Constructive Logic (15-317), Fall 2022
Assignment 6: Cut Admissibiity and Classical Logic

Constructive Logic Staff
(Instructor: Karl Crary)

Due: Wednesday, October 12, 2022, 11:59 pm

This assignment will have a written portion and a coding portion. You will submit both portions through Gradescope, to the assignments labelled "Homework 6 (written)" and "Homework 6 (code)." Please submit a file named "`hw.pdf`" to the former, and a file named "`hw.deriv`" to the latter.

We recommend that you typeset your written solutions. Most students use LaTeX, but other software is acceptable. (Please put each task on its own page to speed up grading.) If you choose not to typeset your solutions, be aware that you are answerable for your handwriting. Any that the grader has difficulty reading (in the sole judgement of the grader), will be marked wrong.

For the coding portion you will use Dcheck. You can find documentation on Dcheck at `cs.cmu.edu/~crary/dcheck/dcheck.pdf` and a sample file at `cs.cmu.edu/~crary/dcheck/example.deriv`. (Be aware that the sample file uses several logics that we have not seen yet in class.)

1 Cut Admissibility

Task 1 (10 points). Extend the proof of the admissibility of cut from class by filling in the following inductive case. A valid proof should include a detailed English explanation—not only notation. Make sure to cite weakening explicitly if it is used, and to justify any use of the induction hypothesis.

Case: $\mathcal{D}$ ends in $\vee R2$ and $\mathcal{E}$ ends in $\vee L$, where $\vee L$ operates on the cut formula.

2 DeMorgan's Revenge

Provide derivations of the following Classical Logic judgements using Dcheck. Remember that negation ("`~ P`") is primitive in classical logic, not defined in terms of implication and false.

Task 2 (6 points). Define a derivation named `task2` that derives:

$$\neg(A \wedge B) \supset (\neg A \vee \neg B) \;\mathsf{true}$$

Task 3 (6 points). Define a derivation named `task3` that derives:

$$(A \supset B) \supset (\neg A \vee B) \;\mathsf{true}$$

Note that neither of these are constructively true in general.

3 Classical Quantifiers

We can extend classical logic with universal and existential quantifiers by adding the following truth and falsity rules:

$$\frac{\begin{array}{c}[a : \tau] \\ \vdots \\ A(a)\;\mathsf{true}\end{array}}{\forall x{:}\tau.\, A(x)\;\mathsf{true}}\;\forall T^{a} \qquad \frac{m : \tau \quad A(m)\;\mathsf{false}}{\forall x{:}\tau.\, A(x)\;\mathsf{false}}\;\forall F$$

$$\frac{m : \tau \quad A(m)\;\mathsf{true}}{\exists x{:}\tau.\, A(x)\;\mathsf{true}}\;\exists T \qquad \frac{\begin{array}{c}[a : \tau] \\ \vdots \\ A(a)\;\mathsf{false}\end{array}}{\exists x{:}\tau.\, A(x)\;\mathsf{false}}\;\exists F^{a}$$

Note the duality between the $\forall$ and $\exists$.

Task 4 (16 points). Using these rules (and the other rules of classical logic as necessary), show that the usual *elimination* rules for the universal and existential quantifiers are *derivable*. For reference, those rules are:

$$\frac{\forall x{:}\tau.\, A(x)\;\mathsf{true} \quad m : \tau}{A(m)\;\mathsf{true}}\;\forall E \qquad \frac{\exists x{:}\tau.\, A(x)\;\mathsf{true} \qquad \begin{array}{c}[a : \tau] \quad [A(a)\;\mathsf{true}]_u \\ \vdots \\ C\;\mathsf{true}\end{array}}{C\;\mathsf{true}}\;\exists E^{a,u}$$