Lecture Notes on
Natural Deduction

15-317: Constructive Logic
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Lecture 2
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1 Introduction

The goal of this chapter is to develop the two principal notions of logic, namely *propositions* and *proofs*. There is no universal agreement about the proper foundations for these notions. One approach, which has been particularly successful for applications in computer science, is to understand the meaning of a proposition by understanding its proofs. In the words of Martin-Löf [1983, Page 27]:

> *The meaning of a proposition is determined by [. . .] what counts as a verification of it.*

A *verification* may be understood as a certain kind of proof that only examines the constituents of a proposition. This is analyzed in greater detail by Dummett [1991] although with less direct connection to computer science. The system of inference rules that arises from this point of view is *natural deduction*, first proposed by Gentzen [1935] and studied in depth by Prawitz [1965].

In this chapter we apply Martin-Löf's approach, which follows a rich philosophical tradition, to explain the basic propositional connectives. We will see later that universal and existential quantifiers and types such as natural numbers, lists, or trees naturally fit into the same framework.

We will define the meaning of the usual connectives of propositional logic (conjunction, implication, disjunction) by rules that allow us to infer when they should be true, so-called *introduction rules*. From these, we derive rules for the use of propositions, so-called *elimination rules*. The resulting system of *natural deduction* is the foundation of intuitionistic logic which has direct connections to functional programming and logic programming.

2 Judgments and Propositions

The cornerstone of Martin-Löf's foundation of logic is a clear separation of the notions of judgment and proposition. A *judgment* is something we may know, that is, an object of knowledge. A judgment is *evident* if we in fact know it.

We make a judgment such as "*it is raining*", because we have evidence for it. In everyday life, such evidence is often immediate: we may look out the window and see that it is raining. In logic, we are concerned with situation where the evidence is indirect: we deduce the judgment by making correct inferences from other evident judgments.

The most important judgment form in logic is "*A is true*", where A is a proposition. There are many others that have been studied extensively. For example, "*A is false*", "*A is true at time t*" (from temporal logic), "*A is necessarily true*" (from modal logic), "*program M has type τ*" (from programming languages), etc.

Returning to the first judgment, let us try to explain the meaning of conjunction. We write A *true* for the judgment "*A is true*" (presupposing that A is a proposition). Given propositions A and B, we can form the compound proposition "*A and B*", written more formally as $A \wedge B$. But we have not yet specified what conjunction *means*, that is, what counts as a verification of $A \wedge B$. This is accomplished by the following inference rule:

$$\frac{A\ \mathit{true} \quad B\ \mathit{true}}{A \wedge B\ \mathit{true}}\ \wedge I$$

Here the name $\wedge I$ stands for "conjunction introduction", since the conjunction is introduced in the conclusion.

This rule allows us to conclude that $A \wedge B$ *true* if we already know that A *true* and B *true*. In this inference rule, A and B are *schematic variables*, and $\wedge I$ is the name of the rule.

Intuitively, the $\wedge I$ rule says that a proof of $A \wedge B$ *true* consists of a proof of A *true* together with a proof of B *true*. This is precisely the intuitionistic meaning of conjunction we mentioned in Lecture 1.

The general form of an inference rule is

$$\frac{J_1 \ \ldots \ J_n}{J}\ \mathit{name}$$

where the judgments $J_1, \ldots, J_n$ are called the *premises*, the judgment J is called the *conclusion*. In general, we will use letters J to stand for judgments, while A, B, and C are reserved for propositions.

We take conjunction introduction as specifying the meaning of $A \wedge B$ completely. So what can be deduced if we know that $A \wedge B$ is true? By the above rule, to have a verification for $A \wedge B$ means to have verifications for A and B. Hence the following two rules are justified:

$$\frac{A \wedge B\ \mathit{true}}{A\ \mathit{true}}\ \wedge E_1 \qquad\qquad \frac{A \wedge B\ \mathit{true}}{B\ \mathit{true}}\ \wedge E_2$$

The name $\wedge E_1$ stands for "first conjunction elimination", since the conjunction in the premise has been eliminated in the conclusion. Similarly $\wedge E_2$ stands for "second conjunction elimination".

We will later see what precisely is required in order to guarantee that the formation, introduction, and elimination rules for a connective fit together correctly. For now, we will informally argue the correctness of the elimination rules, as we did for the conjunction elimination rules.

As a second example we consider the proposition "*truth*" written as $\top$. Truth should always be true, which means its introduction rule has no premises.

$$\frac{}{\top \; \textit{true}} \; \top I$$

Consequently, we have no information if we know $\top$ *true*, so there is no elimination rule.

A conjunction of two propositions is characterized by one introduction rule with two premises, and two corresponding elimination rules. We may think of truth as a conjunction of zero propositions. By analogy it should then have one introduction rule with zero premises, and zero corresponding elimination rules. This is precisely what we wrote out above.

