Constructive Logic (15-317), Spring 2023
Assignment 1: Natural Deduction (50 points)

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Due: February 2, 2023, 11:59 pm

Welcome to Constructive Logic Spring 2023!

This assignment will have a written portion and a coding portion. You will submit both portions through Gradescope.

We recommend that you typeset your written solutions. Most students use LaTeX, but other software is acceptable. If you choose not to typeset your solutions, be aware that your handwriting must be **legible**.

For the coding portion you will use Dcheck. You can find documentation on Dcheck at `cs.cmu.edu/~crary/dcheck/dcheck.pdf` and a sample file at `cs.cmu.edu/~crary/dcheck/example.deriv`.

1 Natural Deduction (18 points)

Using Dcheck, give derivations of each of the following judgements.

Task 1. (4 points)

$$((A \wedge B) \supset C) \supset (A \supset (B \supset C)) \textit{ true}$$

Task 2. (4 points)

$$((A \vee B) \wedge (A \supset B)) \supset B \textit{ true}$$

Task 3. (5 points)

$$((\neg A) \wedge (\neg B)) \supset (\neg(A \vee B)) \textit{ true}$$

Task 4. (5 points)

$$(\neg(A \vee B)) \supset ((\neg A) \wedge (\neg B)) \textit{ true}$$

2 Harmony (30 points)

Consider a new connective $\spadesuit$ (LaTeX: `\spadesuit`).

$$\dfrac{A\spadesuit B \; true \qquad \begin{array}{c} \dfrac{}{A \; true}\,u \quad \dfrac{}{B \; true}\,v \\ \vdots \\ C \; true \end{array}}{C \; true}\,\spadesuit E^{u,v}$$

Normally when we define a connective, we write its introduction rules first, but for this task we will go in the opposite direction.

Task 5. (2 points) Write a correct set of (zero or more) introduction rules for this connective.

Task 6. (6 points) Show that the connective is locally sound for your choice of introduction rules.

Task 7. (6 points) Show that the connective is locally complete for your choice of introduction rules.

Now we consider another new connective $\triangle$ (LaTeX: `\triangle`).

$$\dfrac{\begin{array}{c} \dfrac{}{A \; true}\,u \\ \vdots \\ \bot \; true \end{array}}{A \triangle B \; true}\,\triangle I_1^u \qquad \dfrac{\begin{array}{c} \dfrac{}{B \; true}\,v \\ \vdots \\ \bot \; true \end{array}}{A \triangle B \; true}\,\triangle I_2^v \qquad \dfrac{A \triangle B \; true \quad A \; true \quad B \; true}{C \; true}\,\triangle E$$

Task 8. (8 points) Is this connective locally sound? If so, provide the local reduction. If not, provide a derivation to show that this connective can prove judgements that are not provable, for instance $\bot$ *true*.

Task 9. (8 points) Is this connective locally complete? If so, provide the local expansion. If not, briefly explain why local expansion fails.

3 Constructive Mathematics (2 points)

In this class we will mostly be exploring constructive logic in a formal way, over an abstract problem domain. In this problem we will explore constructivity in informal proofs of practical mathematics.

When we ask for a constructive proof, we mean a proof that does not use any principle of reasoning that is forbidden in constructive logic. Specifically, you should not use the law of the excluded middle, double-negation elimination, or proof by contradiction.

Task 10. (2 points) Is the following proof that $3 \nmid 10$ constructive? Justify your answer.

> *Proof.* Assume to the contrary that $3 \mid 10$. Then there exists a k such that $10 = 3k$. By the fundamental theorem of arithmetic, k has some unique prime factorisation $k = \prod_{i=1}^{n} p_i$. So 10 factors into primes as $10 = 3 \prod_{i=1}^{n} p_i$. But we also know that 10 factors into primes as $10 = 2 \times 5$. The existence of two distinct prime factorisations for 10 contradicts the uniqueness guaranteed by the fundamental theorem of arithmetic. We thus conclude that $3 \nmid 10$.