Constructive Logic (15-317), Fall 2022
Assignment 10: Linear Logic

Constructive Logic Staff
(Instructor: Karl Crary)

Due: Wednesday, November 16, 2022, 11:59 pm

This assignment will have a written portion and a coding portion. You will submit both portions through Gradescope, to the assignments labelled "Homework 10 (written)" and "Homework 10 (code)." Please submit a file named "`hw.pdf`" to the former, and a file named "`hw.deriv`" to the latter.

We recommend that you typeset your written solutions. Most students use LaTeX, but other software is acceptable. (Please put each task on its own page to speed up grading.) If you choose not to typeset your solutions, be aware that you are answerable for your handwriting. Any that the grader has difficulty reading (in the sole judgement of the grader), will be marked wrong.

For the coding portion you will use Dcheck. You can find documentation on Dcheck at `cs.cmu.edu/~crary/dcheck/dcheck.pdf` and a sample file at `cs.cmu.edu/~crary/dcheck/example.deriv`. (Be aware that the sample file uses several logics that we have not seen yet in class.)

1 Practicing Linear Logic

Task 1 (16 points). Provide derivations for the following Linear Logic judgements using Dcheck syntax. Derivation names are given in the starter code.

a. $\Vdash (A \multimap B \multimap C) \multimap (A \otimes B \multimap C)\ \mathsf{true}$

b. $\Vdash (((A \otimes \top) \,\&\, (B \otimes \top)) \multimap C) \multimap (A \multimap B \multimap C)\ \mathsf{true}$

c. $\Vdash ((A \multimap C) \oplus (B \multimap C)) \multimap (A \,\&\, B) \multimap C\ \mathsf{true}$

d. $\Vdash ((A \multimap 0) \oplus (B \multimap 0)) \multimap (A \otimes B) \multimap 0\ \mathsf{true}$

2 Linear Harmony

As at the beginning of the course, we establish the local correctness of our linear logic rules by proving Harmony. Local soundness and completeness for linear natural deduction is largely the same as for ordinary (persistent) natural deduction, but one subtlety arises:

For local soundness, one must prove that **all** detours can be locally contracted. To ensure that all detours are covered, we must take care not to constrain any of the contexts used in the derivation any further than the rules require. For example, if we assumed that a context split in a particular way, we would be failing to cover any detours that split the context differently.

Similarly, for local completness, one must assume that the conclusion of the derivation to be expanded may have **any** context. If we required that its context had a particular form, we would be failing to cover some derivations.

More generally, be sure that your use of the linear logic rules deal with contexts correctly. Context management is central to linear logic, not incidental.

Remark (Linear substitution principle) When exhibiting local reductions and expansions, you will need to use substitutions $[\mathcal{D}/u]\mathcal{E}$. These are governed by the *linear substitution principle*, which states:

$$\text{If } \Gamma_1 \Vdash \overset{\mathcal{D}}{A} \,\mathsf{true} \text{ and } \Gamma_2, u : A \overset{\mathcal{E}}{\mathsf{true}} \Vdash B \,\mathsf{true} \text{ then } \Gamma_1, \Gamma_2 \overset{[\mathcal{D}/u]\mathcal{E}}{\Vdash} B \,\mathsf{true}.$$

Task 2 (10 points). Verify that tensor $\otimes$ satisfies local soundness and completeness.

Task 3 (10 points). Verify that that 1 satisfies local soundness and completeness.

Task 4 (10 points). Verify that $\top$ satisfies local soundness and completeness.

The relevant rules are:

$$\frac{}{A\,\mathsf{true} \Vdash A\,\mathsf{true}}\; hyp$$

$$\frac{\Gamma_1 \Vdash A\,\mathsf{true} \quad \Gamma_2 \Vdash B\,\mathsf{true}}{\Gamma_1, \Gamma_2 \Vdash A \otimes B\,\mathsf{true}}\; \otimes I \qquad \frac{\Gamma_1 \Vdash A \otimes B\,\mathsf{true} \quad \Gamma_2, u : A\,\mathsf{true}, v : B\,\mathsf{true} \Vdash C\,\mathsf{true}}{\Gamma_1, \Gamma_2 \Vdash C\,\mathsf{true}}\; \otimes E^{u,v}$$

$$\frac{}{\Vdash 1\,\mathsf{true}}\; 1I \qquad \frac{\Gamma_1 \Vdash 1\,\mathsf{true} \quad \Gamma_2 \Vdash C\,\mathsf{true}}{\Gamma_1, \Gamma_2 \Vdash C\,\mathsf{true}}\; 1E \qquad \frac{}{\Gamma \Vdash \top\,\mathsf{true}}\; \top I$$