Available online at www.sciencedirect.com

ScienceDirect

Journal of Combinatorial Theory, Series A 115 (2008) 1008–1020

Journal of Combinatorial Theory Series A

www.elsevier.com/locate/jcta

Erdős–Ko–Rado theorems for permutations and set partitions

Cheng Yeaw Ku, David Renshaw

Department of Mathematics, Caltech, Pasadena, CA 91125, USA

Received 4 June 2007

Available online 30 January 2008

Abstract

Let Sym($[n]$) denote the collection of all permutations of $[n] = \{1, \dots, n\}$. Suppose $\mathcal{A} \subseteq \mathrm{Sym}([n])$ is a family of permutations such that any two of its elements (when written in its cycle decomposition) have at least t cycles in common. We prove that for sufficiently large n, $|\mathcal{A}| \leqslant (n-t)!$ with equality if and only if $\mathcal{A}$ is the stabilizer of t fixed points. Similarly, let $\mathcal{B}(n)$ denote the collection of all set partitions of $[n]$ and suppose $\mathcal{A} \subseteq \mathcal{B}(n)$ is a family of set partitions such that any two of its elements have at least t blocks in common. It is proved that, for sufficiently large n, $|\mathcal{A}| \leqslant B_{n-t}$ with equality if and only if $\mathcal{A}$ consists of all set partitions with t fixed singletons, where B_n is the nth Bell number.

Keywords: Intersecting families; Permutations; Set partitions

1. Introduction

Let $\binom{[n]}{k}$ denote the collection of all k-subsets of $[n] = \{1, \dots, n\}$. A fundamental result in extremal combinatorial set theory is the Erdős–Ko–Rado theorem which asserts that if a family $\mathcal{A} \subseteq \binom{[n]}{k}$ is t-intersecting (i.e. $|A \cap B| \geqslant t$ for any $A, B \in \mathcal{A}$) and $2k - t < n$, then $|\mathcal{A}| \leqslant \binom{n-t}{k-t}$ for $n \geqslant n_0(k,t)$. The smallest $n_0(k,t) = (k-t+1)(t+1)$ has been determined by Frankl [9] for $t \geqslant 15$ and subsequently by Wilson [12] for all t.

E-mail addresses: cyk@caltech.edu (C.Y. Ku), renshaw@caltech.edu (D. Renshaw).

0097-3165/$ – see front matter
doi:10.1016/j.jcta.2007.12.004

Theorem 1.1. *(Erdős, Ko and Rado [5], Frankl [9], Wilson [12].) Suppose $\mathcal{A} \subseteq \binom{[n]}{k}$, $2k - t < n$, is t-intersecting. Then, for $n \geqslant (k - t + 1)(t + 1)$,*

$$|\mathcal{A}| \leqslant \binom{n-t}{k-t}.$$

Moreover, if $n > (k - t + 1)(t + 1)$, equality holds if and only if $\mathcal{A} = \{A \in \binom{[n]}{k}\colon T \subseteq A\}$ for some t-set T.

This paper is motivated by several Erdős–Ko–Rado type results for permutations and set partitions. Deza and Frankl [4] first considered such a problem for permutations in a context of coding theory. Let $\mathrm{Sym}([n])$ denote the collection of all permutations of $[n]$. A family $\mathcal{A} \subseteq \mathrm{Sym}([n])$ is *t-intersecting* if any two elements $g, h \in \mathcal{A}$ have at least t positions in common, i.e. $|\{x\colon g(x) = h(x)\}| \geqslant t$, or equivalently, the Hamming distance between g and h is at most $n - t$. Among other results, they proved that for $t = 1$, the maximum size of such a family is $(n - 1)!$. Later, it was proved by Cameron and Ku [2] and independently by Larose and Malvenuto [10] that the only 1-intersecting families of maximal size are the cosets of point stabilizers.

Theorem 1.2. *(See [2,4,10].) Let $n \geqslant 2$. Suppose $\mathcal{A} \subseteq \mathrm{Sym}([n])$ is 1-intersecting. Then $|\mathcal{A}| \leqslant (n - 1)!$. Moreover, equality holds if and only if $\mathcal{A} = \{g \in \mathrm{Sym}([n])\colon g(x) = y\}$ for some $x, y \in [n]$.*

For general $t > 1$, it was conjectured in [4] that the maximum size of a t-intersecting family is $(n - t)!$ for $n \geqslant n_0(t)$. It was proved in [4] that the conjecture is true when $t = 2$ and n is a prime power; $t = 3$ and n is a prime power plus one. The conjecture remains open. It was stated without proof in [3] that the following holds:

Theorem 1.3. *(Deza and Frankl [3].) Suppose $\mathcal{A} \subseteq \mathrm{Sym}([n])$ such that any three elements of $\mathcal{A}$ have at least t positions in common. Then for $n \geqslant n_0(t)$, $|\mathcal{A}| \leqslant (n - t)!$.*

In this paper, we provide a proof of the above result for $n_0(t) = O(t^2)$. Moreover, we show that, for all such n, equality holds if and only if $\mathcal{A}$ is a coset of the stabilizer of t points (see Corollary 2.8).

We further introduce a new notion of intersection for permutations. This notion seems to be natural when we write permutations in their cycle decomposition. We say that $\mathcal{A} \subseteq \mathrm{Sym}([n])$ is *t-cycle-intersecting* if any two elements of $\mathcal{A}$, when written in their cycle decomposition, have at least t cycles in common. Clearly, if $\mathcal{A}$ is t-cycle-intersecting, then $\mathcal{A}$ is t-intersecting. The converse, however, is not true. For example, consider the families $\mathcal{A} = \{g_1 = (1\ 2\ 3\ 4\ 5),\ g_2 = (1\ 2)(3)(4\ 5),\ g_3 = (1\ 2\ 3)(4\ 5)\}$ and $\mathcal{B} = \{h_1 = (1)(2)(3)(4\ 5),\ h_2 = (1\ 2)(3)(4)(5),\ h_3 = (1\ 2)(3)(4\ 5)\}$ of permutations of $[5]$. Both $\mathcal{A}$ and $\mathcal{B}$ are 1-intersecting. Since g_1 is a cyclic shift, the only permutation which can have a common cycle with g_1 is g_1 itself, so $\mathcal{A}$ cannot be 1-cycle-intersecting. On the other hand, $\mathcal{B}$ is 1-cycle-intersecting since all its elements contain the cycle (3).

