Lecture Notes on
Sequent Calculus

15-317: Constructive Logic
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Lecture 7
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1 Introduction

In this lecture we shift to a different presentation style for proof calculi. We develop the sequent calculus as a formal system for proof search in natural deduction. In addition to enabling an understanding of proof search, sequent calculus leads to a more transparent management of the scope of assumptions during a proof, and also allows us more proof theory, so proofs about properties of proofs.

Sequent calculus was originally introduced by Gentzen [1935], primarily as a technical device for proving consistency of predicate logic. Our goal of describing a proof search procedure for natural deduction predisposes us to a formulation due to Kleene [1952] called G_3.

Our sequent calculus is designed to *exactly* capture the notion of a *verification*, introduced in Lecture 5. Recall that verifications are constructed bottom-up, from the conclusion to the premises using introduction rules, while uses are constructed top-down, from hypotheses to conclusions using elimination rules. They meet in the middle, where a proposition we have deduced from assumptions may be used as a verification. In the sequent calculus, both steps work bottom-up, which will ultimately allows us to prove global versions of the local soundness and completeness properties.

2 Sequents

When constructing a verification, we are generally in a state of the following form

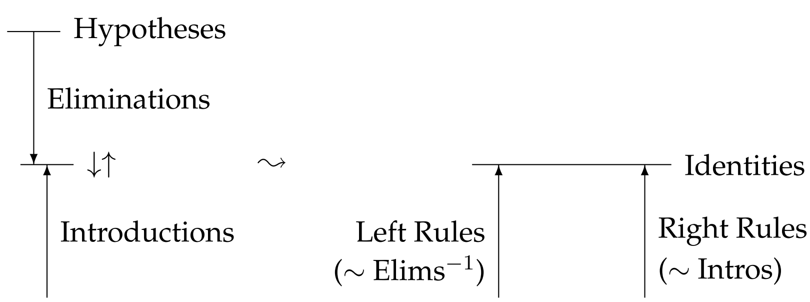
$$\begin{array}{ccc} A_1{\downarrow} & \cdots & A_n{\downarrow} \\ & \vdots & \\ & C{\uparrow} & \end{array}$$

where $A_1, \ldots, A_n$ embody knowledge we may *use*, while C is the conclusion we are trying to *verify*. A *sequent* is just a local notation for such a partially complete verification. We write

$$A_1\ \mathsf{ante}, \ldots, A_n\ \mathsf{ante} \Longrightarrow C\ \mathsf{succ}$$

where the judgments $A\,\mathsf{ante}$ (for "*A is an antecedent*") and $C\,\mathsf{right}$ (for "*C is a succedent*") correspond to $A{\downarrow}$ and $C{\uparrow}$, respectively. Sequent calculus is explicit about the assumptions that are available for use (antecedent) and about the proposition to be verified (succedent). Because they have a fixed position in the sequent, we often omit the judgment ante and succ.

The rules that define the $A\,\mathsf{ante}$ and $A\,\mathsf{succ}$ judgment are systematically constructed from the introduction and elimination rules, keeping in mind their directions in terms of verifications and uses. Introduction rules are translated to corresponding *right rules*. Since introduction rules already work from the conclusion to the premises, this mapping is straightforward. Elimination rules work top-down, so they have to be flipped upside-down in order to work as sequent rules, and are turned into *left rules*. Pictorially:



We now proceed connective by connective, constructing the right and left rules from the introduction and elimination rules. A collection of antecedents $A_1\,\mathsf{ante}, \ldots, A_n\,\mathsf{ante}$ is abbreviated as Γ. The order of the antecedents does not matter, so we will allow them to be implicitly reordered.

Conjunction. We recall the introduction rule first and show the corresponding right rule.

$$\frac{A\uparrow \quad B\uparrow}{A\wedge B\uparrow}\,\wedge I \qquad\qquad \frac{\Gamma \Longrightarrow A \quad \Gamma \Longrightarrow B}{\Gamma \Longrightarrow A\wedge B}\,\wedge R$$

The only difference is that the antecedents Γ are made explicit. Both premises have the same antecedents, because any assumption can be used in both subdeductions.

As discussed in lecture, it is a perfectly sensible and correct rule to write

$$\frac{\Gamma_1 \Longrightarrow A \quad \Gamma_2 \Longrightarrow B}{\Gamma_1\cup\Gamma_2 \Longrightarrow A\wedge B}\,\wedge R'$$

This rule is most naturally read from the premises to the conclusion. Indeed, if we have a verification of A using Γ_1 and a verification of B using Γ_2, then we can obtain a verification of $A\wedge B$ using $\Gamma_1\cup\Gamma_2$. On the other hand, if we think of *bottom-up* proof construction we want to propagate *all* the usable antecedents to both subgoals and so for this purpose the original rule is preferable.

