Constructive Logic (15-317), Spring 2023
Miniprojects (150 points)

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Checkpoint: **Tuesday, April 4, 2023, 11:59 pm** (no late submissions allowed)
Final: **Tuesday, April 11, 2023, 11:59pm**

There are 3 independent miniprojects and you should choose one of them. In Gradescope, they are listed as separate assignments.

For your miniproject, **you may team up with a partner, but you are not required to do so**. If you do team up, make sure to name your partner upon submission and hand in only one solution.

You may use external resources such as reference books and research articles as you see fit, but you must cite them.

In grading, we emphasize correctness, clarity, and elegance. Also, we require that all your proofs should be constructive (except where otherwise noted), and that you clearly state the overall structure of the proof.

There are two deadlines: a checkpoint worth 50 points and a final submission worth 100 points. You may recover up to 20 points from the checkpoint by correcting your errors for your final submission. Each miniproject also has 15 bonus points.

You cannot apply late days to the checkpoint because we will use it to give you quick feedback so it is helpful for your final submission. You may use up to two late days for the final submission.

As always, we recommend that you typeset your solutions. Most students use LaTeX, but other software is acceptable. If you choose not to typeset your solutions, be aware that your handwriting must be **legible**.

1 Axioms and Combinators (150+15 points)

Before Gentzen's natural deduction and sequent calculus, Hilbert defined a system of deduction with logical axioms and, in the propositional case, just a single rule of inference. The judgment $\vdash A$ expresses that A is true according to Hilbert's rules. Considering only implication, the intuitionistic subset his calculus just has axioms K and S and the inference rule of *Modus Ponens* (MP).

$$\frac{}{\vdash A \supset (B \supset A)}\ \mathsf{K} \qquad \frac{}{\vdash (A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))}\ \mathsf{S}$$

$$\frac{}{\vdash A \supset A}\ \mathsf{I} \qquad \frac{\vdash A \supset B \quad \vdash A}{\vdash B}\ \mathsf{MP}$$

In this miniproject we explore the properties of this calculus, its relation to natural deduction, and its computational interpretation.

1.1 Relating Hilbert's Calculus to Natural Deduction (50 pts)

We begin by proving that Hilbert's calculus and natural deduction derive the same true propositions.

Task 1 (15 pts). Prove that if $\vdash A$ then A *true* in natural deduction.

The other direction takes two steps. We first define a *hypothetical Hilbert derivations* that allow assumptions. We write

$$\Gamma \vdash A$$

where $\Gamma = (x_1 : A_1, \ldots, x_n : A_n)$ is an unordered collection of hypotheses. We presuppose that all the x_i are distinct. We define it with the following rules:

$$\frac{}{\Gamma \vdash A \supset (B \supset A)}\ \mathsf{K} \qquad \frac{}{\Gamma \vdash (A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))}\ \mathsf{S}$$

$$\frac{}{\vdash A \supset A}\ \mathsf{I} \qquad \frac{\Gamma \vdash A \supset B \quad \Gamma \vdash A}{\Gamma \vdash B}\ \mathsf{MP} \qquad \frac{}{\Gamma, x : A \vdash A}\ x$$

Task 2 (15 pts). Prove that the following rule is admissible:

$$\frac{\Gamma, x : A \vdash B}{\Gamma \vdash A \supset B}\ \mathsf{Ded}$$

Task 3 (15 pts). Prove that

$$\begin{array}{c} \dfrac{}{A_1\ \mathit{true}}\ x_1 \quad \cdots \quad \dfrac{}{A_n\ \mathit{true}}\ x_n \\ \mathcal{D} \\ A\ \mathit{true} \end{array} \quad \text{implies} \quad \begin{array}{c} \mathcal{E} \\ x_1 : A_1, \ldots, x_n : A_n \vdash A \end{array}$$

Task 4 (5 pts). As a corollary of the preceding theorems, prove that A *true* if and only if $\vdash A$.

