Lecture Notes on Harmony

15-317: Constructive Logic
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1 Introduction

In the verificationist definition of the logical connectives via their introduction rules we have briefly justified the elimination rules. In this lecture, we study the balance between introduction and elimination rules more closely.

We elaborate on the verificationist point of view that logical connectives are defined by their introduction rules. We show that for intuitionistic logic as presented so far, the elimination rules are in harmony with the introduction rules in the sense that they are neither too strong nor too weak. We demonstrate this via local reductions and expansions, respectively. In the second part of the lecture we make more precise what a verification is and state, without proof, the global counterparts of the local soundness and completeness properties used to justify the elimination rules.

2 Local Soundness and Local Completeness

In order to show that introduction and elimination rules are in harmony we establish two properties: *local soundness* and *local completeness*.

Local soundness shows that the elimination rules are not too strong: no matter how we apply elimination rules to the result of an introduction we cannot gain any new information. We demonstrate this by showing that we can find a more direct proof of the conclusion of an elimination than one

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that first introduces and then eliminates the connective in question. This is witnessed by a *local reduction* of the given introduction and the subsequent elimination.

Local completeness shows that the elimination rules are not too weak: there is always a way to apply elimination rules so that we can reconstitute a proof of the original proposition from the results by applying introduction rules. This is witnessed by a *local expansion* of an arbitrary given derivation into one that introduces the primary connective.

Connectives whose introduction and elimination rules are in harmony in the sense that they are locally sound and complete are properly defined from the verificationist perspective. If not, the proposed connective should be viewed with suspicion. Another criterion we would like to apply uniformly is that both introduction and elimination rules do not refer to other propositional constants or connectives (besides the one we are trying to define), which could create a dangerous dependency of the various connectives on each other. As we present correct definitions we will occasionally also give some counterexamples to illustrate the consequences of violating the principles behind the patterns of valid inference.

In the discussion of each individual connective below we use the notation

$$\begin{array}{c}\mathcal{D}\\ A\ \mathit{true}\end{array} \Longrightarrow_R \begin{array}{c}\mathcal{D}'\\ A\ \mathit{true}\end{array}$$

for the local reduction of a deduction $\mathcal{D}$ to another deduction $\mathcal{D}'$ of the same judgment A *true*. In fact, $\Longrightarrow_R$ can itself be a higher level judgment relating two proofs, $\mathcal{D}$ and $\mathcal{D}'$, although we will not directly exploit this point of view. Similarly,

$$\begin{array}{c}\mathcal{D}\\ A\ \mathit{true}\end{array} \Longrightarrow_E \begin{array}{c}\mathcal{D}'\\ A\ \mathit{true}\end{array}$$

is the notation of the local expansion of $\mathcal{D}$ to $\mathcal{D}'$.

Conjunction. We start with local soundness, i.e., locally reducing an elimination of a conjunction that was just introduced. Since there are two elimination rules and one introduction, we have two cases to consider, because there are two different elimination rules $\wedge E_1$ and $\wedge E_2$ that could follow the

$\wedge I$ introduction rule. In either case, we can easily reduce.

$$\dfrac{\dfrac{\begin{array}{cc}\mathcal{D} & \mathcal{E} \\ A\ \mathit{true} & B\ \mathit{true}\end{array}}{A \wedge B\ \mathit{true}}\,\wedge I}{A\ \mathit{true}}\,\wedge E_1 \quad \Longrightarrow_R \quad \begin{array}{c}\mathcal{D} \\ A\ \mathit{true}\end{array}$$

$$\dfrac{\dfrac{\begin{array}{cc}\mathcal{D} & \mathcal{E} \\ A\ \mathit{true} & B\ \mathit{true}\end{array}}{A \wedge B\ \mathit{true}}\,\wedge I}{B\ \mathit{true}}\,\wedge E_2 \quad \Longrightarrow_R \quad \begin{array}{c}\mathcal{E} \\ B\ \mathit{true}\end{array}$$

These two reductions justify that, after we just proved a conjunction $A \wedge B$ to be true by the introduction rule $\wedge I$ from a proof $\mathcal{D}$ of A *true* and a proof $\mathcal{E}$ of B *true*, the only thing we can get back out by the elimination rules is something that we have put into the proof of $A \wedge B$ *true*. This makes $\wedge E_1$ and $\wedge E_2$ locally sound, because the only thing we get out is A *true* which already has the direct proof $\mathcal{D}$ as well as B *true* which has the direct proof $\mathcal{E}$. The above two reductions make $\wedge E_1$ and $\wedge E_2$ locally sound.