3 Hypothetical Judgments

Consider the following deduction, for arbitrary propositions A, B, and C:

$$\dfrac{\dfrac{A \wedge (B \wedge C) \; \textit{true}}{B \wedge C \; \textit{true}} \; \wedge E_2}{B \; \textit{true}} \; \wedge E_1$$

Have we actually proved anything here? At first glance it seems that cannot be the case: B is an arbitrary proposition; clearly we should not be able to prove that it is true. Upon closer inspection we see that all inferences are correct, but the first judgment $A \wedge (B \wedge C)$ *true* has not been justified. We can extract the following knowledge:

From the hypothesis that $A \wedge (B \wedge C)$ *is true, we deduce that* B *must be true.*

This is an example of a *hypothetical judgment*, and the figure above is an *hypothetical deduction*. In general, we may have more than one assumption, so a hypothetical deduction has the form

$$\begin{array}{ccc} J_1 & \cdots & J_n \\ & \vdots & \\ & J & \end{array}$$

where the judgments $J_1, \ldots, J_n$ are unproven assumptions, and the judgment J is the conclusion. All instances of the inference rules are hypothetical judgments as well (albeit possibly with 0 assumptions if the inference rule has no premises).

Many mistakes in reasoning arise because dependencies on some hidden assumptions are ignored. When we need to be explicit, we will write $J_1, \ldots, J_n \vdash J$ for the hypothetical judgment which is established by the hypothetical deduction above. We may refer to $J_1, \ldots, J_n$ as the antecedents and J as the succedent of the hypothetical judgment. For example, the hypothetical judgment $A \wedge (B \wedge C)$ *true* $\vdash B$ *true* is proved by the above hypothetical deduction that B *true* indeed follows from the hypothesis $A \wedge (B \wedge C)$ *true* using inference rules.

The key property, one might say the *defining property*, of a hypothetical judgment is that we can always substitute a proof of A for all uses of a hypothesis A *true*. Formalizing proof substitution is not entirely straightforward, but it should be intuitive so for now we take it as a primitive notion.

With hypothetical judgments, we can now explain the meaning of implication "*A implies B*" or "*if A then B*" (more formally: $A \supset B$). The introduction rule reads: $A \supset B$ is true, if B is true under the assumption that A is true.

$$\dfrac{\dfrac{}{A\ \mathit{true}}\ u \\ \vdots \\ B\ \mathit{true}}{A \supset B\ \mathit{true}}\ {\supset}I^u$$

The tricky part of this rule is the label u and its bar. If we omit this annotation, the rule would read

$$\dfrac{A\ \mathit{true} \\ \vdots \\ B\ \mathit{true}}{A \supset B\ \mathit{true}}\ {\supset}I$$

which would be incorrect: it looks like a derivation of $A \supset B\ \mathit{true}$ from the hypothesis $A\ \mathit{true}$. But the assumption $A\ \mathit{true}$ is introduced in the process of proving $A \supset B\ \mathit{true}$; the conclusion should not depend on it!

Certainly, whether the implication $A \supset B$ is true is independent of the question whether A itself is actually true. Therefore we label uses of the assumption with a new name u, and the corresponding inference which introduced this assumption into the derivation with the same label u.

Note that the hypothesis labeled u is only available in the proof *above* the inference ${\supset}I^u$. We will shortly see that natural deduction would become *inconsistent* if we allowed a violation of the scope of the label u. Also, labels should be distinct in a proof so we can always unambiguously tell where a hypothesis is introduced.

Before we do some examples proofs, we consider what the elimination rule for implication should say. By the only introduction rule, having a proof of $A \supset B\ \mathit{true}$ means that we have a hypothetical proof of $B\ \mathit{true}$ from $A\ \mathit{true}$. By the substitution principle, if we also have a proof of $A\ \mathit{true}$ then we get a proof of $B\ \mathit{true}$.

$$\dfrac{A \supset B\ \mathit{true} \quad A\ \mathit{true}}{B\ \mathit{true}}\ {\supset}E$$

This completes the rules concerning implication.

4 Constructing Proofs

Before moving on to further connectives, let's construct some example proofs in (intuitionistic) natural deduction. As for ordinary mathematics, it is often the process of proof construction that provides the most insight, rather than the finished product. This temporal aspect of conveying intuition is difficult to replicate on paper, but we will try.

We start with the simplest use of implication, namely $A \supset A$ *true*. In the display, we write vertical dots to indicate that we are trying to fill in a missing proof. Hypotheses (if there are any) are shown above.

$$\begin{array}{c} \vdots \\ A \supset A \; \mathit{true} \end{array}$$

Since we have no assumptions to work with, we start with an implication introduction.

$$\cfrac{\begin{array}{c} \cfrac{}{A \; \mathit{true}}\, u \\ \vdots \\ A \; \mathit{true} \end{array}}{A \supset A \; \mathit{true}} \supset\! I^u$$

At this point we see that our only assumptions matches what we are trying to prove and we close the gap.

$$\cfrac{\cfrac{}{A \; \mathit{true}}\, u}{A \supset A \; \mathit{true}} \supset\! I^u$$

Sanity check: u is only used above the $\supset\! I^u$ inference, so the scope is correct.