Checkpoint

1.2 Combinator Terms (35 pts)

We assign proof terms to hypothetical Hilbert derivations as follows, using lowercase letters m, n, p to distinguish them from the natural deduction terms. We call them *combinator terms*.

$$\frac{}{\Gamma \vdash K : A \supset (B \supset A)}\ \mathsf{K} \qquad \frac{}{\Gamma \vdash S : (A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))}\ \mathsf{S}$$

$$\frac{}{\Gamma \vdash I : A \supset A}\ \mathsf{I} \qquad \frac{\Gamma \vdash m : A \supset B \quad \Gamma \vdash n : A}{\Gamma \vdash m\,n : B}\ \mathsf{MP} \qquad \frac{}{\Gamma, x : A \vdash x : A}\ x$$

As usual, application $m\,n$ associates to the left, so $m\,n\,p$ stands for $(m\,n)\,p$. We observe that for $\vdash m : A$ (without hypotheses) the term m does not contain any variables.

Task 5 (10 pts). Instrument the deduction theorem by devising an operation of *abstraction* $\mathsf{abs}(x.\,m)$ such that

$$\frac{\Gamma, x : A \vdash m : B}{\Gamma \vdash \mathsf{abs}(x.\,m) : A \supset B}\ \mathsf{Ded}$$

Note that the $\mathsf{abs}(x.\,m)$ should be a term n that no longer contains the variable x which might occur in m. Your operation should be extracted from the proof in Task 2.

Task 6 (10 pts). Define an operation $\mathsf{compile}(M)$ from natural deduction proof terms to combinator terms such that if $M : A$ then $\vdash \mathsf{compile}(M) : A$. This operation is a translation from closed proof terms in natural deduction into the language of combinators.

Now assume we are working with terms m such that $\vdash m : A$. The following is a natural rule of *proof reduction* for combinator terms:

$$I\,m \quad \longrightarrow \quad m$$

Task 7 (5 pts). Prove that the reduction rule for I is sound for typing, that is, if $\vdash I\,m : A$ then $\vdash m : A$.

Task 8 (10 pts). Devise reduction rules for the combinator K and S and prove that they are sound for typing.

1.3 Local Reductions (30 pts)

Below we show a minor variant of Hilbert's axioms for the remaining connectives of intuitionistic logic. Interestingly, modus ponens remains the only inference rules. We name the rules C for constructor and D for destructor.

$$
\begin{aligned}
\mathsf{AndC} &: A \supset B \supset (A \wedge B) \\
\mathsf{AndD}_1 &: (A \wedge B) \supset A \\
\mathsf{AndD}_2 &: (A \wedge B) \supset B \\
\mathsf{TrueC} &: \top \\
\mathsf{OrC}_1 &: A \supset (A \vee B) \\
\mathsf{OrC}_2 &: B \supset (A \vee B) \\
\mathsf{OrD} &: (A \vee B) \supset ((A \supset C) \supset ((B \supset C) \supset C)) \\
\mathsf{FalseD} &: \bot \supset C
\end{aligned}
$$

Task 9 (15 pts). Write out suitable proof reduction rules for the new set of combinators. You need to make sure they are sound with respect to typing, but you do not need to prove that.

Task 10 (15 pts). Extend your translation from Task 6 to cover conjunction, truth, disjunction, and falsehood.

1.4 Values and Computations (35+15 pts)

Next we design a functional semantics for combinator terms, which requires two judgments, m *value* and $m \longrightarrow m'$. As for natural deduction, we posit that the structure of of functions (type $A \supset B$) and pairs (type $A \wedge B$) cannot be observed, while the structure of injections (type $A \vee B$) can.

Task 11 (20 pts). Write out rules for computing with combinator terms that is suitable for evaluating the result of compiling natural deduction terms. Individual steps may differ, but the *observable* outcomes should coincide. You do not have to prove this property.

Next we define $\mathsf{bool} \triangleq \top \vee \top$, $\mathsf{True} \triangleq \mathsf{OrC}_1\ \mathsf{TrueC}$ and $\mathsf{False} \triangleq \mathsf{OrC}_2\ \mathsf{TrueC}$.

Task 12 (15 pts). Write a function $\mathsf{Neg} : \mathsf{bool} \supset \mathsf{bool}$ as a combinatory term such that it represents the negation function on the underlying Booleans. We suggest you write function in the form of natural deduction and compile it using the result from Tasks 6 and 10. You may need to optimize your translation for this to be feasible. For example, you might consider optimizing the case $\mathsf{abs}(x.\, m)$ when x does not occur in m.