Local completeness establishes that we are not losing information from the elimination rules. Local completeness requires us to apply eliminations to an arbitrary proof of $A \wedge B$ *true* in such a way that we can reconstitute a proof of $A \wedge B$ from the results.

$$\begin{array}{c}\mathcal{D} \\ A \wedge B\ \mathit{true}\end{array} \quad \Longrightarrow_E \quad \dfrac{\dfrac{\begin{array}{c}\mathcal{D} \\ A \wedge B\ \mathit{true}\end{array}}{A\ \mathit{true}}\,\wedge E_1 \qquad \dfrac{\begin{array}{c}\mathcal{D} \\ A \wedge B\ \mathit{true}\end{array}}{B\ \mathit{true}}\,\wedge E_2}{A \wedge B\ \mathit{true}}\,\wedge I$$

This local expansion shows that, collectively, the elimination rules $\wedge E_1$ and $\wedge E_2$ extract all information from the judgment $A \wedge B$ *true* that is needed to reprove $A \wedge B$ *true* with the introduction rule $\wedge I$. Remember that the hypothesis $A \wedge B$ *true*, once available, can be used multiple times, which is very apparent in the local expansion, because the proof $\mathcal{D}$ of $A \wedge B$ *true* can simply be repeated on the left and on the right premise.

As an example where local completeness fails, consider the case where we "forget" the second/right elimination rule $\wedge E_2$ for conjunction. The remaining rule is still locally sound, because it proves something that was put into the proof of $A \wedge B$ *true*, but not locally complete because we cannot extract a proof of B from the assumption $A \wedge B$. Now, for example, we cannot prove $(A \wedge B) \supset (B \wedge A)$ even though this should clearly be true.

Substitution Principle. We need the defining property for hypothetical judgments before we can discuss implication. Intuitively, we can always substitute a deduction of $A\ \mathit{true}$ for any use of a hypothesis $A\ \mathit{true}$. In order to avoid ambiguity, we make sure assumptions are labelled and we substitute for all uses of an assumption with a given label. Note that we can only substitute for assumptions that are not discharged in the subproof we are considering. The substitution principle then reads as follows:

> If
>
> $$\begin{array}{c} \dfrac{}{A\ \mathit{true}}\ u \\ \mathcal{E} \\ B\ \mathit{true} \end{array}$$
>
> is a hypothetical proof of $B\ \mathit{true}$ under the undischarged hypothesis $A\ \mathit{true}$ labelled u, and
>
> $$\begin{array}{c} \mathcal{D} \\ A\ \mathit{true} \end{array}$$
>
> is a proof of $A\ \mathit{true}$ then
>
> $$\begin{array}{c} \dfrac{\mathcal{D}}{A\ \mathit{true}}\ u \\ \mathcal{E} \\ B\ \mathit{true} \end{array}$$
>
> is our notation for substituting $\mathcal{D}$ for all uses of the hypothesis labelled u in $\mathcal{E}$. This deduction, also sometime written as $[\mathcal{D}/u]\mathcal{E}$ no longer depends on u.

Implication. To witness local soundness, we reduce an implication introduction followed by an elimination using the substitution operation.

$$\dfrac{\dfrac{\begin{array}{c}\dfrac{}{A\ \mathit{true}}\ u \\ \mathcal{E} \\ B\ \mathit{true}\end{array}}{A \supset B\ \mathit{true}}\ \supset I^u \qquad \begin{array}{c}\mathcal{D} \\ A\ \mathit{true}\end{array}}{B\ \mathit{true}}\ \supset E \quad \Longrightarrow_R \quad \begin{array}{c}\dfrac{\mathcal{D}}{A\ \mathit{true}}\ u \\ \mathcal{E} \\ B\ \mathit{true}\end{array}$$

The conditions on the substitution operation is satisfied, because u is introduced at the $\supset I^u$ inference and therefore not discharged in $\mathcal{E}$.