In view of Theorem 1.2, we shall be interested in t-cycle-intersecting families of maximum size for $t \geqslant 2$. One of our main results is the following:

Theorem 1.4. *Let $t \geqslant 2$. Suppose $\mathcal{A} \subseteq \mathrm{Sym}([n])$ is t-cycle-intersecting and $n \geqslant n_0(t)$ where $n_0(t) = O(t^2)$. Then $|\mathcal{A}| \leqslant (n-t)!$ with equality if and only if $\mathcal{A}$ is the stabilizer of t fixed points.*

Note that Theorem 1.4 is not true if n is too small compared to t. For example, take $n = 8$ and $t = 4$. The stabilizer of 4 points has size 24 while the family consisting of the identity and all transpositions interchanging i and j, where $i \neq j$, has size 29.

Similar problems for set partitions have also received some attention. Recall that a set partition of $[n]$ is a collection of pairwise disjoint non-empty subsets (called *blocks*) of $[n]$ whose union is $[n]$. Let $\mathcal{B}(n)$ denote the set of all set partitions of $[n]$. Then $|\mathcal{B}(n)|$ is the nth Bell number B_n. A family $\mathcal{A} \subseteq \mathcal{B}(n)$ is said to be *t-intersecting* if any two elements of $\mathcal{A}$ have at least t blocks in common.

A set partition is called a *k-partition* if it has k blocks. Denote by P_k^n the set of all k-partitions of $[n]$. Further, denote by U_k^n the set of all k-partitions of $[n]$ such that every block has the same size. Two simple constructions of t-intersecting families in P_k^n and U_k^n are:

$$\mathcal{P} = \left\{P \in P_k^n \colon \{1\}, \ldots, \{t\} \in P\right\},$$
$$\mathcal{Q} = \left\{P \in U_k^n \colon [1, c], [c+1, 2c], \ldots, \left[(t-1)c+1, tc\right] \in P\right\},$$

where $[a, b] = \{a, a+1, \ldots, b\}$, $c = n/k$. The maximum size of a t-intersecting family in P_k^n and U_k^n respectively has been determined:

Theorem 1.5. *(Erdős and Székely [6].) Let $n \geqslant k \geqslant t \geqslant 1$. Suppose $\mathcal{A} \subseteq P_k^n$ is t-intersecting. If $n \geqslant n_0(k, t)$, then $|\mathcal{A}| \leqslant |\mathcal{P}|$.*

Theorem 1.6. *(Meagher and Moura [11].) Let $n \geqslant k \geqslant t \geqslant 1$. Suppose $\mathcal{A} \subseteq U_k^n$ is t-intersecting. Let $c = n/k$ be the size of a block in each set partition. If $n \geqslant n_0(k, t)$ or $n \geqslant n_0(c, t)$ when $c \geqslant t+2$, then $|\mathcal{A}| \leqslant |\mathcal{Q}|$ with equality if and only if $\mathcal{A}$ is isomorphic to $\mathcal{Q}$.*

Here, we prove similar results for t-intersecting families in $\mathcal{B}(n)$:

Theorem 1.7. *Let $n \geqslant 2$. Suppose $\mathcal{A} \subseteq \mathcal{B}(n)$ is 1-intersecting. Then $|\mathcal{A}| \leqslant B_{n-1}$ with equality if and only if $\mathcal{A}$ consists of all set partitions with a fixed singleton* (*block of size* 1).

Theorem 1.8. *Let $t \geqslant 2$. Let $\mathcal{A} \subseteq \mathcal{B}(n)$ be t-intersecting. If $n \geqslant n_0(t)$, then $|\mathcal{A}| \leqslant B_{n-t}$ with equality if and only if $\mathcal{A}$ consists of all set partitions with t fixed singletons.*

As in Theorem 1.4, the condition that n being sufficiently large compared to t is also necessary in Theorem 1.8. For example, take $n = 6$ and $t = 2$. Then the family consisting of all set partitions with 2 fixed singletons has $B_4 = 15$ elements while the family consisting of all set partitions with 4 or more singletons has 16 elements.

The rest of the paper is organized as follows. In Section 2, we prove Theorem 1.3 and characterize the case of equality. This is followed by a proof of Theorem 1.4. Next, proofs of Theorems 1.7 and 1.8 are presented in Section 3. An important tool in the study of intersecting families of finite sets is the well-known shifting operation. Our approach uses an analogue of such operation for permutations and set partitions. Our methods are similar to those given in [2].

2. Cycle-intersecting family of permutations

2.1. Fixing operation

We introduce an analogue of shifting operation for permutations and prove some properties which are useful for the study of cycle-intersecting families of permutations. The following operation was first introduced in [2].

Let $i, j \in [n]$, $i \neq j$, and $g \in \mathrm{Sym}([n])$. We define the *ij-fixing* of g to be the permutation ${}_{[ij]}g$ defined as follows:

- if $g(i) \neq j$, then ${}_{[ij]}g = g$,
- if $g(i) = j$, then

$$
{}_{[ij]}g(x) = \begin{cases} i & \text{if } x = i, \\ j & \text{if } x = g^{-1}(i), \\ g(x) & \text{otherwise.} \end{cases}
$$

For example, $g = (2\,3\,1\,4)(5\,6)(7) \in \mathrm{Sym}([7])$. Then ${}_{[12]}g = g$ and ${}_{[14]}g = (1)(2\,3\,4)(5\,6)(7)$. When applying the fixing operation, it is often helpful to think of g in terms of its cycle decomposition. Suppose $i \neq j$ and $g = c_1 \oplus \cdots \oplus c_q$ is the cycle decomposition of g such that the cycle $c_p = (x_1 \cdots x_l)$, $1 \leqslant p \leqslant q$, contains $x_k = i$ and $x_{k+1} = j$ (i.e. $g(i) = j$). Then

$$
{}_{[ij]}g = (x_k) \oplus c_1 \oplus \cdots \oplus c_{p-1} \oplus c'_p \oplus c_{p+1} \oplus \cdots \oplus c_q,
$$

where c'_p denotes the cycle $(x_1 \cdots x_{k-1}\; x_{k+1} \cdots x_l)$.