Task 13 (15 pts). (bonus) Reduce $\mathsf{Neg}\,\mathsf{True}$ to a *normal form*, that is, a term that cannot be reduced any further. Show the intermediate steps of reduction. Is the result your representation of False?

2 Truth and Necessity (150+15 points)

In this miniproject we explore the interpretation of the (intuitionistic, of course!) modal logic S4 and its connection to quotation in programming languages.

A key concept in modal logic is that of *validity*. We say a proposition A is *valid* if A is true without any hypotheses, sometimes stated as "A is always true". For example, for any proposition A, $A \supset A$ is valid. On the other hand, if we assume A *true* and $A \supset B$ *true* then B *true* but certainly not B *valid*. This notion is expressed as a proposition by the modal operator of *necessity* $\Box A$ (often pronounced "box A"). The defining property is that $\Box A$ is true if A is valid. We follow the common syntactic convention that $\Box$ binds more tightly than the logical connectives so that, for example, $\Box A \supset B$ is parsed as $(\Box A) \supset B$.

Necessity may not sound very interesting, but it is a rich subject. We have theorems such as $\Box A \supset A$ *true* (*"if A is valid it is certainly true"*), $\Box(A \wedge B) \supset (\Box A \wedge \Box B)$ (*"if $A \wedge B$ is valid, so are A and B"*) and also properties that are *not* derivable such as $P \supset \Box P$ *true* (*"just because P happens to be true that doesn't mean that P is* always *true"*).

We can use necessity to model *quotation* in programming languages: $M : \Box A$ will evaluate to **quote** N where N is a closed, quoted expression of type A. Among other things, this can be used for efficient code generation at runtime.

2.1 Natural Deduction (40 pts)

We start with the two judgments A *true* and A *valid* and the following rules:

$$\frac{A\ \mathit{true}}{\Box A\ \mathit{true}}\ \Box I^{\dagger} \qquad \frac{\Box A\ \mathit{true} \quad \begin{array}{c}\overline{A\ \mathit{valid}}\ u \\ \vdots \\ C\ \mathit{true}\end{array}}{C\ \mathit{true}}\ \Box E^{u}$$

$$\frac{A\ \mathit{valid}}{A\ \mathit{true}}\ \mathsf{VT}$$

A crucial aspect of the rules is hidden in the notation $\Box I^{\dagger}$. The $\dagger$ annotation expresses that the derivation of the premise may only depend on hypotheses B *valid* and not on any hypotheses B *true*. Here is a correct derivation, followed by an incorrect one.

$$\dfrac{\dfrac{\dfrac{\overline{\Box A\ \mathit{true}}\ x \quad \dfrac{\overline{A\ \mathit{valid}}\ u}{A\ \mathit{true}}\ \mathsf{VT}}{A\ \mathit{true}}\ \Box E^{u}}{\Box A \supset A\ \mathit{true}}\ {\supset} I^{x}}{\Box(\Box A \supset A)\ \mathit{true}}\ \Box I^{\dagger}$$

The rule application $\Box I^{\dagger}$ is legal because there are no (undischarged) hypotheses in the derivation of the premises (and so also none about truth). On the other hand, the following is is illegal:

$$\dfrac{\dfrac{\dfrac{}{\Box A\ \mathit{true}}\,x}{\Box\Box A\ \mathit{true}}\,{\color{red}\Box I^\dagger}}{\Box A \supset \Box\Box A\ \mathit{true}}\,{\supset} I^x$$

The $\Box I^\dagger$ inference (in red) is illegal because the derivation of its premise depends up the hypothesis x which is of the form C *true* rather than C *valid*.

Task 1 (20 pts). Give derivation of the following judgments

(a) $\Box A \supset A$ *true*

(b) $\Box A \supset \Box\Box A$ *true* (correcting the one above)

(c) $\Box(A \supset B) \supset (\Box A \supset \Box B)$ *true*.