Next we try to prove something slightly more complicated, namely $A \supset (B \supset A)$ *true*. Intuitively, this is true because we can conclude A from the first assumption and don't need the assumption B. Here is how this works out.

$$\begin{array}{c} \vdots \\ A \supset (B \supset A) \; \mathit{true} \end{array}$$

We again start with implication introduction.

$$\cfrac{\begin{array}{c} \cfrac{}{A \; \mathit{true}}\, u \\ \vdots \\ B \supset A \; \mathit{true} \end{array}}{A \supset (B \supset A) \; \mathit{true}} \supset\! I^u$$

Once again we use implication introduction with a new label w.

$$\cfrac{\cfrac{\begin{array}{c} \cfrac{}{A \; \mathit{true}}\, u \quad \cfrac{}{B \; \mathit{true}}\, w \\ \vdots \\ A \; \mathit{true} \end{array}}{B \supset A \; \mathit{true}} \supset\! I^w}{A \supset (B \supset A) \; \mathit{true}} \supset^u$$

Here we have two hypotheses to work with, but we only need the first labeled u to close the cap and complete the proof.

$$\cfrac{\cfrac{\cfrac{}{A \; \mathit{true}}\, u}{B \supset A \; \mathit{true}} \supset\! I^w}{A \supset (B \supset A) \; \mathit{true}} \supset\! I^u$$

This is an illustration of the fact that we don't need to use hypotheses, which which case they do not appear in the final proof.

Next, let's do an example that requires using hypotheses more than once, and also reasoning with elimination rules.

$$\begin{array}{c}\vdots\\ (A \wedge B) \supset (B \wedge A)\ \mathit{true}\end{array}$$

As usual, we start with implication introduction:

$$\dfrac{\begin{array}{c}\overline{A \wedge B\ \mathit{true}}\ u\\ \vdots\\ B \wedge A\ \mathit{true}\end{array}}{(A \wedge B) \supset (B \wedge A)\ \mathit{true}}\ {\supset}I^u$$

We are trying to prove a conjunction, so we can reduce it to proving the two conjuncts.

$$\dfrac{\dfrac{\begin{array}{c}\overline{A \wedge B\ \mathit{true}}\ u\\ \vdots\\ B\ \mathit{true}\end{array} \quad \begin{array}{c}\overline{A \wedge B\ \mathit{true}}\ u\\ \vdots\\ A\ \mathit{true}\end{array}}{B \wedge A\ \mathit{true}}\ {\wedge}I}{(A \wedge B) \supset (B \wedge A)\ \mathit{true}}\ {\supset}I^u$$

We have written the hypothesis u twice here, because it is available in both subproofs. In lecture we used a slightly different form where we had a "scratch area" on the side in which we recorded assumptions, and there we'd have to write it only once.

Let's look at the first subgoal, proving B *true* from $A \wedge B$ *true*. We cannot proceed with an introduction rule here (because B is arbitrary), but we can exploit the knowledge of $A \wedge B$ *true* by applying an elimination rule.

$$\dfrac{\dfrac{\begin{array}{c}\dfrac{\overline{A \wedge B\ \mathit{true}}\ u}{B\ \mathit{true}}\ {\wedge}E_2\\ \vdots\\ B\ \mathit{true}\end{array} \quad \begin{array}{c}\overline{A \wedge B\ \mathit{true}}\ u\\ \vdots\\ A\ \mathit{true}\end{array}}{B \wedge A\ \mathit{true}}\ {\wedge}I}{(A \wedge B) \supset (B \wedge A)\ \mathit{true}}\ {\supset}I^u$$

Now we can close the gap because we have a proof of B *true*, constructed by an elimination rule.

$$\dfrac{\dfrac{\dfrac{\overline{A \wedge B\ \mathit{true}}\ u}{B\ \mathit{true}}\ {\wedge}E_2 \quad \begin{array}{c}\overline{A \wedge B\ \mathit{true}}\ u\\ \vdots\\ A\ \mathit{true}\end{array}}{B \wedge A\ \mathit{true}}\ {\wedge}I}{(A \wedge B) \supset (B \wedge A)\ \mathit{true}}\ {\supset}I^u$$

The proof of the second premise of $\wedge I$ is entirely symmetric, so we won't go through each individual step of construction.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \wedge B\ \mathit{true}}\,u}{B\ \mathit{true}}\,\wedge E_2 \qquad \dfrac{\dfrac{}{A \wedge B\ \mathit{true}}\,u}{A\ \mathit{true}}\,\wedge E_1}{B \wedge A\ \mathit{true}}\,\wedge I}{(A \wedge B) \supset (B \wedge A)\ \mathit{true}}\,\supset I^u$$

This completes the proof. Let's make sure all the uses of u are above the rule where u is introduced ($\supset I^u$), and they are. So there are no scope violations and the proof is valid.

Finally, for this section, we construct a "counterexamples" which shows the necessity of the scope restrictions. Clearly, $(A \supset A) \wedge A$ should not be true for arbitrary A. We construct a "proof", starting with

$$\begin{array}{c} \vdots \\ (A \supset A) \wedge A\ \mathit{true} \end{array}$$

and applying conjunction introduction:

$$\dfrac{\begin{array}{c}\vdots \\ A \supset A\ \mathit{true}\end{array} \quad \begin{array}{c}\vdots \\ A\ \mathit{true}\end{array}}{(A \supset A) \wedge A\ \mathit{true}}\,\wedge I$$

In two steps we can fill in the previous proof of $A \supset A$.