There are two elimination rules, so we have two corresponding left rules.

$$\frac{A \wedge B\downarrow}{A\downarrow}\ \wedge E_1 \qquad \frac{\Gamma, A \wedge B, A \Longrightarrow C}{\Gamma, A \wedge B \Longrightarrow C}\ \wedge L_1$$

$$\frac{A \wedge B\downarrow}{B\downarrow}\ \wedge E_2 \qquad \frac{\Gamma, A \wedge B, B \Longrightarrow C}{\Gamma, A \wedge B \Longrightarrow C}\ \wedge L_2$$

We preserve the *principal formula* $A \wedge B$ of the left rule in the premise. This is because we are trying to model proof construction in natural deduction where assumptions can be used multiple times. If we temporarily ignore the copy of $A \wedge B$ in the premise, it is easier to see how the rules correspond.

Truth. Truth is defined just by an introduction rule and has no elimination rule. Consequently, there is only a right rule in the sequent calculus and no left rule.

$$\frac{}{\top\uparrow}\ \top I \qquad \frac{}{\Gamma \Longrightarrow \top}\ \top R$$

Implication. Again, the right rule for implication is quite straightforward, because it models the introduction rule directly.

$$\frac{\begin{array}{c}\overline{A\downarrow}\ u \\ \vdots \\ B\uparrow\end{array}}{A \supset B\uparrow}\ \supset^u \qquad \frac{\Gamma, A \Longrightarrow B}{\Gamma \Longrightarrow A \supset B}\ \supset R$$

We see here one advantage of the sequent calculus over natural deduction: the scoping for additional assumptions is simple. The new antecedent A left is available anywhere in the deduction of the premise, because in the sequent calculus we only work bottom-up. Moreover, we arrange all the rules so that antecedents are *persistent*: they are always propagated from the conclusion to all premises.

The elimination rule is trickier, because it involves a more complicated combination of verifications and uses.

$$\frac{A \supset B\downarrow \quad A\uparrow}{B\downarrow}\ \supset E \qquad \frac{\Gamma, A \supset B \Longrightarrow A \quad \Gamma, A \supset B, B \Longrightarrow C}{\Gamma, A \supset B \Longrightarrow C}\ \supset L$$

In words: in order to use $A \supset B$ to verify C we have to produce a verification of A, in which case we can then use B in the verification of C. The antecedent $A \supset B$ is carried over to both premises to maintain persistence. Note that the premises of the left rule are reversed, when compared to the elimination rule to indicate that we do not want to make the assumption B unless we have already established A.

In terms of provability, there is some redundancy in the $\supset L$ rule. For example, once we know B, we no longer need $A \supset B$, because B is a stronger assumption. As stressed above, we try to maintain the correspondence to natural deductions and postpone these kinds of optimization until later.

Disjunction. The right rules correspond directly to the introduction rules, as usual.

$$\frac{A\uparrow}{A \vee B\uparrow}\ \vee I_1 \qquad \frac{\Gamma \Longrightarrow A}{\Gamma \Longrightarrow A \vee B}\ \vee R_1$$

$$\frac{B\uparrow}{A \vee B\uparrow}\ \vee I_2 \qquad \frac{\Gamma \Longrightarrow B}{\Gamma \Longrightarrow A \vee B}\ \vee R_2$$

The disjunction elimination rule was somewhat odd, because it introduced two new assumptions, one for each case of the disjunction. The left rule for disjunction actually has a simpler form that is more consistent with all the other rules we have shown so far.

$$\frac{A \vee B\downarrow \quad \begin{matrix}\overline{A\downarrow}^{\,u} \\ \vdots \\ C\uparrow\end{matrix} \quad \begin{matrix}\overline{B\downarrow}^{\,w} \\ \vdots \\ C\uparrow\end{matrix}}{C\uparrow}\ \vee E^{u,w} \qquad \frac{\Gamma, A \vee B, A \Longrightarrow C \quad \Gamma, A \vee B, B \Longrightarrow C}{\Gamma, A \vee B \Longrightarrow C}\ \vee L$$

As for implication, scoping issues are more explicit and simplified because the new assumptions A and B in the first and second premise, respectively, are available anywhere in the deduction above. But the assumption A is only available in the deduction for the left premise, while B is only available in the right premise. Sequent calculus is explicit about that. The sequent calculus formulation also makes it more transparent what the appropriate verification/uses assignment is.

