Constructive Logic (15-317), Spring 2023
Assignment 4: Cut Elimination (70 points)

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Due: February 23, 2023, 11:59 pm

1 Cut Elimination (30 points)

Theorem 1 (Cut Admissibility) *If $\Gamma \Longrightarrow A$ and $\Gamma, A \Longrightarrow C$ then $\Gamma \Longrightarrow C$. Alternatively, the rule*

$$\dfrac{\Gamma \Longrightarrow A \quad \Gamma, A \Longrightarrow C}{\Gamma \Longrightarrow C}\ \mathsf{cut}$$

is admissible.

The proof goes by nested inductions on the structure of A, the derivation $\mathcal{D}$ of $\Gamma \Longrightarrow A$ and $\mathcal{E}$ of $\Gamma, A \Longrightarrow C$. The proof is constructive: we show how to construct $\mathcal{F}$ of $\Gamma \Longrightarrow C$ from $\mathcal{D}$ and $\mathcal{E}$:

$$\dfrac{\overset{\mathcal{D}}{\Gamma \Longrightarrow A} \quad \overset{\mathcal{E}}{\Gamma, A \Longrightarrow C}}{\Gamma \Longrightarrow C}\ \mathsf{cut} \qquad \leadsto \qquad \overset{\mathcal{F}}{\Gamma \Longrightarrow C}$$

Let us add a new connective $\mathsf{opt}[A]$, "optional A", with the following rules:

$$\dfrac{\Gamma \Longrightarrow A}{\Gamma \Longrightarrow \mathsf{opt}[A]}\ \mathsf{opt}R_1 \qquad \dfrac{}{\Gamma \Longrightarrow \mathsf{opt}[A]}\ \mathsf{opt}R_2$$

$$\dfrac{\Gamma, \mathsf{opt}[A], A \Longrightarrow C \quad \Gamma, \mathsf{opt}[A] \Longrightarrow C}{\Gamma, \mathsf{opt}[A] \Longrightarrow C}\ \mathsf{opt}L$$

Adding a new connective means we must extend our proofs of admissibility with a few cases.

To extend our proof of *cut admissiblity* so that it holds for our logic with $\mathsf{opt}[A]$, we need to check four cases:

- *Principal cases*: $\mathcal{D}$ ends with an $\mathsf{opt}R_1$ (or $\mathsf{opt}R_2$) and $\mathcal{E}$ ends with an $\mathsf{opt}L$, where $\mathsf{opt}L$ operates on the *principal formula* (the one the cut introduces).
- $\mathcal{D}$ ends with an $\mathsf{opt}L$ and $\mathcal{E}$ is arbitrary.
- $\mathcal{D}$ is arbitrary and $\mathcal{E}$ ends with an $\mathsf{opt}R_1$ (or $\mathsf{opt}R_2$)
- $\mathcal{D}$ is arbitrary and $\mathcal{E}$ ends with an $\mathsf{opt}L$, where $\mathsf{opt}L$ operates on a *side formula* (a formula that is not the principal one).

Task 1 (30 points) *Show the two principal cases and show one of the other (commuting) cases (you choose which one).*

2 Rule Induction (40 points)

Rule induction is when we use induction on the structure of a derivation. It allows us to apply the induction hypothesis whenever a derivation's size decreases. In the following tasks, we want to prove that

$$\textit{If } \overset{\mathcal{D}}{A\uparrow} \textit{ then there is an } M \textit{ such that } M \Leftarrow A$$

by rule induction (induction on the derivation $\mathcal{D}$ of $A\uparrow$).

Task 2 (10 points) *Because we define $A\uparrow$ and $A\downarrow$ mutually, the induction hypothesis we get directly from the proposition is not enough to prove it. Extend the induction hypothesis so that it is strong enough.*

Task 3 (10 points) *Check the case of the proof where the last rule in $\mathcal{D}$ is $\supset I$.*

Task 4 (10 points) *Check the case of the proof where the last rule applied to the judgment is $\supset E$.*

Task 5 (10 points) *Check the case of the proof where the last rule in $\mathcal{D}$ is $\downarrow\uparrow$.*