$$\dfrac{\dfrac{\dfrac{}{A\ \mathit{true}}\,u}{A \supset A\ \mathit{true}}\,\supset I^u \qquad \begin{array}{c}\vdots \\ A\ \mathit{true}\end{array}}{(A \supset A) \wedge A\ \mathit{true}}\,\wedge I$$

Now we "notice" (incorrectly!) that we have an assumption *A true* so we use it to complete the proof attempt.

$$\left.\dfrac{\dfrac{\dfrac{}{A\ \mathit{true}}\,u}{A \supset A\ \mathit{true}}\,\supset I^u \qquad \dfrac{}{A\ \mathit{true}}\,u}{(A \supset A) \wedge A\ \mathit{true}}\,\wedge I\right\}\text{Incorrect!}$$

If this were a correct proof, we could now take a final step to deduce that A is true for an arbitrary A!

$$\left.\dfrac{\dfrac{\dfrac{\dfrac{}{A\ \mathit{true}}\,u}{A \supset A\ \mathit{true}}\,\supset I^u \qquad \dfrac{}{A\ \mathit{true}}\,u}{(A \supset A) \wedge A\ \mathit{true}}\,\wedge I}{A\ \mathit{true}}\,\wedge E_2\right\}\text{Incorrect!}$$

Of course, the incorrect inference here is using u to justify A *true* in the second premise of the $\wedge I$ rule. This use of u is *not above the rule* $\supset I^u$ *that introduces* u and therefore out of scope.

We call a system of deduction *inconsistent* if we can derive that any arbitrary proposition A is true. Such a system is useless for proof. This is example shows that natural deduction would be inconsistent if we do not respect scope boundaries because in the erroneous proof above, A is in fact arbitrary.

5 Disjunction and Falsehood

So far we have explained the meaning of conjunction, truth, and implication. The disjunction "A *or* B" (written as $A \vee B$) is more difficult, but does not require any new judgment forms. Disjunction is characterized by two introduction rules: $A \vee B$ is true, if either A or B is true.

$$\frac{A\ \mathit{true}}{A \vee B\ \mathit{true}}\ \vee I_1 \qquad \frac{B\ \mathit{true}}{A \vee B\ \mathit{true}}\ \vee I_2$$

Now it would be incorrect to have an elimination rule such as

$$\frac{A \vee B\ \mathit{true}}{A\ \mathit{true}}\ \vee E_1?$$

because even if we know that $A \vee B$ is true, we do not know whether the disjunct A or the disjunct B is true. Concretely, with such a rule we could derive the truth of an arbitrary proposition A as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{B\ \mathit{true}}\ u}{B \supset B\ \mathit{true}}\ \supset I^u}{A \vee (B \supset B)\ \mathit{true}}\ \vee I_2}{A\ \mathit{true}}\ \vee E_1?$$

Thus we take a different approach. If we know that $A \vee B$ is true, we must consider two cases: A *true* and B *true*. If we can prove a conclusion C *true* in both cases, then C must be true! Written as an inference rule:

$$\frac{A \vee B\ \mathit{true} \qquad \begin{array}{c} \overline{A\ \mathit{true}}\ u \\ \vdots \\ C\ \mathit{true} \end{array} \qquad \begin{array}{c} \overline{B\ \mathit{true}}\ w \\ \vdots \\ C\ \mathit{true} \end{array}}{C\ \mathit{true}}\ \vee E^{u,w}$$

If we know that $A \vee B$ *true* then we also know C *true*, if that follows both in the case where $A \vee B$ *true* because A is true and in the case where $A \vee B$ *true* because B is true. Note that we use once again the mechanism of hypothetical judgments. In the proof of the second premise we may use the assumption A *true* labeled u, in the proof of the third premise we may use the assumption B *true* labeled w. Both are discharged at the disjunction elimination rule.

Let us justify the conclusion of this rule more explicitly. By the first premise we know $A \vee B$ *true*. The premises of the two possible introduction rules are A *true* and B *true*. In case A *true* we conclude C *true* by the substitution principle and the second premise: we substitute the proof of $A\ true$ for any use of the assumption labeled u in the hypothetical derivation. The case for B *true* is symmetric, using the hypothetical derivation in the third premise.

In lecture, it was proposed to have the following rule instead:

$$\frac{A \vee B\ \mathit{true} \quad A \supset C\ \mathit{true} \quad B \supset C\ \mathit{true}}{C\ \mathit{true}}\ \vee E?$$

This is closely related to the rule we chose and it is in fact *derivable* and therefore correct. But it has a drawback: in our verificationist enterprise we would like to explain the meaning of the connectives completely independently of each other. That is, the meaning of $A \vee B$ should only depend on the meanings of A and B, and not on any other connectives. The above rule does not satisfy this principle because it uses implication in the premises. While this may seem like a small pedantic point, allowing such a rule as an elimination rule would have significant negative consequences once we start to investigate principles of proof construction. Fortunately, it does not matter in the sense that anyone wishing to use $\vee E$? can do so safely because it can be derived.

