Constructive Logic (15-317), Fall 2021
Assignment 6: Cut Admissibility and Classical Logic

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Due: Submit to Gradescope by Friday, October 20, 11:59 pm

This assignment contains both coding and written portions. Written PDFs and Dcheck code files will both go to Gradescope. The written is required to be typeset. We recommend using LaTeX but other suitable tools are also acceptable.

- `hw6.deriv` (your coding solutions)
- `hw6.pdf` (your written solutions)

The coding portion will use the experimental Dcheck derivation checker. You can find documentation and examples on the Software page at the course web site (`cs.cmu.edu/~crary/317-f21/software.html`). That document has been updated with information on preparing classical logic derivations. In particular, be sure to use the primitive not ("`not P`"), rather than the defined not ("`~ P`"), for classical logic problems.

1 Admissibility of Cut

Task 1 (10 points)**.** Extend the proof of the admissibility of cut from class by filling in the following inductive case. A valid proof should include a detailed English explanation—not only notation. Make sure to cite weakening explicitly if it is used.

Case: $\mathcal{D}$ ends in $\vee\mathsf{R}_2$ and $\mathcal{E}$ ends in $\vee\mathsf{L}$, where $\vee\mathsf{L}$ is applied on the principal formula of the cut.

2 DeMorgan's Revenge

Provide derivations of the following Classical Logic judgements using Dcheck syntax.

Task 2 (6 points)**.** Define a derivation named `task2` that derives:

$$\neg(A \wedge B) \supset (\neg A \vee \neg B)\ \mathsf{true}$$

Task 3 (6 points)**.** Define a derivation named `task3` that derives:

$$(A \supset B) \supset (\neg A \vee B)\ \mathsf{true}$$

Note that neither of these are constructively true in general.

3 Classical Quantifiers

We can extend classical logic with universal and existential quantifiers by adding the following truth and falsity rules:

$$\dfrac{\begin{array}{c}[a:\tau] \\ \vdots \\ A(a)\ \mathsf{true}\end{array}}{\forall x{:}\tau.\ A(x)\ \mathsf{true}}\ \forall T^a \qquad\qquad \dfrac{t:\tau \quad A(t)\ \mathsf{false}}{\forall x{:}\tau.\ A(x)\ \mathsf{false}}\ \forall F$$

$$\dfrac{t:\tau \quad A(t)\ \mathsf{true}}{\exists x{:}\tau.\ A(x)\ \mathsf{true}}\ \exists T \qquad\qquad \dfrac{\begin{array}{c}[a:\tau] \\ \vdots \\ A(a)\ \mathsf{false}\end{array}}{\exists x{:}\tau.\ A(x)\ \mathsf{false}}\ \exists F^a$$

Note the duality between the $\forall$ and $\exists$.

Task 4 (16 pts)**.** Using these rules, show that the usual *elimination* rules for the universal and the existential quantifier are derivable. For reference, those rules are:

$$\dfrac{t:\tau \quad \forall x{:}\tau.\ C(x)\ \mathsf{true}}{C(t)\ \mathsf{true}}\ \forall\mathsf{E} \qquad\qquad \dfrac{\exists x{:}\tau.\ A(x)\ \mathsf{true} \qquad \begin{array}{c}[a:\tau] \quad [A(a)\ \mathsf{true}]_u \\ \vdots \\ C\ \mathsf{true}\end{array}}{C\ \mathsf{true}}\ \exists\mathsf{E}^{a,u}$$