For a family $\mathcal{A} \subseteq \mathrm{Sym}([n])$, let ${}_{[ij]}\mathcal{A} = \{{}_{[ij]}g\colon\ g \in \mathcal{A}\}$. Given $i, j \in [n]$, $i \neq j$, and a family $\mathcal{A} \subseteq \mathrm{Sym}([n])$, decompose $\mathcal{A}$ as follows:

$$
\mathcal{A} = (\mathcal{A} \setminus \mathcal{A}_{ij}) \cup \mathcal{A}_{ij},
$$

where $\mathcal{A}_{ij} = \{g \in \mathcal{A}\colon\ {}_{[ij]}g \notin \mathcal{A}\}$. Now, define the *$ij$-fixing* of $\mathcal{A}$ to be

$$
\lhd_{ij}(\mathcal{A}) = (\mathcal{A} \setminus \mathcal{A}_{ij}) \cup {}_{[ij]}\mathcal{A}_{ij}.
$$

Clearly the fixing operator $\lhd_{ij}$ preserves the size of the family, i.e. $|\lhd_{ij}(\mathcal{A})| = |\mathcal{A}|$.

Proposition 2.1. *Let $n \geqslant t + 1$. Suppose $\mathcal{A} \subseteq \mathrm{Sym}([n])$ is t-cycle-intersecting. If $\lhd_{ij}(\mathcal{A})$ is the stabilizer of t points, then so is $\mathcal{A}$.*

Proof. If $n = t+1$, then both $\lhd_{ij}(\mathcal{A})$ and $\mathcal{A}$ consist of just the identity permutation. Let $n > t+1$. Suppose $\lhd_{ij}(\mathcal{A})$ is the stabilizer of the points $x_1, \ldots, x_t$. Note that $|\mathcal{A}| = |\lhd_{ij}(\mathcal{A})| = (n-t)! \geqslant 2$. Assume that $\mathcal{A} \neq \lhd_{ij}(\mathcal{A})$. Consider the permutation $g = (x_1) \oplus \cdots \oplus (x_t) \oplus (x_{t+1} \cdots x_n)$ where $\{x_1, \ldots, x_n\} = [n]$. Suppose $g \in \mathcal{A}$ and h is another permutation in $\mathcal{A}$ which contains the cycle $(x_{t+1} \cdots x_n)$. Since h must have at least another $t-1$ cycles in common with g, we must have $h = g$, which is a contradiction. So all permutations in $\mathcal{A}$ must fix $x_1, \ldots, x_t$, i.e. $\mathcal{A}$ has the required form.

Therefore, we may assume that $g \notin \mathcal{A}$. The fact that $g \in \lhd_{ij}(\mathcal{A})$ implies that $g = {}_{[ij]}h$ for some $h \in \mathcal{A}$. Since h has exactly t cycles and $\mathcal{A}$ is t-cycle-intersecting, we deduce that $\mathcal{A} = \{h\}$, which is a contradiction. $\square$

Let $I(n, t)$ denote the set of all t-cycle-intersecting families of permutations of $[n]$.

Proposition 2.2. *Let $i, j \in [n]$, $i \neq j$. Let $\mathcal{A} \in I(n,t)$. Then $\triangleleft_{ij}(\mathcal{A}) \in I(n,t)$.*

Proof. Clearly $\mathcal{A} \setminus \mathcal{A}_{ij}$ and ${}_{[ij]}\mathcal{A}_{ij}$ are t-cycle-intersecting (in fact, ${}_{[ij]}\mathcal{A}_{ij}$ is even $(t+1)$-cycle-intersecting).

Let $g \in \mathcal{A} \setminus \mathcal{A}_{ij}$ and $h \in {}_{[ij]}\mathcal{A}_{ij}$. We will show that g and h have at least t cycles in common. Let $h' \in \mathcal{A}_{ij}$ such that ${}_{[ij]}h' = h$.

Case I. $g(i) = j$.

The fact that $g \notin \mathcal{A}_{ij}$ implies that $g' = {}_{[ij]}g \in \mathcal{A}$. Then g' and h' (and hence g and h') must have at least t common cycles which do not involve i or j. These t cycles also belong to g and h.

Case II. $g(i) \neq j$.

Clearly g and h' have at least t common cycles which do not involve i or j. These cycles also belong to h. □

A family $\mathcal{A}$ of permutations is said to be *compressed* if for any $i, j \in [n]$, $i \neq j$, we have $\triangleleft_{ij}(\mathcal{A}) = \mathcal{A}$.

Proposition 2.3. *Given a family $\mathcal{A} \in I(n,t)$, by repeatedly applying fixing operations, we eventually obtain a compressed family $\mathcal{A}^* \in I(n,t)$ with $|\mathcal{A}^*| = |\mathcal{A}|$.*

Proof. For a permutation g, let $\mathrm{fix}(g)$ denote the number of fixed points of g, that is the number of cycles of length 1 in g. For a family $\mathcal{A}$ of permutations, let $w(\mathcal{A}) = \sum_{g \in \mathcal{A}} \mathrm{fix}(g)$.

We construct a sequence of families $\mathcal{A}_0 = \mathcal{A}, \mathcal{A}_1, \ldots$ as follows: if there exist $i, j \in [n]$, $i \neq j$, such that $\triangleleft_{ij}(\mathcal{A}_k) \neq \mathcal{A}_k$, then set $\mathcal{A}_{k+1} = \triangleleft_{ij}(\mathcal{A}_k)$.

We observe that $w(\mathcal{A}_0) < w(\mathcal{A}_1) < \cdots$. Since this sequence cannot continue indefinitely, there must exist a positive integer q such that $\triangleleft_{ij}(\mathcal{A}_q) = \mathcal{A}_q$ for all $i, j \in [n]$, $i \neq j$. Moreover $|\mathcal{A}_q| = |\mathcal{A}|$ and $\mathcal{A}_q \in I(n,t)$ by Proposition 2.2. Therefore $\mathcal{A}^* = \mathcal{A}_q$ is the required family. □

For a family $\mathcal{A}$, let $\mathrm{Fix}(\mathcal{A}) = \{\mathrm{Fix}(g)\colon\ g \in \mathcal{A}\}$, where $\mathrm{Fix}(g) = \{x\colon\ g(x) = x\}$.

