Constructive Logic (15-317), Spring 2023
Assignment 6: Deciding Propositional Logic (60 points)

Instructor: Frank Pfenning

Due: Tuesday, March 28, 2023, 11:59 pm

This assignment has a written portion and an SML programming portion. You will submit all portions through Gradescope.

We recommend that you typeset your written solutions. Most students use LaTeX, but other software is acceptable. If you choose not to typeset your solutions, be aware that your handwriting must be **legible**.

1 Inversion for the Contraction-Free Sequent Calculus (30 points)

In Lecture 14 we introduced the contraction-free sequent calculus and in Lecture 15 we introduced the inversion calculus. In this problem you will combine the two.

In the context of the contraction-free sequent calculus, we say *the inversion property holds for a rule* if we can apply the rule in bottom-up proof construction without backtracking whenever it contains the principal formula in the conclusion of the rule. For most rules it should be clear what the principal formula is (for example, $A \vee B$ in $\vee R_1$ and $(A_1 \supset A_2) \supset B$ in $\supset\supset L$). For id^* we consider the succedent principal, and for $P \supset L$ it is $P \supset B$.

The *contraction-free sequent calculus with inversion* should satisfy the following properties:

1. It should be immediately *sound* with respect to the contraction-free sequent calculus in the sense that it has analogous rules, plus some structural rules.
2. When inversion applies as defined above, exactly one rule should be applicable.
3. When inversion does not apply, it should offer a complete set of choices among the remaining rules. By "complete" we mean that if a sequent is provable in the contraction-free sequent calculus, it should be provable in your calculus.

Task 1. (20 points) Spell out your judgments and write out all the rules. You may find the rules in the LaTeX source of this assignment a time-saving starting point. We suggest at least three judgments: right inversion, left inversion, and choice.

Task 2. (5 points) Characterize the components of your judgments as in the inversion calculus in Appendix B where we define possible antecedents and succedents.

Task 3. (5 points) Provide a counterexample for the rules that are not part of inversion. In each case, this should be a sequent (possibly involving atoms $P, Q, R, \ldots$) for which the principal formula is in the sequent, but either the rule is not applicable or one of the premises is not derivable.

2 Implementation (30 points)

Implement your set of rules as a collection of functions in SML. You structure should satisfy the signature

```
signature DECIDE =
sig
  val prove : Prop.prop -> bool
end

structure Decide :> DECIDE =
struct
  (* your code here *)
end
```

where the structure `Prop` that defines the language of propositions is given in the starter code in the file `prop.sml`.

We have provided some test cases with the code from Lecture 16 where we wrote a loop-checking decision procedure loop.sml. These are in no way exhaustive.

Keep in mind that your implementation *should not do loop checking* since termination of the prover should be guaranteed by the multiset ordering defined in Lecture 14.

Task 4. (30 points) Implement the structure `Decide` in the file `hw6.sml`.

A Contraction-Free Sequent Calculus

$$\frac{}{\Gamma, P \longrightarrow P}\ \mathsf{id}^*$$

$$\frac{\Gamma \longrightarrow A \quad \Gamma \longrightarrow B}{\Gamma \longrightarrow A \wedge B}\ \wedge R \qquad \frac{\Gamma, A, B \longrightarrow C}{\Gamma, A \wedge B \longrightarrow C}\ \wedge L$$

$$\frac{}{\Gamma \longrightarrow \top}\ \top R \qquad \frac{\Gamma \longrightarrow C}{\Gamma, \top \longrightarrow C}\ \top L$$

$$\frac{\Gamma \longrightarrow A}{\Gamma \longrightarrow A \vee B}\ \vee R_1 \qquad \frac{\Gamma \longrightarrow B}{\Gamma \longrightarrow A \vee B}\ \vee R_2 \qquad \frac{\Gamma, A \longrightarrow C \quad \Gamma, B \longrightarrow C}{\Gamma, A \vee B \longrightarrow C}\ \vee L$$

no $\bot R$ rule $\qquad \dfrac{}{\Gamma, \bot \longrightarrow C}\ \bot L$

$$\frac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \supset B}\ \supset R$$

$$\frac{\Gamma, P, B \longrightarrow C}{\Gamma, P, P \supset B \longrightarrow C}\ P \supset L$$

$$\frac{\Gamma, A_1 \supset (A_2 \supset B) \longrightarrow C}{\Gamma, (A_1 \wedge A_2) \supset B \longrightarrow C}\ \wedge \supset L \qquad \frac{\Gamma, B \longrightarrow C}{\Gamma, \top \supset B \longrightarrow C}\ \top \supset L$$

$$\frac{\Gamma, A_1 \supset B, A_2 \supset B \longrightarrow}{\Gamma, (A_1 \vee A_2) \supset B \longrightarrow C}\ \vee \supset L \qquad \frac{\Gamma \longrightarrow C}{\Gamma, \bot \supset B \longrightarrow C}\ \bot \supset L$$

$$\frac{\Gamma, A_2 \supset B, A_1 \longrightarrow A_2 \quad \Gamma, B \longrightarrow C}{\Gamma, (A_1 \supset A_2) \supset B \longrightarrow C}\ \supset \supset L$$

B Inversion Calculus

Antecedents and Succedents

Antecedents, not left invertible, unordered	Γ	::=	$A \supset B \mid P \mid \cdot \mid \Gamma_1, \Gamma_2$
Antecedents, ordered	Ω	::=	$A \cdot \Omega \mid \epsilon$
Succedent, not right invertible	C	::=	$A \vee B \mid \bot \mid P$

Judgments.