We write $\mathcal{D}^\dagger$ for a derivation all of whose (undischarged) hypotheses are of the form A *valid*. We can substitute such a derivation $\mathcal{D}^\dagger$ of A *true* for uses of the hypotheses A *valid*. We write

Substitution for Validity.

$$\text{Given}\quad \begin{array}{c}\mathcal{D}^\dagger\\ A\ \mathit{true}\end{array}\quad\text{and}\quad \begin{array}{c}\overline{A\ \mathit{valid}}\,u\\ \mathcal{E}\\ C\ \mathit{true}\end{array}\qquad\text{construct}\quad \begin{array}{c}\mathcal{D}^\dagger\\ \overline{A\ \mathit{valid}}\,u\\ \mathcal{E}\\ C\ \mathit{true}\end{array}$$

This may look like it should not work because the judgment A *true* does not match the hypothesis A *valid*, but since the only way to use such a hypothesis is in an inference

$$\dfrac{\dfrac{}{A\ \mathit{valid}}\,u}{A\ \mathit{true}}\,\mathsf{VT}$$

we can just erase the use of the VT rule.

Task 2 (10 pts). For the correctness of modal substitution is it important that $\mathcal{D}^\dagger$ does not depend on any hypotheses B *true*. Show an example illustrating that the resulting derivation may otherwise not be valid.

Task 3 (5 pts). Show the local reduction(s) for $\Box A$.

Task 4 (5 pts). Show the local expansion(s) for $\Box A$.

2.2 Verifications (10 pts)

Because the validity judgment is only used as a hypothesis, it becomes $A\Downarrow$.

Task 5 (10 pts). Show the rules for verifications and uses, relevant to the judgments A *valid* and A *true* where A is restricted to implication $A_1 \supset A_2$ and necessity $\Box A'$.

Checkpoint

2.3 Sequent Calculus (65 pts)

For the sequent calculus, we now use sequents $\Gamma \Longrightarrow A\ \textit{true}$ that have mixed antecedents and allow only a succedent $A\ \textit{true}$.

$$\text{Antecedents} \quad \Gamma \quad ::= \quad A\ \textit{true} \mid A\ \textit{valid} \mid \cdot \mid \Gamma_1, \Gamma_2$$

Task 6 (15 pts). Design rules corresponding to verifications in the sequent calculus. You only need to show the rules relevant to implication and necessity, as well as rules relevant to the judgment $A\ \textit{valid}$.

Task 7 (25 pts). Prove or refute each of the following:

(a) $\Longrightarrow P \supset \Box P$

(b) $\Longrightarrow \Box(A \wedge B) \supset \Box A \wedge \Box B$

(c) $\Longrightarrow \Box(A \vee B) \supset \Box A \vee \Box B$

(d) $\Longrightarrow (\Box A \wedge \Box B) \supset \Box(A \wedge B)$

(e) $\Longrightarrow (\Box A \vee \Box B) \supset \Box(A \vee B)$

In addition to the usual cut, we have the following *modal cut*

$$\frac{\overset{\mathcal{D}}{\Gamma^{\dagger} \Longrightarrow A\ \textit{true}} \quad \overset{\mathcal{E}}{\Gamma, A\ \textit{valid} \Longrightarrow C\ \textit{true}}}{\Gamma \Longrightarrow C\ \textit{true}}\ \mathsf{mcut}$$

Here, $\Gamma^{\dagger}$ is the restriction of Γ to antecedents of the form $A\ \textit{valid}$. We have to prove, simultaneously, that cut and mcut are admissible rules.

Task 8 (5 pts). State the form of induction necessary to prove the admissibility of cut and mcut.

Task 9 (10 pts). Show the principal case for cut when $A = \Box A'$ is just introduced in the last inference in $\mathcal{D}$ and $\mathcal{E}$.

Task 10 (10 pts). Show a case for mcut where the induction hypothesis appeals to a cut.

2.4 Natural Deductions and Sequent Calculus (15 pts)

As in Lecture 9, we can prove a sequence of theorems to show that natural deductions, sequent calculus, and verifications derive the truth of the same propositions. We examine the generalization of one of the steps.