In lecture we proved

$$((A \vee B) \supset C) \supset ((A \supset C) \wedge (B \supset C))\ \mathit{true}$$

Let's go through this.

$$\begin{array}{c} \vdots \\ ((A \vee B) \supset C) \supset ((A \supset C) \wedge (B \supset C))\ \mathit{true} \end{array}$$

We combine the first two steps, which are implication introduction followed by conjunction introduction.

$$\frac{\dfrac{\begin{array}{c}\dfrac{}{(A \vee B) \supset C\ \mathit{true}}\ u \\ \vdots \\ A \supset C\end{array} \qquad \begin{array}{c}\dfrac{}{(A \vee B) \supset C\ \mathit{true}}\ u \\ \vdots \\ B \supset C\end{array}}{(A \supset C) \wedge (B \supset C)\ \mathit{true}}\ \wedge I}{((A \vee B) \supset C) \supset ((A \supset C) \wedge (B \supset C))\ \mathit{true}}\ \supset I^u$$

We work on the first open subgoals, once again applying implication introduction.

$$\frac{\dfrac{\dfrac{\begin{array}{c}\dfrac{}{(A \vee B) \supset C\ \mathit{true}}\ u \quad \dfrac{}{A\ \mathit{true}}\ w \\ \vdots \\ C\ \mathit{true}\end{array}}{A \supset C\ \mathit{true}}\ \supset I^w \qquad \begin{array}{c}\dfrac{}{(A \vee B) \supset C\ \mathit{true}}\ u \\ \vdots \\ B \supset C\end{array}}{(A \supset C) \wedge (B \supset C)\ \mathit{true}}\ \wedge I}{((A \vee B) \supset C) \supset ((A \supset C) \wedge (B \supset C))\ \mathit{true}}\ \supset I^u$$

At this point we can no longer work upwards with introduction rules in the first subgoal because the conclusion $C\ \mathit{true}$ is general. But we have an assumption labeled u which is an implication, so we can try implication elimination.

$$
\dfrac{\dfrac{\dfrac{}{(A \lor B) \supset C\ \mathit{true}}\,u \qquad \begin{array}{c}\dfrac{}{(A \lor B) \supset C\ \mathit{true}}\,u \quad \dfrac{}{A\ \mathit{true}}\,w \\ \vdots \\ A \lor B\ \mathit{true}\end{array}}{\dfrac{C\ \mathit{true}}{A \supset C\ \mathit{true}}\supset I^w}\supset E \qquad \begin{array}{c}\dfrac{}{(A \lor B) \supset C\ \mathit{true}}\,u \\ \vdots \\ B \supset C\end{array}}{\dfrac{(A \supset C) \land (B \supset C)\ \mathit{true}}{((A \lor B) \supset C) \supset ((A \supset C) \land (B \supset C))\ \mathit{true}}\supset I^u}\land I
$$

In the first remaining subgoal we no longer need the hypothesis labeled u, because the assumptions $A\ \mathit{true}$ (labeled w) is sufficient to complete that part of the proof using one of the two disjunction introduction rules.

$$
\dfrac{\dfrac{\dfrac{}{(A \lor B) \supset C\ \mathit{true}}\,u \qquad \dfrac{\dfrac{}{A\ \mathit{true}}\,w}{A \lor B\ \mathit{true}}\lor I_1}{\dfrac{C\ \mathit{true}}{A \supset C\ \mathit{true}}\supset I^w}\supset E \qquad \begin{array}{c}\dfrac{}{(A \lor B) \supset C\ \mathit{true}}\,u \\ \vdots \\ B \supset C\end{array}}{\dfrac{(A \supset C) \land (B \supset C)\ \mathit{true}}{((A \lor B) \supset C) \supset ((A \supset C) \land (B \supset C))\ \mathit{true}}\supset I^u}\land I
$$

The only remaining subgoal is symmetric to one we already proved, so we'll just fill it in.

$$
\dfrac{\dfrac{\dfrac{}{(A \lor B) \supset C\ \mathit{true}}\,u \qquad \dfrac{\dfrac{}{A\ \mathit{true}}\,w}{A \lor B\ \mathit{true}}\lor I_1}{\dfrac{C\ \mathit{true}}{A \supset C\ \mathit{true}}\supset I^w}\supset E \qquad \dfrac{\dfrac{}{(A \lor B) \supset C\ \mathit{true}}\,u \qquad \dfrac{\dfrac{}{B\ \mathit{true}}\,x}{A \lor B\ \mathit{true}}\lor I_2}{\dfrac{C\ \mathit{true}}{A \supset C\ \mathit{true}}\supset I^x}\supset E}{\dfrac{(A \supset C) \land (B \supset C)\ \mathit{true}}{((A \lor B) \supset C) \supset ((A \supset C) \land (B \supset C))\ \mathit{true}}\supset I^u}\land I
$$

We should go over the proof once more to make sure there are no scope violations and we see, indeed, u, w, and x are only used above the inference that introduced it into the proof.

This concludes the discussion of disjunction. Falsehood (written as $\bot$, sometimes called absurdity) is a proposition that should have no proof! Therefore there are no introduction rules.

Since there cannot be a proof of $\bot\ \mathit{true}$, it is sound to conclude the truth of any arbitrary proposition if we know $\bot\ \mathit{true}$. This justifies the elimination rule

$$
\dfrac{\bot\ \mathit{true}}{C\ \mathit{true}}\,\bot E
$$

We can also think of falsehood as a disjunction between zero alternatives. By analogy with the binary disjunction, we therefore have zero introduction rules, and an elimination rule in which we have to consider zero cases. This is precisely the $\bot E$ rule above.