Falsehood. Falsehood has no introduction rule, and therefore no right rule in the sequent calculus. To arrive at the left rule, we need to pay attention to the distinction between uses and verifications, or we can construct the 0-ary case of disjunction from above.

$$\frac{\bot\downarrow}{C\uparrow}\ \bot E \qquad \frac{}{\Gamma, \bot \Longrightarrow C}\ \bot L$$

Completing verifications. Recall that we cannot use an introduction rule to verify atomic propositions P because they cannot be broken down further. The only possible verification of P is directly via a use of P. In the version of verifications we have presented, we can complete the construction of a verification whenever $P\downarrow$ is available to conclude $P\uparrow$.[1] This turns into a so-called *initial sequent*, also called and application of the *identity rule*.

$$\frac{P\downarrow}{P\uparrow}\ \downarrow\uparrow^{*} \qquad \frac{}{\Gamma, P \Longrightarrow P}\ \mathsf{id}$$

This rule has a special status in that it does not break down any proposition, but establishes a connection between two judgments. In natural deduction, it is the connection between

[1] There are laxer version of this, where the $\downarrow\uparrow$ rule can be applied to all propositions A, not just variables. We will return to this point in the next lecture.

uses and verifications; in sequent calculus, it is the connection between the left and right judgments.

As a simple example, we consider the proof of $(A \lor B) \supset (B \lor A)$. Since A and B are variables we may complete the derivations with identity rules.

$$\dfrac{\dfrac{\dfrac{}{A \lor B, A \Longrightarrow A}\,\mathsf{id}}{A \lor B, A \Longrightarrow B \lor A}\,\lor R_2 \quad \dfrac{\dfrac{}{A \lor B, B \Longrightarrow B}\,\mathsf{id}}{A \lor B, B \Longrightarrow B \lor A}\,\lor R_1}{\dfrac{A \lor B \Longrightarrow B \lor A}{\Longrightarrow (A \lor B) \supset (B \lor A)}\,\supset R}\,\lor L$$

Observe that sequent calculus proofs are always constructed bottom-up, with the desired conclusion at the bottom, working upwards using the respective left or right proof rules in the antecedent or succedent.

3 Observations on Sequent Proofs

We have already mentioned that antecedents in sequent proofs are *persistent*: once an assumption is made, it is henceforth usable above the inference that introduces it. Sequent proofs also obey the important *subformula property*: if we examine the complete or partial proof above a sequent, we observe that all sequents are made up of subformulas of the sequent itself. This is consistent with the design criteria for the verifications: the verification of a proposition A may only contain subformulas of A. This is important from multiple perspectives. Foundationally, we think of verifications as defining the meaning of the propositions, so a verification of a proposition should only depend on its constituents. For proof search, it means we do not have to try to resort to some unknown formula, but can concentrate on subformulas of our proof goal.

If we trust for the moment that a proposition A is true if and only if it has a deduction in the sequent calculus (as $\Longrightarrow A$), we can use the sequent calculus to formally prove that some proposition can *not* be true in general. For example, we can prove that intuitionistic logic is *consistent*.

Theorem 1 (Consistency) *It is not the case that* $\Longrightarrow \bot$.

Proof: No left rule is applicable, since there is no antecedent. No right rule is applicable, because there is no right rule for falsehood. The identity rule is not applicable either. Therefore, there cannot be a proof of $\Longrightarrow \bot$. □

Theorem 2 (Disjunction Property) *If* $\Longrightarrow A \lor B$ *then either* $\Longrightarrow A$ *or* $\Longrightarrow B$.

Proof: No left rule is applicable, since there is no antecedent. The only right rules that are applicable are $\lor R_1$ and $\lor R_2$. In the first case, we have $\Longrightarrow A$, in the second $\Longrightarrow B$. □

Theorem 3 (Failure of Excluded Middle) *It is not the case that* $\Longrightarrow A \lor \neg A$ *for arbitrary* A.

Proof: From the disjunction property, either $\Longrightarrow A$ or $\Longrightarrow \neg A$. For the first sequent, no rule applies. For the second sequent, only $\supset R$ applies and we would have to have a deduction of $A \Longrightarrow \bot$. But for this sequent no rule applies. $\square$

Of course, there are still specific formulas A for which $\Longrightarrow A \vee \neg A$ will be provable, such as $\Longrightarrow \top \vee \neg \top$ or $\Longrightarrow \bot \vee \neg \bot$, but not generally for any A.

4 Provability and Unprovability

We have already seen that neither $\bot$ for $A \vee \neg A$ are provable in the sequent calculus from the empty set of antecedents.