Theorem If $\begin{array}{c}\Gamma\\ \mathcal{D}\\ A\ \mathit{true}\end{array}$ then $\Gamma \overset{\mathcal{E}}{\Longrightarrow} A\ \mathit{true}$, where Γ is a mixed collection of hypotheses B *true* and B *valid*. The proof is by induction on $\mathcal{D}$.

Task 11 (15 pts). Show the cases for $\Box I^{\dagger}$, $\Box E$, and VT in the translation from natural deduction to the sequent calculus. These are part of the *completeness* proof of the sequent calculus and should be formulated as such. You may use the admissibility of cut, modal cut, and identity.

2.5 Quotation (35+15 points)

We annotate the judgment A *true* as usual with $M : A$ and A *valid* as $u :: A$. The latter is always just a variable since the judgment A *valid* arises only as a hypothesis.

$$\frac{M : A}{\mathbf{quote}\ M : \Box A}\ \Box I^{\dagger} \qquad \frac{M : \Box A \quad \begin{array}{c}\overline{u :: A}\ u\\ \vdots\\ N : C\end{array}}{\mathbf{unquote}(M, u.N) : C\ \mathit{true}}\ \Box E^{u}$$

$$\frac{u :: A}{u : A}\ \mathsf{VT}$$

Task 12 (10 pts). Show the proof terms for the derivations in Task 1.

Computationally, we decide to that **quote** M *value* regardless whether M is a value.

Task 13 (10 pts). Write a complete set of reduction rules concerning **quote** and **unquote** such that preservation, progress, and determinism hold.

Task 14 (15 pts). (bonus) Consider the program

$$\begin{array}{lcl}\mathsf{dec} & : & \mathsf{bool} \supset A \supset B \supset A \vee B\\ \mathsf{dec} & = & \lambda b.\, \lambda x.\, \lambda y.\, \mathbf{case}(b, u.\, \mathbf{inl}\, x, w.\, \mathbf{inr}\, y)\end{array}$$

Write a corresponding program

$$\mathsf{dec\_code} \quad : \quad \Box\mathsf{bool} \supset \Box A \supset \Box B \supset \Box(A \vee B)$$

such that the resulting quoted term no longer contains a case construct.

3 Multiple Worlds (150+15 points)

Our main judgment so far has been A *true*, and others like $A\uparrow$ have been considered in relation to truth. This is the view prevalent in the study of mathematics. In philosophy and computer science there are richer judgments of interest. For example, in computer science we might be interested in A *true at time* t which could give some insight into the computational cost of computing a value of type A. In philosophy we might study judgments such as P *knows* A. In this miniproject we study an abstraction of those two examples where the basic judgment is A *true in world* w. This could then be applied or refined further in more specific circumstances. For example, a world w refers to a time or place or state of memory, etc.

A crucial part of reasoning with worlds is an abstract *accessibility relation* $w \leq w'$. As examples, for time we might think of w' to be in the future of w and for locations, we might think of there being a path to reach w' from w. So modal logic with an accessibility relation actually uses two judgments A *true in world* w and $w \leq w'$.

We assume that the logical connectives have their usual meaning at each world. Since the meaning is given by introduction and elimination rules, these rules are localized to a particular world. We show some sample rules. We abbreviate the judgment A *true in world* w as $A[w]$ which we read as "A at w".

$$\dfrac{\begin{array}{c}\overline{A\,[w]}^{\;x}\\ \vdots\\ B\,[w]\end{array}}{A \supset B\,[w]}\;{\supset}I^x \qquad \dfrac{A \vee B\,[w] \quad \begin{array}{c}\overline{A\,[w]}^{\;x}\\ \vdots\\ C\,[w']\end{array} \quad \begin{array}{c}\overline{B\,[w]}^{\;y}\\ \vdots\\ C\,[w']\end{array}}{C\,[w']}\;{\vee}E^{x,y}$$

The locality of inference in the $\vee E^{x,y}$ rule is evident from the transition of $A \vee B$ at w to A at w and B at w. Nevertheless, we can make this case distinction even if we ultimately want to prove what may be true at world w'. We can interpret this by saying that we are *omniscient*: we can see and reason from everything we happen to know at any world, but in general truth at one world does not imply truth at another.

