases, specify A, $\mathcal{D}$ and $\mathcal{E}$ where A is the cut formula. Use as few distinct cases as possible. For example, when the final rule of $\mathcal{D}$ is *init*, it does not matter what $\mathcal{E}$'s final rule is, so it should be listed as one case, not as several cases. You do not need to prove any of these additional cases; you need only list them.

3 Applications of Theorems

The central theorem of structural proof theory is the closure of sequent calculus under the principle of *cut*. In sequent calculus we state it as follows:

Theorem 1 (Cut)**.** If $\Gamma \Longrightarrow A$ and $\Gamma, A \Longrightarrow C$ then $\Gamma \Longrightarrow C$.

This theorem can be used to prove many difficult properties about a proof system, including consistency, constructivity, and others. In mathematics, the same technique is also used to establish difficult coherence theorems for higher-dimensional structures.

Another important theorem is identity:

Theorem 2 (Identity)**.** For any proposition A, we can show that $\Gamma, A \Rightarrow A$ holds.

A useful lemma is closure under the principle of *weakening*, stated as follows.

Lemma 3 (Weakening)**.** If $\Gamma \Longrightarrow C$ then $\Gamma, A \Longrightarrow C$.

Using Theorems 1 and 2 and Lemma 3, we can prove some results that relate the sequent calculus back to natural deduction:

Task 6 (5 points)**.** Using Theorem 1, Theorem 2, and/or Lemma 3, prove:

If $\Gamma, A \wedge B \Longrightarrow C$ *then* $\Gamma, A, B \Longrightarrow C$.

Task 7 (5 points)**.** Using Theorem 1, Theorem 2, and/or Lemma 3, prove:

If $\Gamma \Longrightarrow A \supset B$ *and* $\Gamma \Longrightarrow A$*, then* $\Gamma \Longrightarrow B$.

The purpose here is to use the theorems. **Do not use any induction in your argument for either of these tasks.**

Note the resemblance of these two theorems to the $\wedge E$ and $\supset E$ rules of natural deduction. Indeed, it is possible to prove similar variants all of the elimination rules of natural deduction admissible in the sequent calculus in much the same way. Likewise, if we slightly modify the rules of natural deduction to make hypotheses explicit, as they are in the sequent calculus, we can translate in the other direction as well.

4 Admissibility and Derivability

Task 8 (16 points)**.** In the following question you'll be considering whether a given rule is *derivable*, *admissible*, or neither. It's important to understand the difference. Given a set of inference rules, a rule is *derivable* if the conclusion of the rule can be *derived* from its premises using only the other rules. That is, a derivable rule could be thought of as a definition that stands for the use of other rules. You saw an example on Homework 5: the derived rules for negation in sequent calculus. Admissibility is a weaker claim. A rule is *admissible* if the conclusion holds whenever the premises hold. All derivable rules are admissible, but not all admissible rules are derivable. It may be impossible to reach the conclusion of an admissible rule from the premises using only other rules. You saw an example of an admissible rule like this in class: the Cut rule for sequent calculus. The admissibility of Cut can only be proven by induction on sequent calculus derivations. An important practical consequence of the distinction is this: Derivable rules remain derivable even when one adds new primitive rules to the system. However, admissible rules can be lost when one adds new primitive rules to the system.

For each of the following rules (with A, B, C atomic) in the cut-free sequent calculus, indicate which are derivable, admissible or neither. If a rule is derivable, you must supply the derivation; if it is not derivable but is admissible, you must include a proof that it is admissible.[1] If it is neither admissible or derivable, please just indicate why you believe this (but no rigorous proof is required).

[1]You may use lemmas that we have proved in class, including the admissibility of cut.

$$\frac{\Gamma \Longrightarrow A \quad \Gamma, B \Longrightarrow \bot}{\Gamma \Longrightarrow A \wedge \neg B} \tag{1}$$

$$\frac{\Gamma, A \Longrightarrow B \vee C}{\Gamma, A \wedge \neg B \Longrightarrow C} \tag{2}$$

$$\frac{\Gamma, B \Longrightarrow A}{\Gamma \Longrightarrow B \supset (C \wedge A)} \tag{3}$$

$$\frac{}{\Gamma, A \vee (B \supset (C \wedge B)) \Longrightarrow A \vee (B \supset (C \wedge B))} \tag{4}$$