Proposition 2.4. *If $\mathcal{A} \in I(n,t)$ is compressed, then* $\mathrm{Fix}(\mathcal{A})$ *is a t-intersecting family of subsets of $[n]$.*

Proof. Assume, for a contradiction, that there exist $g, h \in \mathcal{A}$ such that $|\mathrm{Fix}(g) \cap \mathrm{Fix}(h)| < t$. Since $g, h \in \mathcal{A} \in I(n,t)$, there are at least $t - |\mathrm{Fix}(g) \cap \mathrm{Fix}(h)|$ common cycles in g and h which do not involve any points in $\mathrm{Fix}(g) \cup \mathrm{Fix}(h)$. Let $c_1, \ldots, c_s$ be these cycles, where $s \geqslant t - |\mathrm{Fix}(g) \cap \mathrm{Fix}(h)|$. Note that these cycles have length at least 2. For $i = 1, \ldots, s$, we may assume that $x_i \mapsto y_i$ occurs in the cycle c_i, where $x_1, y_1, \ldots, x_s, y_s$ are all distinct.

The idea is to use the fixing operation $[x_i y_i]$ to destroy the cycle c_i which involves them in h so that the resulting permutation and g would no longer have c_i in common. More precisely, the permutation $h^* = {}_{[x_s y_s]}(\cdots({}_{[x_1 y_1]}(h))\cdots)$ and g have less than t common cycles, contradicting the fact that both g and h^* belong to $\mathcal{A}$ (since $\mathcal{A}$ is compressed). □

2.2. Proof of Theorem 1.4

We require the following well-known results in extremal set theory.

Proposition 2.5 *(LYM Inequality). Let $\mathcal{F}$ be an antichain of subsets of $[n]$. Then*

$$\sum_{F\in\mathcal{F}} |F|!\big(n-|F|\big)! \leqslant n!.$$

Lemma 2.6. *If $\mathcal{F}$ is an antichain of subsets of $[n]$ such that $|F|\geqslant k$ for all $F\in\mathcal{F}$, then*

$$\sum_{F\in\mathcal{F}} \big(n-|F|\big)! \leqslant n!/k!.$$

Proof.

$$\sum_{F\in\mathcal{F}} \big(n-|F|\big)! \leqslant \sum_{F\in\mathcal{F}} \frac{|F|!}{k!}\big(n-|F|\big)! \leqslant n!/k!,$$

by applying the LYM Inequality. □

We proceed to prove the main theorem in this section:

Theorem 2.7. *Let $\mathcal{A}\subseteq \mathrm{Sym}([n])$ such that $|\mathcal{A}|\geqslant (n-t)!$ and $\mathrm{Fix}(\mathcal{A})$ is a t-intersecting family of subsets of $[n]$. Then, for every t there exists $n_0(t)=O(t^2)$ such that for all $n\geqslant n_0(t)$, $\mathcal{A}$ is the stabilizer of t fixed points.*

Proof. Let $\mathcal{F}$ be the set of all minimal elements in $\mathrm{Fix}(\mathcal{A})$ partially ordered by inclusion. Then

$$|\mathcal{A}| \leqslant \sum_{F\in\mathcal{F}} \big(n-|F|\big)!,$$

where $\mathcal{F}$ is a t-intersecting antichain with $|F|\geqslant t$ for all $F\in\mathcal{F}$.

Clearly if $\mathcal{F}$ contains an element of size t or $|\bigcap_{F\in\mathcal{F}} F|\geqslant t$, then the theorem holds. Also, note that $|F|\leqslant n-2$ since $\mathcal{F}$ is an antichain and no permutation fixes exactly $n-1$ points. Thus, we may assume that $t+1\leqslant |F|\leqslant n-2$ for all $F\in\mathcal{F}$ and $|\bigcap_{F\in\mathcal{F}} F|<t$. For a contradiction, we shall prove that $|\mathcal{A}|<(n-t)!$ if $n\geqslant c_0t^2$ for some absolute constant c_0. Consider the following cases:

Case I. $|F|\geqslant t+2$ for all $F\in\mathcal{F}$.

Let $\mathcal{F}_i=\mathcal{F}\cap\binom{[n]}{i}$ denote the set of elements of size i in $\mathcal{F}$. Then, applying Lemma 2.6 to $\bigcup_{i>\lfloor\frac{n}{t+1}+t-1\rfloor}\mathcal{F}_i$ and the Erdős–Ko–Rado theorem (Theorem 1.1) to each $\mathcal{F}_i$ with $i\leqslant\lfloor\frac{n}{t+1}+t-1\rfloor$, we have

$$\begin{aligned}
|\mathcal{A}| &\leqslant \sum_{\substack{F\in\mathcal{F}\\ t+2\leqslant|F|\leqslant\lfloor\frac{n}{t+1}+t-1\rfloor}} \big(n-|F|\big)! + \sum_{\substack{F\in\mathcal{F}\\ |F|>\lfloor\frac{n}{t+1}+t-1\rfloor}} \big(n-|F|\big)!\\
&= \sum_{t+2\leqslant i\leqslant\lfloor\frac{n}{t+1}+t-1\rfloor} |\mathcal{F}_i|(n-i)! + \sum_{\substack{F\in\mathcal{F}\\ |F|>\lfloor\frac{n}{t+1}+t-1\rfloor}} \big(n-|F|\big)!
\end{aligned}$$

$$\leqslant \sum_{t+2\leqslant i\leqslant \lfloor \frac{n}{t+1}+t-1\rfloor} \binom{n-t}{i-t}(n-i)! + \frac{n!}{(\lfloor \frac{n}{t+1}+t-1\rfloor+1)!}.$$

Consequently,

$$|\mathcal{A}| \leqslant \sum_{t+2\leqslant i\leqslant \lfloor \frac{n}{t+1}+t-1\rfloor} \frac{(n-t)!}{(i-t)!} + \frac{n!}{(\lfloor \frac{n}{t+1}+t-1\rfloor+1)!}.$$

Since $\sum_{i=t+2}^{\lfloor \frac{n}{t+1}+t-1\rfloor} \frac{1}{(i-t)!} < e-2 < 0.75$ (where $e = 2.718\ldots$ is the base of the natural logarithm), it suffices to show that

$$\frac{n!}{(\lfloor \frac{n}{t+1}+t-1\rfloor+1)!} < 0.2(n-t)!, \tag{1}$$

for $n \geqslant c_0t^2$ for some absolute constant c_0. Indeed, (1) holds if

$$n^t < 0.2\left(\left\lfloor \frac{n}{t+1}+t-1\right\rfloor+1\right)!.$$

Using the fact that $n! > (n/e)^n$, the above holds if

$$n^t < 0.2\left(\frac{n}{e(t+1)}\right)^{\frac{n}{t+1}},$$

which, after a simple calculation, holds for all $n \geqslant c_0t^2$ for some absolute constant c_0.