There are two modal operators that let us reason within the logic about what may be true in other worlds. $\Box A$ is true at w if A is true at every world reachable from w. We refer to $\Box$ as *necessity*. Dually, $\Diamond A$ is true at w if there is a reachable world w' such that A is true at w'. We refer to $\Diamond$ as *possibility*.

As a syntactic convention, the $\Box$ and $\Diamond$ modalities bind more tightly than the logical connectives so that, for example, $\Box A \supset A$ is parsed as $(\Box A) \supset A$.

3.1 Natural Deduction (50 pts)

We show the rules for $\Box A$ and leave the rules for $\Diamond A$ for you to design and test.

$$\dfrac{\begin{array}{c}\overline{w \leq \alpha}\\ \vdots\\ A\,[\alpha]\end{array}}{\Box A\,[w]}\;\Box I^{\alpha} \qquad \dfrac{\Box A\,[w] \quad w \leq w'}{A\,[w']}\;\Box E$$

In $\Box I^\alpha$ we introduce a new parameter α into the derivation whose scope is limited to be above this inference. This is analogous to the $\forall I^a$ rule. We further introduce the hypothesis that α is accessible from w with the same scope.

Task 1 (5 pts). Show the local soundness of $\Box E$. Which properties related to the judgment $w \leq w'$ do you need?

Task 2 (5 pts). Show the local completeness of $\Box E$. Again, which properties related to the judgment $w \leq w'$ do you need?

Task 3 (5 pts). Prove $\Box(A \supset B) \supset (\Box A \supset \Box B)\,[\alpha]$. Just like A and B stand for arbitrary propositions, α here stands for an arbitrary world.

We obtain different modal logics depending on which reasoning principles we allow for the accessibility relation, or which worlds we may explicitly reference. Of particular significance are the following rules:

$$\frac{}{w \leq w}\ \mathsf{refl} \qquad \frac{w_1 \leq w_2 \quad w_2 \leq w_3}{w_1 \leq w_3}\ \mathsf{trans} \qquad \frac{w \leq w'}{w' \leq w}\ \mathsf{sym}$$

Assumptions about the properties of accessibility will never be formally made *inside* the logic, but we will explicitly or disallow allow the use of these rules in our study of modal logics.

Task 4 (10 pts). Prove each of the following and state for each which properties of the accessibility relation you need.

(a) $\Box A \supset A\,[\alpha]$

(b) $\Box A \supset \Box\Box A\,[\alpha]$

Task 5 (15 pts). Design introduction and elimination rules for $\Diamond A$ that satisfy the intended meaning of possibility explained above.

Task 6 (5 pts). Show that your elimination rules is locally sound. If this requires a property of the accessibility relation, please state it explicitly.

Task 7 (5 pts). Show that the elimination is locally complete. If this requires a property of the accessibility relation, please state it explicitly.

Checkpoint

3.2 Verifications (30 points)

Task 8 (10 pts). Give the rules for *verifications* $A\,[\alpha]\!\uparrow$ and *uses* $A\,[\alpha]\!\downarrow$. We do not annotate the accessibility judgment.

Task 9 (20 pts). Prove each of the following and state which properties of accessibility you needed (none, reflexivity, transitive, both). Try to use as few as possible.

(a) $\Diamond\bot \supset \bot\,[\alpha]\!\uparrow$

(b) $(\Diamond P \supset \Box Q) \supset \Box(P \supset Q)\,[\alpha]\!\uparrow$

(c) $P \supset \Box\Diamond P\,[\alpha]\!\uparrow$

(d) $\Box\Diamond\Box\Diamond P \supset \Box\Diamond P\,[\alpha]\!\uparrow$

3.3 Sequent Calculus (45 pts)

In the sequent calculus for modal logic with explicit words we have we two forms of succedents: $A\,[w]$ and $w \leq w'$. Both of these can also be antecedents.

$$\text{Antecedents} \quad \Gamma \quad ::= \quad A\,[w] \mid w \leq w' \mid \cdot \mid \Gamma_1, \Gamma_2$$

In a sequent $\Gamma \Longrightarrow w \leq w'$ only assumptions $w_i \leq w_j$ may be used, so we write $\Gamma^{\leq}$ for the result of erasing all antecedents $A\,[w]$ from Γ.

