Analysis of Algorithms: Solutions 1

```
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                     X
number of            X
homeworks            X
                  X  X
               X  X  X
               X  X  X
         X  X  X  X  X
      X  X  X  X  X  X
      ----------------
      5  6  7  8  9  10
           grades
```

The histogram shows the distribution of grades, from 0 to 10.

Problem 1

Let $A[1..n]$ be a sorted array of n distinct numbers. Write an efficient algorithm BINARY-SEARCH(A, n, k) that finds a given value k in the array $A[1..n]$. It should return the index of the found element; if the array does not include k, the algorithm should return 0.

```
BINARY-SEARCH(A, n, k)
p ← 1
r ← n
while p < r
    do q = ⌊(p + r)/2⌋
       if k ≤ A[q]
          then r ← q
          else p ← q + 1
if k = A[p]
   then return p
   else return 0
```

Problem 2

(a) $1 + 2 + 3 + 4 + ... + n = \frac{n \cdot (n+1)}{2}$.

We use a proof by induction. Clearly, the equality holds for $n = 1$; we now show that, if it holds for n, then it also holds for $n + 1$:

$$\begin{aligned} 1 + 2 + ... + n + (n+1) &= (1 + 2 + ... + n) + (n + 1) \\ &= \frac{n \cdot (n+1)}{2} + (n+1) \\ &= \frac{n \cdot (n+1) + 2 \cdot (n+1)}{2} \\ &= \frac{(n+2) \cdot (n+1)}{2} \\ &= \frac{(n+1) \cdot ((n+1)+1)}{2} \end{aligned}$$

(b) $1 + x + x^2 + x^3 + ... + x^n = \frac{x^{n+1}-1}{x-1}$ (where $x \neq 1$).

We observe that the equality holds for $n = 1$, and apply induction to "step" from n to $n+1$:

$$\begin{aligned} 1 + x + ... + x^n + x^{n+1} &= \frac{x^{n+1} - 1}{x - 1} + x^{n+1} \\ &= \frac{x^{n+1} - 1 + x^{n+1} \cdot (x-1)}{x - 1} \\ &= \frac{x^{n+1} - 1 + x^{n+2} - x^{n+1}}{x - 1} \\ &= \frac{x^{(n+1)+1} - 1}{x - 1} \end{aligned}$$