Right inversion	$\Gamma \,;\, \Omega \xrightarrow{\mathsf{R}} A$
Left inversion	$\Gamma \,;\, \Omega \xrightarrow{\mathsf{L}} C$
Choice	$\Gamma \,;\, \epsilon \xrightarrow{\mathsf{C}} C$

Rules.

Right Inversion.

$$\frac{\Gamma\ ;\ \Omega \xrightarrow{\mathsf{R}} A \quad \Gamma\ ;\ \Omega \xrightarrow{\mathsf{R}} B}{\Gamma\ ;\ \Omega \xrightarrow{\mathsf{R}} A \wedge B}\ \wedge R \qquad \frac{\Gamma\ ;\ A \cdot \Omega \xrightarrow{\mathsf{R}} B}{\Gamma\ ;\ \Omega \xrightarrow{\mathsf{R}} A \supset B}\ \supset R \qquad \frac{}{\Gamma\ ;\ \Omega \xrightarrow{\mathsf{R}} \top}\ \top R$$

$$\frac{\Gamma\ ;\ \Omega \xrightarrow{\mathsf{L}} A \vee B}{\Gamma\ ;\ \Omega \xrightarrow{\mathsf{R}} A \vee B}\ \mathsf{LR} \qquad \frac{\Gamma\ ;\ \Omega \xrightarrow{\mathsf{L}} \bot}{\Gamma\ ;\ \Omega \xrightarrow{\mathsf{R}} \bot}\ \mathsf{LR} \qquad \frac{\Gamma\ ;\ \Omega \xrightarrow{\mathsf{L}} P}{\Gamma\ ;\ \Omega \xrightarrow{\mathsf{R}} P}\ \mathsf{LR}$$

Left Inversion.

$$\frac{\Gamma\ ;\ A \cdot B \cdot \Omega \xrightarrow{\mathsf{L}} C}{\Gamma\ ;\ (A \wedge B) \cdot \Omega \xrightarrow{\mathsf{L}} C}\ \wedge L \qquad \frac{\Gamma\ ;\ A \cdot \Omega \xrightarrow{\mathsf{L}} C \quad \Gamma\ ;\ B \cdot \Omega \xrightarrow{\mathsf{L}} C}{\Gamma\ ;\ (A \vee B) \cdot \Omega \xrightarrow{\mathsf{L}} C}\ \vee L$$

$$\frac{}{\Gamma\ ;\ \bot \cdot \Omega \xrightarrow{\mathsf{L}} C}\ \bot L \qquad \frac{\Gamma\ ;\ \Omega \xrightarrow{\mathsf{L}} C}{\Gamma\ ;\ \top \cdot \Omega \xrightarrow{\mathsf{L}} C}\ \top L$$

$$\frac{\Gamma, A \supset B\ ;\ \Omega \xrightarrow{\mathsf{L}} C}{\Gamma\ ;\ (A \supset B) \cdot \Omega \xrightarrow{\mathsf{L}} C}\ \mathsf{LL} \qquad \frac{\Gamma, P\ ;\ \Omega \xrightarrow{\mathsf{L}} C}{\Gamma\ ;\ P \cdot \Omega \xrightarrow{\mathsf{L}} C}\ \mathsf{LL}$$

$$\frac{\Gamma\ ;\ \epsilon \xrightarrow{\mathsf{C}} C}{\Gamma\ ;\ \epsilon \xrightarrow{\mathsf{L}} C}\ \mathsf{CL}$$

Choice.

$$\frac{\Gamma\ ;\ \epsilon \xrightarrow{\mathsf{R}} A}{\Gamma\ ;\ \epsilon \xrightarrow{\mathsf{C}} A \vee B}\ \vee R_1 \qquad \frac{\Gamma\ ;\ \epsilon \xrightarrow{\mathsf{R}} B}{\Gamma\ ;\ \epsilon \xrightarrow{\mathsf{C}} A \vee B}\ \vee R_2$$

$$\frac{}{\Gamma, P\ ;\ \epsilon \xrightarrow{\mathsf{C}} P}\ \mathsf{id}^* \qquad \frac{\Gamma, A \supset B\ ;\ \epsilon \xrightarrow{\mathsf{R}} A \quad \Gamma, [A \supset B]\ ;\ B \xrightarrow{\mathsf{R}} C}{\Gamma, A \supset B\ ;\ \epsilon \xrightarrow{\mathsf{C}} C}\ \supset L$$