As a start, we show the counterparts of the two natural deduction rules from above.

$$\frac{\Gamma, A\,[w] \Longrightarrow B\,[w]}{\Gamma \Longrightarrow A \supset B\,[w]} \supset\! R \qquad \frac{\Gamma, A\,[w] \Longrightarrow C\,[w'] \quad \Gamma, B\,[w] \Longrightarrow C\,[w']}{\Gamma, A \vee B\,[w] \Longrightarrow C\,[w']} \vee L$$

In the left rule we have omitted the redundant antecedent $A \vee B\,[w]$ in the premise, and you may do the same (but only if they are truly redundant!).

Task 10 (5 pts). Write out the left and right rules for the $\Box$ modality.

Task 11 (5 pts). Write out the left and right rules for the $\Diamond$ modality.

Task 12 (5 pts). Derive $\Diamond(P \vee Q) \supset (\Diamond P \vee \Diamond Q)\,[\alpha]$ without any assumptions about properties of the accessibility relation.

Task 13 (5 pts). Prove that $P \supset \Box P$ is not derivable unless we have a degenerate accessibility relation, and state what that would be.

Task 14 (5 pts). Prove that $\Box P \supset P$ is not derivable unless we assume the property of the accessibility relation you identified in Task 4.

The cut rule has the form

$$\frac{\overset{\mathcal{D}}{\Gamma \Longrightarrow A\,[w]} \quad \overset{\mathcal{E}}{\Gamma, A\,[w] \Longrightarrow C\,[w']}}{\Gamma \Longrightarrow C\,[w']} \ \mathsf{cut}$$

Task 15 (10 pts). Show the principal case in the proof of cut either for $\Box A$ or $\Diamond A$. Which admissible rules for the accessibility judgment do you need? (You do not need to prove them.)

Task 16 (10 pts). Show a case in the proof of cut where either $\mathcal{D}$ ends in a left rule for $\Box B$ or $\Diamond B$ or $\mathcal{E}$ ends in a right rule for $C = \Box C'$ or $C = \Diamond C'$. Again, clarify which admissible rules for the accessibility judgment you need for the case you are considering.

3.4 Natural Deduction and Sequent Calculus (10 pts)

As in Lecture 9, we can show that $A\,[\alpha]$ iff $\implies A\,[\alpha]$ iff $A\,[\alpha]\!\uparrow$.

Task 17 (10 pts). Show the cases for $\Box I$ and $\Box E$ in the proof that the sequent calculus is complete with respect to natural deduction. You may use the admissibility of cut and identity.

3.5 Proof Terms (30+15 pts)

For the proof terms and functional computation we restrict ourselves to the fragment with $\Box A$, $A \supset B$, and $A \wedge B$. For the particular operational interpretation we have in mind, we think of $M : A\,[w]$ as a computation of M of type A that takes place at location w. Furthermore, $w \leq w'$ means that there is a path from w to w'.

$$\dfrac{\begin{array}{c}\overline{w \leq \alpha}\\ \vdots \\ M : A\,[\alpha]\end{array}}{\mathbf{box}(\alpha.\, M) : \Box A\,[w]}\ \Box I^{\alpha} \qquad \dfrac{M : \Box A\,[w] \quad w \leq w'}{\mathbf{fetch}\,[w']\,M : A\,[w']}\ \Box E$$

We interpret $\mathbf{box}(\alpha.\, M)$ as a mobile value and $\mathbf{fetch}\,[w']\,M$ can fetch M from w provided M is a value and $w \leq w'$.

Task 18 (15 pts). Write out the rules for M *value* and $M \longrightarrow M'$ for **box** and **fetch**. They should satisfy preservation, progress, and determinism, but you do not need to prove that.

Task 19 (15 pts). (bonus) Write out the proof terms for

(a) $\Box(A \supset B) \supset (\Box A \supset \Box B)$

(b) $\Box A \supset A$

(c) $\Box A \supset \Box\Box A$

In each case, explain the intuitive computational meaning of the programs. Also, if you needed assumptions about the accessibility relation for the proof, explain what they mean in this context of location-aware computation.