Constructive Logic (15-317), Fall 2022
Assignment 5: Sequent Calculus, Cut, etc.

Constructive Logic Staff
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Due: Wednesday, October 5, 2022, 11:59 pm

This assignment will have a written portion and a coding portion. You will submit both portions through Gradescope, to the assignments labelled "Homework 5 (written)" and "Homework 5 (code)." Please submit a file named "`hw.pdf`" to the former, and a file named "`hw.deriv`" to the latter.

We recommend that you typeset your written solutions. Most students use LaTeX, but other software is acceptable. (Please put each task on its own page to speed up grading.) If you choose not to typeset your solutions, be aware that you are answerable for your handwriting. Any that the grader has difficulty reading (in the sole judgement of the grader), will be marked wrong.

For the coding portion you will use Dcheck. You can find documentation on Dcheck at `cs.cmu.edu/~crary/dcheck/dcheck.pdf` and a sample file at `cs.cmu.edu/~crary/dcheck/example.deriv`. (Be aware that the sample file uses several logics that we have not seen yet in class.)

1 Sequent Calculus

Using Dcheck, give derivations of the following judgements. Use the names `task1`, `task2`, etc.

Task 1 (4 points). $\Longrightarrow ((P \supset F) \vee Q) \supset (P \supset Q)$

Task 2 (4 points). $\Longrightarrow ((P \supset R) \wedge (Q \supset R)) \supset ((P \vee Q) \supset R)$

Task 3 (4 points). $\Longrightarrow (P \supset (Q \supset R)) \supset ((P \supset Q) \supset (P \supset R))$

Task 4 (4 points). $(R \supset (P \vee Q)), ((P \wedge Q) \supset R) \Longrightarrow P \supset (Q \supset R)$

Remember that Dcheck takes P, Q, and R to be atomic propositions.

2 Cut for a New Connective

Recall the $\heartsuit$ connective from HW3:

$$\dfrac{\begin{array}{c}[A\ \mathsf{true}]_v \\ \vdots \\ B\ \mathsf{true}\end{array} \quad \begin{array}{c}[A\ \mathsf{true}]_w \\ \vdots \\ C\ \mathsf{true}\end{array}}{\heartsuit(A,B,C)\ \mathsf{true}}\ \heartsuit I^{v,w}$$

$$\dfrac{\heartsuit(A,B,C)\ \mathsf{true} \quad A\ \mathsf{true} \quad \begin{array}{c}[B\ \mathsf{true}]_u \\ \vdots \\ D\ \mathsf{true}\end{array}}{D\ \mathsf{true}}\ \heartsuit E1^u \qquad \dfrac{\heartsuit(A,B,C)\ \mathsf{true} \quad A\ \mathsf{true} \quad \begin{array}{c}[C\ \mathsf{true}]_u \\ \vdots \\ D\ \mathsf{true}\end{array}}{D\ \mathsf{true}}\ \heartsuit E2^u$$

In sequent calculus, $\heartsuit$ would have the following rules:

$$\dfrac{\Delta, A \Longrightarrow B \quad \Delta, A \Longrightarrow C}{\Delta \Longrightarrow \heartsuit(A,B,C)}\ \heartsuit R \qquad \dfrac{\Delta, \heartsuit(A,B,C) \Longrightarrow A \quad \Delta, \heartsuit(A,B,C), B \Longrightarrow D}{\Delta, \heartsuit(A,B,C) \Longrightarrow D}\ \heartsuit L1$$

$$\dfrac{\Delta, \heartsuit(A,B,C) \Longrightarrow A \quad \Delta, \heartsuit(A,B,C), C \Longrightarrow D}{\Delta, \heartsuit(A,B,C) \Longrightarrow D}\ \heartsuit L2$$

Task 5 (5 points). If we wanted to add this connective (and the sequent calculus rules describing it above) to our logic, what would be the additional cases we have to prove for cut to remain admissible? Please list the additional cut cases that would need to be added to the cut proof from class. For each of these cases, specify A, $\mathcal{D}$ and $\mathcal{E}$ where A is the cut formula. Use as few distinct cases as possible. For example, when the final rule of $\mathcal{D}$ is *init*, it does not matter what $\mathcal{E}$'s final rule is, so it would be listed as one case, not as several cases. You do not need to prove any of these additional cases; you need only list them.

3 Applications of Cut and Friends

The central theorem of proof theory is cut admissibility:

Lemma 1 (Cut) *If $\Delta \longrightarrow A$ and $\Delta, A \longrightarrow C$ then $\Delta \longrightarrow C$.*

This lemma can be used to prove many important properties about a proof system, including consistency. In mathematics, the same technique is also used to establish difficult coherence lemmas for higher-dimensional structures.

Another important lemma is identity:

Lemma 2 (Identity) *For any proposition A, we can show that $\Delta, A \Longrightarrow A$ holds.*

A useful lemma is *weakening*:

Lemma 3 (Weakening) *If $\Delta \longrightarrow C$ then $\Delta, A \Longrightarrow C$.*

Using these, we can prove some results that relate the sequent calculus back to natural deduction:

Task 6 (5 points). Using lemmas 1–3, prove: if $\Delta, A \wedge B \Longrightarrow C$ then $\Delta, A, B \Longrightarrow C$.

Task 7 (5 points). Using lemmas 1–3, prove: if $\Delta \Longrightarrow A \supset B$ and $\Delta \Longrightarrow A$, then $\Delta \Longrightarrow B$.

The purpose here is to use the lemmas. **Do not use any induction in your argument for either of these tasks.**

Note the resemblance of these two lemmas to the $\wedge E$ and $\supset E$ rules of natural deduction. Indeed, it is possible to prove similar variants all of the elimination rules of natural deduction admissible in the sequent calculus in much the same way. Likewise, if we slightly modify the rules of natural deduction to make hypotheses explicit, as they are in the sequent calculus, we can translate in the other direction as well.

4 Admissibility and Derivability

In the following question you'll be considering whether a given rule is *derivable*, *admissible*, or neither. It's important to understand the difference:

Given a set of inference rules, a rule is *derivable* if the conclusion of the rule can be obtained (*derived*) from its premises using only the other rules. That is, a derivable rule could be thought of as a definition that stands for the use of other rules.

Admissibility is a weaker claim. A rule is *admissible* if the conclusion holds whenever the premises hold. All derivable rules are admissible, but not vice versa. It may be impossible to reach the conclusion of an admissible rule from its premises. An example is the cut principle.

An important practical consequence of the distinction is this: Derivable rules remain derivable even when one adds additional rules to the system. However, admissible rules can be lost when one adds additional rules to the system.

Task 8 (16 points). For each of the following rules (taking A, B, C to be atomic) in the (cut-free) sequent calculus, indicate which are derivable, admissible, or neither. If a rule is derivable, supply the derivation. Iif it is not derivable but is admissible, supply a proof that it is admissible.[1] If it is neither admissible or derivable, please just indicate why you believe this (but no rigorous proof is required).

a.

$$\frac{\Delta \Longrightarrow A \quad \Delta, B \Longrightarrow F}{\Delta \Longrightarrow A \wedge \neg B}$$

b.

$$\frac{\Delta, A \Longrightarrow B \vee C}{\Delta, A \wedge \neg B \Longrightarrow C}$$

c.

$$\frac{\Delta, B \Longrightarrow A}{\Delta \Longrightarrow B \supset (C \wedge A)}$$

d.

$$\frac{}{\Delta, A \vee (B \supset (C \wedge B)) \Longrightarrow A \vee (B \supset (C \wedge B))}$$

[1] You may—and probably should—use lemmas from section 3.