Local completeness is witnessed by the following expansion.

$$\dfrac{\mathcal{D}}{A \supset B \ \mathit{true}} \quad \Longrightarrow_E \quad \dfrac{\dfrac{\mathcal{D}}{A \supset B \ \mathit{true}} \quad \dfrac{}{A \ \mathit{true}}\,u}{\dfrac{B \ \mathit{true}}{A \supset B \ \mathit{true}}\,\supset I^u}\,\supset E$$

Here u must be chosen fresh: it only labels the new hypothesis A *true* which is used only once.

Disjunction. For disjunction we also employ the substitution principle because the two cases we consider in the elimination rule introduce hypotheses. Also, in order to show local soundness we have two possibilities for the introduction rule, in both situations followed by the only elimination rule.

$$\dfrac{\dfrac{\mathcal{D}}{A \ \mathit{true}}}{A \vee B \ \mathit{true}}\,\vee I_L \qquad \dfrac{}{A \ \mathit{true}}\,u \quad \dfrac{\mathcal{E}}{C \ \mathit{true}} \qquad \dfrac{}{B \ \mathit{true}}\,w \quad \dfrac{\mathcal{F}}{C \ \mathit{true}}}{C \ \mathit{true}}\,\vee E^{u,w} \quad \Longrightarrow_R \quad \dfrac{\dfrac{\mathcal{D}}{A \ \mathit{true}}\,u}{\mathcal{E}\ C \ \mathit{true}}$$

$$\dfrac{\dfrac{\mathcal{D}}{B \ \mathit{true}}}{A \vee B \ \mathit{true}}\,\vee I_R \qquad \dfrac{}{A \ \mathit{true}}\,u \quad \dfrac{\mathcal{E}}{C \ \mathit{true}} \qquad \dfrac{}{B \ \mathit{true}}\,w \quad \dfrac{\mathcal{F}}{C \ \mathit{true}}}{C \ \mathit{true}}\,\vee E^{u,w} \quad \Longrightarrow_R \quad \dfrac{\dfrac{\mathcal{D}}{B \ \mathit{true}}\,w}{\mathcal{F}\ C \ \mathit{true}}$$

An example of a rule that would not be locally sound is

$$\dfrac{A \vee B \ \mathit{true}}{A \ \mathit{true}}\,\vee E_1?$$

and, indeed, we would not be able to reduce

$$\dfrac{\dfrac{B \ \mathit{true}}{A \vee B \ \mathit{true}}\,\vee I_R}{A \ \mathit{true}}\,\vee E_1?$$

In fact we can now derive a contradiction from no assumption, which means the whole system is incorrect.

$$\dfrac{\dfrac{\dfrac{}{\top \ \mathit{true}}\,\top I}{\bot \vee \top \ \mathit{true}}\,\vee I_R}{\bot \ \mathit{true}}\,\vee E_1?$$

Local completeness of disjunction distinguishes cases on the known $A \vee B$ *true*, using $A \vee B$ *true* as the conclusion.

$$\begin{array}{c}\mathcal{D}\\ A \vee B \text{ true}\end{array} \quad \Longrightarrow_E \quad \dfrac{\begin{array}{c}\mathcal{D}\\ A \vee B \text{ true}\end{array} \quad \dfrac{\overline{A \text{ true}}\,^{u}}{A \vee B \text{ true}} \vee I_L \quad \dfrac{\overline{B \text{ true}}\,^{w}}{A \vee B \text{ true}} \vee I_R}{A \vee B \text{ true}} \vee E^{u,w}$$

Visually, this looks somewhat different from the local expansions for conjunction or implication. It looks like the elimination rule is applied last, rather than first. Mostly, this is due to the notation of natural deduction: the above represents the step from using the knowledge of $A \vee B$ *true* and eliminating it to obtain the hypotheses A *true* and B *true* in the two cases.