Case II. $\mathcal{F}$ contains an element of size $t+1$.

Let $\mathcal{F}_{t+1} = \{F \in \mathcal{F}\colon |F| = t+1\}$ be the collection of all the sets of size $t+1$ in $\mathcal{F}$. Since $\mathcal{F}_{t+1}$ is t-intersecting, there are two possibilities: either all the sets in $\mathcal{F}_{t+1}$ contain t fixed points (see Subcase (i) below) or there exist three sets F_1, F_2 and F_3 such that $F_1 \cap F_2 \nsubseteq F_3$. In the later, F_3 must contain the symmetric difference $F_1 \triangle F_2$, and since $|F_3 \cap F_i| \geqslant t$ for $i = 1, 2$, F_3 must take the form $(F_1 \cup F_2) \setminus \{x\}$ for some $x \in F_1 \cap F_2$. Indeed, all sets in $\mathcal{F}_{t+1}$ other than F_1 and F_2 must also have this form (see Subcase (ii) below).

Without loss of generality, we may consider the following subcases:

Subcase (i). $\mathcal{F}_{t+1}$ consists of $(t+1)$-sets of the following form:

$$\begin{aligned}\mathcal{F}_{t+1}\colon\quad &\bigl\{\{1,2,3,\ldots,t-1,t,(t+1)\},\\ &\{2,3,4,\ldots,t,(t+1),(t+2)\},\\ &\{2,3,4,\ldots,t,(t+1),(t+3)\},\\ &\ldots\\ &\{2,3,4,\ldots,t,(t+1),(t+c)\}\bigr\}\end{aligned}$$

for some $c \in \{1, 2, \ldots, n-t\}$.

Suppose $F \in \mathcal{F} \setminus \mathcal{F}_{t+1}$. Consider a permutation $g \in \mathcal{A}$ such that $\mathrm{Fix}(g) \supseteq F$.

If $2 \notin F$, then

$$F \supseteq \{1,3,4,\ldots,t,t+1,\ldots,t+c\} = A, \tag{2}$$

so that $|A| = t + c - 1$. Hence any permutation $g \in \mathcal{A}$ such that $\mathrm{Fix}(g) \supseteq F$ must lie in the set of all permutations h with $h(a) = a$ for all $a \in A$. There are at most $(n - (t + c - 1))!$ such permutations.

If $2 \in F$ but $1 \notin F$, then $F \supseteq \{2, 3, \dots, t, t+1\}$ and $t+2, t+3, \dots, t+c \notin F$ (since $\mathcal{F}$ is an antichain). Since $|F| \geqslant t + 2$, F contains two more points a and b where

$$a, b \notin \{1, 2, \dots, t, t+1, \dots, t+c\}. \tag{3}$$

There are $\binom{n-(t+c)}{2}$ choices for such $\{a, b\}$, and so there are at most $\binom{n-(t+c)}{2}(n - (t + 2))!$ such permutations.

If $2 \in F$ but $i \notin F$ for some $i \in \{3, 4, \dots, t+1\}$, then

$$F \supseteq \{1, 2, \dots, t+c\} \setminus \{i\}, \tag{4}$$

and thus $|F| \geqslant t + c - 1$. As there are $(t - 1)$ choices for i, there are at most $(t - 1)(n - (t + c - 1))!$ such permutations.

Therefore,

$$|\mathcal{A}| \leqslant c\big(n - (t+1)\big)! + \big(n - (t + c - 1)\big)! + \binom{n - (t+c)}{2}\big(n - (t+2)\big)! \\ + (t-1)\big(n - (t + c - 1)\big)!,$$

where the first term on the right-hand side is the upper bound for the number of permutations whose fixed-point sets contain some element of $\mathcal{F}_{t+1}$. Meanwhile, the rest of the terms come from the possibilities discussed in the preceding three paragraphs.

After simplifications, we have

$$|\mathcal{A}| = c\big(n - (t+1)\big)! + t\big(n - (t + c - 1)\big)! + \binom{n - (t+c)}{2}\big(n - (t+2)\big)!. \tag{5}$$

Suppose that $c = n - t$. Let $F \in \mathcal{F} \setminus \mathcal{F}_{t+1}$. If $2 \notin F$, then by (2), $|F| > n - 2$, which is impossible. So $2 \in F$. But this is again impossible by (3) and (4). Hence $\mathcal{F} = \mathcal{F}_{t+1}$. Consequently, $|\bigcap_{F \in \mathcal{F}} F| = |\bigcap_{F \in \mathcal{F}_{t+1}} F| \geqslant t$, contradicting our assumption.

So we may assume that $1 \leqslant c \leqslant n - t - 1$. Suppose for a moment that $2 \leqslant c \leqslant n - t - 1$. Choosing $n > 5t$ and using (5), it is readily checked that $|\mathcal{A}| < (n - t)!$.

We are left to consider the case when $c = 1$. Using the argument in Case I, we deduce that for $n \geqslant O(t^2)$,

$$|\mathcal{A}| \leqslant (n - t - 1)! + \sum_{\substack{F \\ |F| \geqslant t+2}} \big(n - |F|\big)! \leqslant (n - t - 1)! + 0.95(n - t)! < (n - t)!,$$

Subcase (ii). $\mathcal{F}_{t+1}$ consists of $(t + 1)$-sets of the following form:

$$\begin{aligned} \mathcal{F}_{t+1}\colon \quad & \big\{\{1, 2, 3, \dots, t-1, t, t+1\}, \\ & \{1, 2, 3, \dots, t, t+1, t+2\} \setminus \{1\}, \\ & \{1, 2, 3, \dots, t, t+1, t+2\} \setminus \{2\}, \\ & \dots \\ & \{1, 2, 3, \dots, t, t+1, t+2\} \setminus \{c\}\big\}, \end{aligned}$$

for some $c \in \{1, 2, \dots, t+1\}$.