From this is might seem that falsehood it useless: we can never prove it. This is correct, except that we might reason from contradictory hypotheses! We will see some examples when we discuss negation, since we may think of the proposition "*not* A" (written $\neg A$) as $A \supset \bot$. In other words, $\neg A$ is true precisely if the assumption A *true* is contradictory because we could derive $\bot$ *true*.

Let's define

$$\neg A \triangleq A \supset \bot$$

and prove that it is not possible for A and $\neg A$ to both be true.

$$\begin{array}{c} \vdots \\ (A \land \neg A) \supset \bot\ \mathit{true} \end{array}$$

As usual, we start with implication introduction.

$$\dfrac{\begin{array}{c}\dfrac{}{A \land \neg A\ \mathit{true}}\,u \\ \vdots \\ \bot\ \mathit{true}\end{array}}{(A \land \neg A) \supset \bot\ \mathit{true}}\,\supset I^u$$

Now we proceed by reasoning from our hypothesis using elimination rules. Let's just extract both components of the conjunction.

$$\dfrac{\begin{array}{c}\dfrac{\dfrac{}{A \land \neg A\ \mathit{true}}\,u}{\neg A\ \mathit{true}}\,\land E_2 \quad \dfrac{\dfrac{}{A \land \neg A\ \mathit{true}}\,u}{A\ \mathit{true}}\,\land E_1 \\ \vdots \\ \bot\ \mathit{true}\end{array}}{(A \land \neg A) \supset \bot\ \mathit{true}}\,\supset I^u$$

But we made a definition in our language of discourse, so $\neg A$ is really the same as $A \supset \bot$. So below we have exactly the same incomplete proof, written in an equivalent way.

$$\dfrac{\begin{array}{c}\dfrac{\dfrac{}{A \land \neg A\ \mathit{true}}\,u}{A \supset \bot\ \mathit{true}}\,\land E_2 \quad \dfrac{\dfrac{}{A \land \neg A\ \mathit{true}}\,u}{A\ \mathit{true}}\,\land E_1 \\ \vdots \\ \bot\ \mathit{true}\end{array}}{(A \land \neg A) \supset \bot\ \mathit{true}}\,\supset I^u$$

Now it is easy to recognize that we can complete the proof with implication elimination.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \land \neg A\ \mathit{true}}\,u}{A \supset \bot\ \mathit{true}}\,\land E_2 \quad \dfrac{\dfrac{}{A \land \neg A\ \mathit{true}}\,u}{A\ \mathit{true}}\,\land E_1}{\bot\ \mathit{true}}\,\supset E}{(A \land \neg A) \supset \bot\ \mathit{true}}\,\supset I^u$$

Uses of u are all in scope, so this is a valid proof. It illustrates that even though $\bot$ is never true, we might nevertheless prove it from contradictory assumptions.

6 Summary: Natural Deduction

The judgments, propositions, and inference rules we have defined so far collectively form a system of *natural deduction*. It is a minor variant of a system introduced by Gentzen [1935] and studied in depth by Prawitz [1965]. One of Gentzen's main motivations was to devise rules that model mathematical reasoning as directly as possible, although clearly in much more detail than in a typical mathematical argument.

The specific interpretation of the truth judgment underlying these rules is *intuitionistic* or *constructive*. This differs from the *classical* or *Boolean* interpretation of truth. For example, classical logic accepts the proposition $A \vee (A \supset B)$ as true for arbitrary A and B, although in the system we have presented so far this would have no proof. Classical logic is based on the principle that every proposition must be true or false. If we distinguish these cases we see that $A \vee (A \supset B)$ should be accepted, because in case that A is true, the left disjunct holds; in case A is false, the right disjunct holds. In contrast, intuitionistic logic is based on explicit evidence, and evidence for a disjunction requires evidence for one of the disjuncts. We will return to classical logic and its relationship to intuitionistic logic later; for now our reasoning remains intuitionistic since, as we will see, it has a direct connection to functional computation, which classical logic lacks.

We summarize the rules of inference for the truth judgment introduced so far in Figure 1.

7 Notational Definition

So far, we have defined the meaning of the logical connectives by their introduction rules, which is the so-called *verificationist* approach. Another common way to define a logical connective is by a *notational definition*. A notational definition gives the meaning of the general form of a proposition in terms of another proposition whose meaning has already been defined. For example, we can define *logical equivalence*, written $A \leftrightarrow B$ as $(A \supset B) \wedge (B \supset A)$. This definition is justified, because we already understand implication and conjunction.

As mentioned before, another common notational definition in intuitionistic logic is $\neg A \triangleq (A \supset \bot)$. Several other, more direct definitions of intuitionistic negation also exist, and we will see some of them later in the course. Perhaps the most intuitive one is to say that $\neg A$ *true* if A *false*, but this requires the new judgment of falsehood.