Truth. The local constant $\top$ has only an introduction rule, but no elimination rule. Consequently, there are no cases to check for local soundness: any introduction followed by any elimination can be reduced, because $\top$ has no elimination rules.

However, local completeness still yields a local expansion: Any proof of $\top$ *true* can be trivially converted to one by $\top I$.

$$\begin{array}{c}\mathcal{D}\\ \top \text{ true}\end{array} \quad \Longrightarrow_E \quad \dfrac{}{\top \text{ true}}\,\top I$$

Falsehood. As for truth, there is no local reduction because local soundness is trivially satisfied since we have no introduction rule.

Local completeness is slightly tricky. Literally, we have to show that there is a way to apply an elimination rule to any proof of $\bot$ *true* so that we can reintroduce a proof of $\bot$ *true* from the result. However, there will be zero cases to consider, so we apply no introductions. Nevertheless, the following is the right local expansion.

$$\begin{array}{c}\mathcal{D}\\ \bot \text{ true}\end{array} \quad \Longrightarrow_E \quad \dfrac{\begin{array}{c}\mathcal{D}\\ \bot \text{ true}\end{array}}{\bot \text{ true}}\,\bot E$$

Reasoning about situation when falsehood is true may seem vacuous, but is common in practice because it corresponds to reaching a contradiction. In intuitionistic reasoning, this occurs when we prove $A \supset \bot$ which is often abbreviated as $\neg A$. In classical reasoning it is even more frequent, due to the rule of proof by contradiction.

$$
\begin{array}{lcl}
M : A \wedge B & \Longrightarrow_E & \langle \mathbf{fst}\, M, \mathbf{snd}\, M \rangle \\
M : A \supset B & \Longrightarrow_E & \lambda u{:}A.\ M\, u \quad \text{for } u \text{ not free in } M \\
M : \top & \Longrightarrow_E & \langle\,\rangle \\
M : A \vee B & \Longrightarrow_E & \mathbf{case}\, M\ \mathbf{of}\ \mathbf{inl}\, u \Rightarrow \mathbf{inl}^B\, u \mid \mathbf{inr}\, w \Rightarrow \mathbf{inr}^A\, w \\
M : \bot & \Longrightarrow_E & \mathbf{abort}^{\bot}\, M
\end{array}
$$

Figure 1: Proof term expansions

3 Revisiting Proof Terms

We saw in the last lecture, that eliminations (destructors) applied to the result of introductions (constructor) give rise to computation in the form of a reduction. We invite you to go back and verify that these computational reductions are *exactly* the witnesses of the local reductions on proofs shown in this lecture! In other words, computational reductions on proof terms witness local soundness of the rules!

What about local completeness? It turns out that the local expansions are less relevant to computation. What they tell us, for example, is that if we need to return a pair from a function, we can always construct it as $\langle M, N \rangle$ for some M and N. Another example would be that whenever we need to return a function, we can always construct it as $\mathsf{fn}\, u \Rightarrow .\ M$ for some M.

We can derive what the local expansion must be by annotating the deductions witnessing local expansions *on proofs* from this lecture with proof terms. We leave this as an exercise to the reader. The left-hand side of each expansion has the form $M : A$, where M is an arbitrary term and A is a logical connective or constant applied to arbitrary propositions. On the right hand side we have to apply a destructor to M and then reconstruct a term of the original type. The resulting rules can be found in Figure 1.

4 Logical Equivalence as a Connective

As another example we would now like to define a new connective, develop introduction and elimination rules, and check their local soundness and completeness (if they hold). First, the proposed introduction rule to

define the connective:

$$\dfrac{\dfrac{}{A\ true}\,x \quad \dfrac{}{B\ true}\,y \\ \vdots \qquad\quad \vdots \\ B\ true \qquad A\ true}{A \equiv B\ true}\,{\equiv}I^{x,y}$$

This suggests the two eliminations rules below. If we omitted one of them, we would expect the eliminations *not* to be locally complete.