Clearly $|\mathcal{F}_{t+1}| = c + 1$. Suppose $c \geqslant 2$ and let $F \in \mathcal{F} \setminus \mathcal{F}_{t+1}$. Since $\mathcal{F}$ is a t-intersecting antichain, $F \supseteq \{1, 2, \ldots, t, t+1, t+2\} \setminus \{i\}$ for some $i \in \{3, \ldots, t+1\}$ and so $|F| \geqslant t + 1$. Hence, for $n > 3t + 1$,

$$\begin{aligned} |\mathcal{A}| &\leqslant (c+1)\big(n-(t+1)\big)! + (t-1)\big(n-(t+1)\big)! \\ &\leqslant (c+t)(n-t-1)! < (n-t)!. \end{aligned}$$

So $c = 1$. Again, by the argument in Case I, we have, for $n \geqslant O(t^2)$,

$$|\mathcal{A}| \leqslant 2(n-t-1)! + 0.95(n-t)! < (n-t)!$$

This concludes the proof. □

Corollary 2.8. *Suppose $\mathcal{A} \subseteq \mathrm{Sym}([n])$ and any three elements of $\mathcal{A}$ have at least t positions in common. Then for $n \geqslant n_0(t) = O(t^2)$, $|\mathcal{A}| \leqslant (n-t)!$ with equality if and only if $\mathcal{A}$ is a coset of the stabilizer of t points.*

Proof. Without loss of generality, we may assume that the identity permutation $\mathrm{Id} \in \mathcal{A}$ and $|\mathcal{A}| \geqslant (n-t)!$. Let $\mathrm{Id} \neq g, h \in \mathcal{A}$, $g \neq h$. Then $|\mathrm{Fix}(g) \cap \mathrm{Fix}(h)| = |\{x\colon\ g(x) = h(x) = \mathrm{Id}(x) = x\}| \geqslant t$. The result holds by Theorem 2.7. □

Proof of Theorem 1.4. By Proposition 2.3, repeated application of fixing operations starting from $\mathcal{A}$ yields a compressed t-cycle-intersecting $\mathcal{A}^*$ with $|\mathcal{A}^*| = |\mathcal{A}|$. The result now follows immediately from Theorem 2.7, Propositions 2.1 and 2.4. □

3. Intersecting family of set partitions

3.1. Splitting operation

Let $i, j \in [n]$, $i \neq j$, and $P \in \mathcal{B}(n)$. Denote by P_i the block of P which contains i. We define the *ij-split* of P to be the following set partition:

$$s_{ij}(P) = \begin{cases} P \setminus \{P_i\} \cup \big\{\{i\}, P_i \setminus \{i\}\big\} & \text{if } j \in P_i, \\ P & \text{otherwise.} \end{cases}$$

For a family $\mathcal{A}$ of set partitions, let $s_{ij}(\mathcal{A}) = \{s_{ij}(P)\colon\ P \in \mathcal{A}\}$. Any family $\mathcal{A}$ of set partitions can be decomposed with respect to given $i, j \in [n]$ as follows:

$$\mathcal{A} = (\mathcal{A} \setminus \mathcal{A}_{ij}) \cup \mathcal{A}_{ij},$$

where $\mathcal{A}_{ij} = \{P \in \mathcal{A}\colon\ s_{ij}(P) \notin \mathcal{A}\}$. Define the *ij-splitting* of $\mathcal{A}$ to be the family

$$S_{ij}(\mathcal{A}) = (\mathcal{A} \setminus \mathcal{A}_{ij}) \cup s_{ij}(\mathcal{A}_{ij}).$$

Let $I_s(n,t)$ denote the set of all t-intersecting families of set partitions of $[n]$. Say that a family of set partitions of $[n]$ is *trivially* t-intersecting if it consists of all set partitions containing t fixed singletons. It is readily checked that the following statements hold for set partitions. Since the proofs are similar to that of Propositions 2.1, 2.2, 2.3 and 2.4, respectively, we omit some of the details.

Proposition 3.1. *Let $n \geqslant t + 1$. Suppose $\mathcal{A} \subseteq \mathcal{B}(n)$ is t-intersecting. If $S_{ij}(\mathcal{A})$ is trivially t-intersecting, then $\mathcal{A}$ is trivially t-intersecting.*

Proof. We may assume that $n > t + 1$ and $S_{ij}(\mathcal{A})$ consists of all set partitions containing t singletons $\{x_1\}, \ldots, \{x_t\}$. We may also assume, for a contradiction, that the set partition $P = \{\{x_1\}, \{x_2\}, \ldots, \{x_t\}, \{x_{t+1}, \ldots, x_n\}\} \notin \mathcal{A}$ where $\{x_1, \ldots, x_n\} = [n]$. Then $P = s_{ij}(Q)$ for some $Q \in \mathcal{A}$. So Q must have exactly t blocks. Since $\mathcal{A}$ is t-intersecting, we deduce that $\mathcal{A} = \{Q\}$, contradicting the fact that $|\mathcal{A}| = |S_{ij}(\mathcal{A})| > 1$. □

Proposition 3.2. *Let $i, j \in [n]$, $i \neq j$. Let $\mathcal{A} \in I_s(n,t)$. Then $S_{ij}(\mathcal{A}) \in I_s(n,t)$.*

Proof. Let $P \in \mathcal{A} \setminus \mathcal{A}_{ij}$ and $Q \in s_{ij}(\mathcal{A})$. We want to show that P and Q have at least t blocks in common. Let $Q' \in \mathcal{A}_{ij}$ such that $s_{ij}(Q') = Q$. If $j \in P_i$, then $s_{ij}(P) \in \mathcal{A}$ (since $P \notin \mathcal{A}_{ij}$) and so $s_{ij}(P)$ and Q' have at least t common blocks which do not involve i or j. These blocks also belong to P and Q. If $j \notin P_i$, then P and Q' would have t common blocks which belong to P and Q. □

A family $\mathcal{A}$ of set partitions is *compressed* if for any $i, j \in [n]$, $i \neq j$, we have $S_{ij}(\mathcal{A}) = \mathcal{A}$. For a set partition P, let $\sigma(P) = \{x\colon \{x\} \in P\}$ denote the union of its singletons (block of size 1). For a family $\mathcal{A}$ of set partitions, let $\sigma(\mathcal{A}) = \{\sigma(P)\colon P \in \mathcal{A}\}$.