Notational definitions can be convenient, but they can be a bit cumbersome at times. We sometimes give a notational definition and then derive introduction and elimination rules for the connective. It should be understood that these rules, even if they may be called introduction or elimination rules, have a different status from those that define a

Introduction Rules	**Elimination Rules**
$\dfrac{A\ \textit{true} \quad B\ \textit{true}}{A \wedge B\ \textit{true}}\ \wedge I$	$\dfrac{A \wedge B\ \textit{true}}{A\ \textit{true}}\ \wedge E_1 \quad \dfrac{A \wedge B\ \textit{true}}{B\ \textit{true}}\ \wedge E_2$
$\dfrac{}{\top\ \textit{true}}\ \top I$	*no* $\top E$ *rule*
$\begin{array}{c}\dfrac{}{A\ \textit{true}}\ u \\ \vdots \\ \dfrac{B\ \textit{true}}{A \supset B\ \textit{true}}\ \supset I^u\end{array}$	$\dfrac{A \supset B\ \textit{true} \quad A\ \textit{true}}{B\ \textit{true}}\ \supset E$
$\dfrac{A\ \textit{true}}{A \vee B\ \textit{true}}\ \vee I_1 \quad \dfrac{B\ \textit{true}}{A \vee B\ \textit{true}}\ \vee I_2$	$\dfrac{A \vee B\ \textit{true} \quad \begin{array}{c}\dfrac{}{A\ \textit{true}}\ u \\ \vdots \\ C\ \textit{true}\end{array} \quad \begin{array}{c}\dfrac{}{B\ \textit{true}}\ w \\ \vdots \\ C\ \textit{true}\end{array}}{C\ \textit{true}}\ \vee E^{u,w}$
no $\perp I$ *rule*	$\dfrac{\perp\ \textit{true}}{C\ \textit{true}}\ \perp E$

Figure 1: Rules for intuitionistic natural deduction

connective. In this particular case, we get the derived rules

$$\begin{array}{c}\dfrac{}{A\ \textit{true}}\ u \\ \vdots \\ \dfrac{\perp\ \textit{true}}{\neg A\ \textit{true}}\ \neg I^u\end{array} \qquad \dfrac{\neg A\ \textit{true} \quad A\ \textit{true}}{\perp\ \textit{true}}\ \neg E$$

You should convince yourself that these are indeed derived rules under the notational definition of $\neg A$. They also *almost* have the form of introduction and elimination rules, except that we use $\perp$ to define $\neg A$, while previously we avoided using other connectives besides the one we are defining.

8 Further Examples[1]

[1] not covered in lecture

We show a few more examples of proofs, starting with $A \supset (B \supset (A \wedge B))$.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A\ \mathit{true}}\,u \quad \dfrac{}{B\ \mathit{true}}\,w}{A \wedge B\ \mathit{true}}\,\wedge I}{B \supset (A \wedge B)\ \mathit{true}}\,{\supset}I^w}{A \supset (B \supset (A \wedge B))\ \mathit{true}}\,{\supset}I^u$$

Note that this derivation is not hypothetical (it does not depend on any assumptions). The assumption A *true* labeled u is discharged in the last inference, and the assumption B *true* labeled w is discharged in the second-to-last inference. It is critical that a discharged hypothesis is no longer available for reasoning, and that all labels introduced in a derivation are distinct.

When we construct such a derivation, we generally proceed by a combination of bottom-up and top-down reasoning. The next example is a distributivity law, allowing us to move implications over conjunctions. This time, we show the partial proofs in each step. Of course, other sequences of steps in proof constructions are also possible.

$$\begin{array}{c} \vdots \\ (A \supset (B \wedge C)) \supset ((A \supset B) \wedge (A \supset C))\ \mathit{true} \end{array}$$

First, we use the implication introduction rule bottom-up.

$$\dfrac{\begin{array}{c} \dfrac{}{A \supset (B \wedge C)\ \mathit{true}}\,u \\ \vdots \\ (A \supset B) \wedge (A \supset C)\ \mathit{true} \end{array}}{(A \supset (B \wedge C) \supset ((A \supset B) \wedge (A \supset C))\ \mathit{true}}\,{\supset}I^u$$

Next, we use the conjunction introduction rule bottom-up, copying the available assumptions to both branches in the scope.

$$\dfrac{\dfrac{\begin{array}{c} \dfrac{}{A \supset (B \wedge C)\ \mathit{true}}\,u \\ \vdots \\ A \supset B\ \mathit{true} \end{array} \quad \begin{array}{c} \dfrac{}{A \supset (B \wedge C)\ \mathit{true}}\,u \\ \vdots \\ A \supset C\ \mathit{true} \end{array}}{(A \supset B) \wedge (A \supset C)\ \mathit{true}}\,\wedge I}{(A \supset (B \wedge C)) \supset ((A \supset B) \wedge (A \supset C))\ \mathit{true}}\,{\supset}I^u$$

We now pursue the left branch, again using implication introduction bottom-up.

$$\dfrac{\dfrac{\dfrac{\begin{array}{c} \dfrac{}{A \supset (B \wedge C)\ \mathit{true}}\,u \quad \dfrac{}{A\ \mathit{true}}\,w \\ \vdots \\ B\ \mathit{true} \end{array}}{A \supset B\ \mathit{true}}\,{\supset}I^w \quad \begin{array}{c} \dfrac{}{A \supset (B \wedge C)\ \mathit{true}}\,u \\ \vdots \\ A \supset C\ \mathit{true} \end{array}}{(A \supset B) \wedge (A \supset C)\ \mathit{true}}\,\wedge I}{(A \supset (B \wedge C)) \supset ((A \supset B) \wedge (A \supset C))\ \mathit{true}}\,{\supset}I^u$$