$$\dfrac{A \equiv B\ true \quad A\ true}{B\ true}\,{\equiv}E_1 \qquad \dfrac{A \equiv B\ true \quad B\ true}{A\ true}\,{\equiv}E_2$$

There is one introduction and two eliminations, so we have to check two cases for local soundness. The first case:

$$\dfrac{\dfrac{\dfrac{}{A\ true}\,x \quad \dfrac{}{B\ true}\,y \\ \mathcal{D} \qquad\quad \mathcal{E} \\ B\ true \qquad A\ true}{A \equiv B\ true}\,{\equiv}I^{x,y} \qquad \mathcal{F} \\ A\ true}{B\ true}\,{\equiv}E_1$$

We see that $B\ true$ is justified, because the proof $\mathcal{D}$ ends in $B\ true$ and its hypothesis is proved by $\mathcal{F}$:

$$\Longrightarrow_R \quad \dfrac{\dfrac{\mathcal{F}}{A\ true}\,x \\ \mathcal{D} \\ B\ true}{}$$

The other reduction is entirely symmetric.

$$\dfrac{\dfrac{\dfrac{}{A\ true}\,x \quad \dfrac{}{B\ true}\,y \\ \mathcal{D} \qquad\quad \mathcal{E} \\ B\ true \qquad A\ true}{A \equiv B\ true}\,{\equiv}I^{x,y} \qquad \mathcal{F} \\ B\ true}{A\ true}\,{\equiv}E_2 \quad \Longrightarrow_R \quad \dfrac{\dfrac{\mathcal{F}}{B\ true}\,y \\ \mathcal{D} \\ A\ true}{}$$

The local expansion will exhibit the necessity of both elimination rules. You should go through this and construct it in stages—the final result of expansion may otherwise be a bit hard to understand.

$$\mathcal{D} \quad A \equiv B \text{ true} \Longrightarrow_E \quad \dfrac{\dfrac{\mathcal{D} \quad A \equiv B \text{ true} \quad \overline{A \text{ true}}\,x}{B \text{ true}} \equiv E_1 \quad \dfrac{\mathcal{D} \quad A \equiv B \text{ true} \quad \overline{B \text{ true}}\,y}{A \text{ true}} \equiv E_2}{A \equiv B \text{ true}} \equiv I^{x,y}$$

At this point we know that, logically, the connective makes sense: it is both locally sound and complete.

Next, we should carry out a proof term assignment and the re-expression local reduction and expansions on proof terms. The local reduction should give us a rule of computation; the local expansion an extensional equality principle.

$$\dfrac{\begin{array}{cc}\overline{x : A \text{ true}}\,x & \overline{y : B \text{ true}}\,y \\ \vdots & \vdots \\ N : B \text{ true} & M : A \text{ true}\end{array}}{(\!|x \Rightarrow N, y \Rightarrow M|\!) : A \equiv B \text{ true}} \equiv I^{x,y}$$

$$\dfrac{M : A \equiv B \text{ true} \quad N : A \text{ true}}{\smile\, M\ N : B \text{ true}} \equiv E_1 \qquad \dfrac{M : A \equiv B \text{ true} \quad N : B \text{ true}}{\frown\, M\ N : A \text{ true}} \equiv E_2$$

We can now annotate the local reductions and expansion with proof terms and read off:

$$\begin{array}{l}\smile\, (\!|x \Rightarrow N, y \Rightarrow M|\!)\ P \Longrightarrow_R [P/x]N \\ \frown\, (\!|x \Rightarrow N, y \Rightarrow M|\!)\ P \Longrightarrow_R [P/y]M \\ M : A \equiv B \Longrightarrow_E (\!|x \Rightarrow \smile\, M\ x, y \Rightarrow \frown\, N\ y|\!)\end{array}$$

Introducing new syntax for new connectives and programs can be tedious and difficult to use. Therefore, in practice, we probably wouldn't define logical equivalence as a new primitive, but use *notational definition* (as we did for negation):

$$A \equiv B \quad \triangleq \quad (A \supset B) \wedge (B \supset A)$$

whose meaning as a type is simply a pair of functions between the types A and B.