Proposition 3.3. *Given a family $\mathcal{A} \in I_s(n,t)$, by repeatedly applying splitting operations, we eventually obtain a compressed family $\mathcal{A}^* \in I_s(n,t)$ with $|\mathcal{A}^*| = |\mathcal{A}|$.*

Proof. For a family $\mathcal{A}$ of set partitions, let $w_s(\mathcal{A}) = \sum_{P \in \mathcal{A}} |\sigma(P)|$. It is enough to observe that if $S_{ij}(\mathcal{A}) \neq \mathcal{A}$, then $w_s(\mathcal{A}) < w_s(S_{ij}(\mathcal{A}))$. □

Proposition 3.4. *If $\mathcal{A} \in I_s(n,t)$ is compressed, then $\sigma(\mathcal{A})$ is a t-intersecting family of subsets of $[n]$.*

Proof. Assume, for a contradiction, that there exist $P, Q \in \mathcal{A}$ such that $|\sigma(P) \cap \sigma(Q)| < t$. Suppose there are $s \geqslant t - |\sigma(P) \cup \sigma(Q)|$ common blocks of P and Q (each of size at least 2), say $B_1, \ldots, B_s$, which are disjoint from $\sigma(P) \cup \sigma(Q)$. Fix two distinct points x_i, y_i from each B_i. Then $Q^* = s_{x_s y_s}(\cdots(s_{x_1 y_1}(Q))\cdots) \in \mathcal{A}$ since $\mathcal{A}$ is compressed. But both P and Q^* have less than t common blocks, which is a contradiction. □

3.2. Proof of Theorem 1.7

For a proof of Theorem 1.7, we require a known solution to a combinatorial optimization problem for set systems. Let w be a function from the set $2^{[n]}$ of all subsets of $[n]$ into the reals, and for any family $\mathcal{A}$ of subsets, define $w(\mathcal{A})$ by

$$w(\mathcal{A}) = \sum_{A \in \mathcal{A}} w(A).$$

Then, given a collection L of families of subsets of $[n]$, a general problem is to determine which $\mathcal{A} \in L$ maximizes $w(\mathcal{A})$ over L. There are some interesting conjectures on these problems, to which [1,7,8] provides a good introduction. Here, we are interested in the special case $L = M_n$, where M_n denote the set of all maximal (with respect to inclusion) 1-intersecting families of subsets of $[n]$. Note that every element of M_n contains $[n]$ and has exactly 2^{n-1} subsets of $[n]$. In particular, the following result is useful.

Theorem 3.5. *(See [1,8].) Suppose w is a function from $2^{[n]}$ into the reals such that $w(A) > w(B)$ if and only if $|A| < |B|$. Then $\mathcal{A}$ maximizes $w(\mathcal{A})$ over M_n if and only if $\mathcal{A}$ consists of all subsets of $[n]$ containing a fixed $x \in [n]$.*

Proof of Theorem 1.7. By repeated applications of the splitting operations, we obtain a compressed family $\mathcal{A}^*$ with $|\mathcal{A}^*| = |\mathcal{A}|$. By Proposition 3.4, $\sigma(\mathcal{A}^*)$ is 1-intersecting. So $\sigma(\mathcal{A}^*) \subseteq \mathcal{F}$ for some $\mathcal{F} \in M_n$.

Let w be the function from $2^{[n]}$ into the reals defined by $w(A) = |\{P \in \mathcal{B}(n)\colon \sigma(P) = A\}|$ for every proper subset $A \subseteq [n]$, and $w([n]) = -1$. (For example, $w(\emptyset)$ is the number of set partitions of $[n]$ which are singleton-free.) Clearly, $w(A) > w(B)$ if and only if $|A| < |B|$. Since $|\mathcal{A}^*| \leqslant \sum_{A \in \mathcal{F}} w(A)$, it follows immediately from Theorem 3.5 that the right-hand side of the preceding inequality is maximized if and only if $\mathcal{F}$ consists of all subsets containing a fixed point. Therefore, $|\mathcal{A}^*| \leqslant B_{n-1}$ with equality if and only if $\mathcal{A}^*$ consists of all set partitions containing a fixed singleton.

It remains to show that $\mathcal{A}$ has the same structure as $\mathcal{A}^*$. This follows immediately from Proposition 3.1. $\square$

3.3. Proof of Theorem 1.8

Let $\tilde{B}_n$ denote the number of singleton-free set partitions of $[n]$. Note that the sequence $\{\tilde{B}_n\}$ obeys the recurrence:

$$\tilde{B}_{n+1} = \sum_{k=0}^{n-1} \binom{n}{i} \tilde{B}_i. \tag{6}$$

An easy consequence of (6) is that for any given $r > 0$,

$$\frac{\tilde{B}_{n+2}}{\tilde{B}_n} > r, \tag{7}$$

for all $n \geqslant r$.

We may assume that $\mathcal{A} \subseteq \mathcal{B}(n)$ is a t-intersecting family of maximum size. By Propositions 3.1, 3.3 and 3.4, we may further assume that $\mathcal{A}$ is compressed and $\sigma(\mathcal{A})$ is not trivially t-intersecting, i.e. $|\bigcap_{F \in \sigma(\mathcal{A})} F| < t$. We want to show that $|\mathcal{A}| < B_{n-t}$ for sufficiently large n.