Note that the hypothesis $A\ \mathit{true}$ is available only in the left branch and not in the right one: it is discharged at the inference ${\supset}I^w$. We now switch to top-down reasoning, taking advantage of implication elimination.

$$\dfrac{\dfrac{\begin{array}{c}\dfrac{\dfrac{}{A \supset (B \wedge C)\ \mathit{true}}\,u \quad \dfrac{}{A\ \mathit{true}}\,w}{B \wedge C\ \mathit{true}}\,{\supset}E \\ \vdots \\ \dfrac{B\ \mathit{true}}{A \supset B\ \mathit{true}}\,{\supset}I^w\end{array} \qquad \begin{array}{c}\dfrac{}{A \supset (B \wedge C)\ \mathit{true}}\,u \\ \vdots \\ A \supset C\ \mathit{true}\end{array}}{(A \supset B) \wedge (A \supset C)\ \mathit{true}}\,{\wedge}I}{(A \supset (B \wedge C)) \supset ((A \supset B) \wedge (A \supset C))\ \mathit{true}}\,{\supset}I^u$$

Now we can close the gap in the left-hand side by conjunction elimination.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \supset (B \wedge C)\ \mathit{true}}\,u \quad \dfrac{}{A\ \mathit{true}}\,w}{B \wedge C\ \mathit{true}}\,{\supset}E}{B\ \mathit{true}}\,{\wedge}E_1}{A \supset B\ \mathit{true}}\,{\supset}I^w \qquad \begin{array}{c}\dfrac{}{A \supset (B \wedge C)\ \mathit{true}}\,u \\ \vdots \\ A \supset C\ \mathit{true}\end{array}}{(A \supset B) \wedge (A \supset C)\ \mathit{true}}\,{\wedge}I}{(A \supset (B \wedge C)) \supset ((A \supset B) \wedge (A \supset C))\ \mathit{true}}\,{\supset}I^u$$

The right premise of the conjunction introduction can be filled in analogously. We skip the intermediate steps and only show the final derivation.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \supset (B \wedge C)\ \mathit{true}}\,u \quad \dfrac{}{A\ \mathit{true}}\,w}{B \wedge C\ \mathit{true}}\,{\supset}E}{B\ \mathit{true}}\,{\wedge}E_1}{A \supset B\ \mathit{true}}\,{\supset}I^w \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A \supset (B \wedge C)\ \mathit{true}}\,u \quad \dfrac{}{A\ \mathit{true}}\,v}{B \wedge C\ \mathit{true}}\,{\supset}E}{C\ \mathit{true}}\,{\wedge}E_2}{A \supset C\ \mathit{true}}\,{\supset}I^v}{(A \supset B) \wedge (A \supset C)\ \mathit{true}}\,{\wedge}I}{(A \supset (B \wedge C)) \supset ((A \supset B) \wedge (A \supset C))\ \mathit{true}}\,{\supset}I^u$$

Because of the complex nature of the elimination rule, reasoning with disjunction is more difficult than with implication and conjunction. As a simple example, we prove the commutativity of disjunction.

$$\begin{array}{c}\vdots \\ (A \vee B) \supset (B \vee A)\ \mathit{true}\end{array}$$

We begin with an implication introduction.

$$\dfrac{\begin{array}{c}\dfrac{}{A \vee B\ \mathit{true}}\,u \\ \vdots \\ B \vee A\ \mathit{true}\end{array}}{(A \vee B) \supset (B \vee A)\ \mathit{true}}\,{\supset}I^u$$

At this point we cannot use either of the two disjunction introduction rules. The problem is that neither B nor A follow from our assumption $A \vee B$! So first we need to distinguish the two cases via the rule of disjunction elimination.

$$\dfrac{\dfrac{\dfrac{}{A \vee B\ \mathit{true}}\,u \quad \begin{array}{c}\dfrac{}{A\ \mathit{true}}\,v \\ \vdots \\ B \vee A\ \mathit{true}\end{array} \quad \begin{array}{c}\dfrac{}{B\ \mathit{true}}\,w \\ \vdots \\ B \vee A\ \mathit{true}\end{array}}{B \vee A\ \mathit{true}}\,{\vee}E^{v,w}}{(A \vee B) \supset (B \vee A)\ \mathit{true}}\,{\supset}I^u$$

The assumption labeled u is still available for each of the two proof obligations, but we have omitted it, since it is no longer needed.

Now each gap can be filled in directly by the two disjunction introduction rules.

$$\dfrac{\dfrac{\dfrac{}{A \vee B\ \mathit{true}}\,u \quad \dfrac{\dfrac{}{A\ \mathit{true}}\,v}{B \vee A\ \mathit{true}}\,{\vee}I_2 \quad \dfrac{\dfrac{}{B\ \mathit{true}}\,w}{B \vee A\ \mathit{true}}\,{\vee}I_1}{B \vee A\ \mathit{true}}\,{\vee}E^{v,w}}{(A \vee B) \supset (B \vee A)\ \mathit{true}}\,{\supset}I^u$$

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