Clearly, $|\mathcal{A}| \leqslant \sum_{F \in \sigma(\mathcal{A})} \tilde{B}_{n-|F|}$. Let $\mathcal{F}_k = \sigma(\mathcal{A}) \cap \binom{[n]}{k}$. Applying the Erdős–Ko–Rado theorem to $\mathcal{F}_k$ for each $k \leqslant \lfloor \frac{n}{t+1} + t - 1 \rfloor$, we have

$$|\mathcal{A}| \leqslant \sum_{k=t+1}^{\lfloor \frac{n}{t+1} + t - 1 \rfloor} \binom{n-t}{k-t} \tilde{B}_{n-k} + \sum_{k=\lfloor \frac{n}{t+1} + t - 1 \rfloor + 1}^{n} \binom{n}{k} \tilde{B}_{n-k}.$$

Note that for $k \geqslant \lfloor \frac{n}{t+1} + t - 1 \rfloor + 1$ and sufficiently large n,

$$\frac{\binom{n}{k}}{\binom{n-t}{k-t}} < \left(\frac{n}{\lfloor \frac{n}{t+1} + t - 1 \rfloor + 1 - t} \right)^t < (t+2)^t = C + 1,$$

where C is a constant depending only on t. Hence,

$$\begin{aligned}
|\mathcal{A}| &\leqslant \sum_{k=t+1}^{\lfloor \frac{n}{t+1}+t-1\rfloor} \binom{n-t}{k-t}\tilde{B}_{n-k} + (C+1)\left(\sum_{k=\lfloor \frac{n}{t+1}+t-1\rfloor+1}^{n} \binom{n-t}{k-t}\tilde{B}_{n-k}\right) \\
&= B_{n-t} - \tilde{B}_{n-t} + C\cdot\left(\sum_{k=\lfloor \frac{n}{t+1}+t-1\rfloor+1}^{n} \binom{n-t}{k-t}\tilde{B}_{n-k}\right) \\
&< B_{n-t} - \tilde{B}_{n-t} + Cn\binom{n}{\frac{n}{2}}\tilde{B}_{\lfloor \frac{nt}{t+1}\rfloor}.
\end{aligned}$$

Therefore, it remains to show that

$$\tilde{B}_{n-t} > Cn\binom{n}{\frac{n}{2}}\tilde{B}_{\lfloor \frac{nt}{t+1}\rfloor},$$

that is

$$\frac{\tilde{B}_{n-t}}{n\binom{n}{\frac{n}{2}}\tilde{B}_{\lfloor \frac{nt}{t+1}\rfloor}} > C.$$

In general, $\binom{n}{\frac{n}{2}} \leqslant (\frac{en}{\frac{n}{2}})^{\frac{n}{2}} = (\sqrt{2e}\,)^n$. Using the relation (7), we may choose n large enough so that

$$\frac{\tilde{B}_{\lfloor \frac{nt}{t+1}\rfloor+2}}{\tilde{B}_{\lfloor \frac{nt}{t+1}\rfloor}} > (2\sqrt{2e}\,)^{2t+4},$$

and by setting $u = \lfloor \frac{1}{2}(n-t-\lfloor \frac{nt}{t+1}\rfloor)\rfloor$,

$$\begin{aligned}
\frac{\tilde{B}_{n-t}}{n\binom{n}{\frac{n}{2}}\tilde{B}_{\lfloor \frac{nt}{t+1}\rfloor}} &\geqslant \frac{1}{n(\sqrt{2e}\,)^n} \times \frac{\tilde{B}_{\lfloor \frac{nt}{t+1}\rfloor+2u}}{\tilde{B}_{\lfloor \frac{nt}{t+1}\rfloor+2u-2}} \times \cdots \times \frac{\tilde{B}_{\lfloor \frac{nt}{t+1}\rfloor+4}}{\tilde{B}_{\lfloor \frac{nt}{t+1}\rfloor+2}} \times \frac{\tilde{B}_{\lfloor \frac{nt}{t+1}\rfloor+2}}{\tilde{B}_{\lfloor \frac{nt}{t+1}\rfloor}} \\
&> \frac{(2\sqrt{2e}\,)^{(2t+4)u}}{n(\sqrt{2e}\,)^n} > \frac{(2\sqrt{2e}\,)^n}{n(\sqrt{2e}\,)^n} = \frac{2^n}{n} > C,
\end{aligned}$$

as desired.

4. A concluding remark

In principle, having a good approximation for Bell numbers $\{B_n\}$ and using an analysis of the structure of $\sigma(\mathcal{A})$ similar to that given in the proof of Theorem 2.7, one should be able to obtain a better estimation for the constant $n_0(t)$ in Theorem 1.8. However, we have avoided this as it seems harder to deal with B_n in such a calculation.

Acknowledgments

We thank the anonymous referees for their comments that helped us make several improvements to this paper.

References

[1] R. Ahlswede, G.O.H. Katona, Contributions to the geometry of Hamming spaces, Discrete Math. 17 (1977) 1–22.

[2] P.J. Cameron, C.Y. Ku, Intersecting families of permutations, European J. Combin. 24 (2003) 881–890.

[3] M. Deza, P. Frankl, Erdős–Ko–Rado theorem—22 years later, SIAM J. Algebr. Discrete Methods 4 (4) (1983) 419–431.

[4] M. Deza, P. Frankl, On the maximum number of permutations with given maximal or minimal distance, J. Combin. Theory Ser. A 22 (1977) 352–360.

[5] P. Erdős, C. Ko, R. Rado, Intersection theorems for systems of finite sets, Quart. J. Math. Oxford 12 (1961) 313–320.

[6] P.L. Erdős, L.A. Székely, Erdős–Ko–Rado theorems of higher order, in: I. Althöfer, Ning Cai, G. Dueck, L. Khachatrian, M.S. Pinsker, A. Sarközy, I. Wegener, Zhen Zhang (Eds.), Numbers, Information and Complexity, Kluwer Academic, 2000, pp. 117–124.

[7] P.C. Fishburn, Combinatorial optimization problems for systems of subsets, SIAM Rev. 30 (4) (1988) 578–588.

[8] P.C. Fishburn, P. Frankl, D. Freed, J.C. Lagarias, A.M. Odlyzko, Probabilities for intersecting systems and random subsets of finite sets, SIAM J. Algebr. Discrete Methods 7 (1986) 73–79.

[9] P. Frankl, The Erdős–Ko–Rado theorem is true for $n = ckt$, Col. Soc. Math. J. Bolyai 18 (1978) 365–375.

[10] B. Larose, C. Malvenuto, Stable sets of maximal size in Kneser-type graphs, European J. Combin. 25 (2004) 657–673.

[11] K. Meagher, L. Moura, Erdős–Ko–Rado theorems for uniform set-partition systems, Electron. J. Combin. 12 (2005) R40.

[12] R.M. Wilson, The exact bound in the Erdős–Ko–Rado theorem, Combinatorica 4 (1984) 247–257.