Let's investigate whether implication distributes over disjunction. There are two propositions to verify (or deny a verification):

$$((A \supset B) \vee (A \supset C)) \supset (A \supset (B \vee C))\uparrow$$
$$(A \supset (B \vee C)) \supset ((A \supset B) \vee (A \supset C))\uparrow$$

The first one should be intuitively true: if we assume that A implies B or A implies C, then reasoning by cases should yield that A implies B or C. We build a sequent derivation entirely bottom-up.

$$\begin{array}{c} \vdots \\ \Longrightarrow ((A \supset B) \vee (A \supset C)) \supset (A \supset (B \vee C)) \end{array}$$

There is no choice here: we have to apply the right rule for implication.

$$\dfrac{\begin{array}{c} \vdots \\ (A \supset B) \vee (A \supset C) \Longrightarrow A \supset (B \vee C) \end{array}}{\Longrightarrow ((A \supset B) \vee (A \supset C)) \supset (A \supset (B \vee C))} \supset R$$

Now we have a choice the between $\vee L$ and $\supset R$. Let's use the latter:

$$\dfrac{\dfrac{\begin{array}{c} \vdots \\ (A \supset B) \vee (A \supset C), A \Longrightarrow B \vee C \end{array}}{(A \supset B) \vee (A \supset C) \Longrightarrow A \supset (B \vee C)} \supset R}{\Longrightarrow ((A \supset B) \vee (A \supset C)) \supset (A \supset (B \vee C))} \supset R$$

At this point there are three applicable rules: $\vee R_1$, $\vee R_2$, and $\vee L$. Neither of the right rules lead to a provable subgoal, since neither B nor C follow outright. Instead, we first need to apply the $\vee L$ rule to distinguish cases.

$$\dfrac{\dfrac{\dfrac{\begin{array}{c} \vdots \\ (A \supset B) \vee (A \supset C), A \supset B, A \Longrightarrow B \vee C \end{array} \quad \begin{array}{c} \vdots \\ (A \supset B) \vee (A \supset C), A \supset C, A \Longrightarrow B \vee C \end{array}}{(A \supset B) \vee (A \supset C), A \Longrightarrow B \vee C} \vee L(1)}{(A \supset B) \vee (A \supset C) \Longrightarrow A \supset (B \vee C)} \supset R}{\Longrightarrow ((A \supset B) \vee (A \supset C)) \supset (A \supset (B \vee C))} \supset R$$

The parenthetical "(1)" isn't logically necessary, but it is required in Dcheck to identify which antecedent the rule is applied to. In Dcheck, the antecedents are numbered $0, 1, 2, \ldots$ *from right to left*, to (1) refers the only available disjunction which is second from the right.

Now may be a good time to make another observation: once we have distinguished the two cases, we don't really need the disjunction any more. The rule says it can still be there, but the way we write down the derivation allows us to drop it silently because it is entirely redundant. Then the state of affairs becomes:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ A \supset B, A \Longrightarrow B \vee C\end{array} \quad \begin{array}{c}\vdots\\ A \supset C, A \Longrightarrow B \vee C\end{array}}{(A \supset B) \vee (A \supset C), A \Longrightarrow B \vee C}\ \vee L(1)}{(A \supset B) \vee (A \supset C) \Longrightarrow A \supset (B \vee C)}\ {\supset}R}{\Longrightarrow ((A \supset B) \vee (A \supset C)) \supset (A \supset (B \vee C))}\ {\supset}R$$

Looking at the first subgoal, we see we now know if B or C is true under the given antecedents, namely B. So we can use the first right disjunction rule.

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ A \supset B, A \Longrightarrow B\end{array}}{A \supset B, A \Longrightarrow B \vee C}\ \vee R_1 \quad \begin{array}{c}\vdots\\ A \supset C, A \Longrightarrow B \vee C\end{array}}{(A \supset B) \vee (A \supset C), A \Longrightarrow B \vee C}\ \vee L(1)}{(A \supset B) \vee (A \supset C) \Longrightarrow A \supset (B \vee C)}\ {\supset}R}{\Longrightarrow ((A \supset B) \vee (A \supset C)) \supset (A \supset (B \vee C))}\ {\supset}R$$

Next we can use the left implication rule, applied to antecedent number 1 (the second counting from the right).

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ A \supset B, A \Longrightarrow A\end{array} \quad \begin{array}{c}\vdots\\ A \supset B, B \Longrightarrow B\end{array}}{A \supset B, A \Longrightarrow B}\ {\supset}L(1)}{A \supset B, A \Longrightarrow B \vee C}\ \vee R_1 \quad \begin{array}{c}\vdots\\ A \supset C, A \Longrightarrow B \vee C\end{array}}{(A \supset B) \vee (A \supset C), A \Longrightarrow B \vee C}\ \vee L(1)}{(A \supset B) \vee (A \supset C) \Longrightarrow A \supset (B \vee C)}\ {\supset}R}{\Longrightarrow ((A \supset B) \vee (A \supset C)) \supset (A \supset (B \vee C))}\ {\supset}R$$

The implications are redundant in both branches, so we drop them and the conclude these branches with applications of the identity rule.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \Longrightarrow A}\,\mathsf{id}(0) \quad \dfrac{}{B \Longrightarrow B}\,\mathsf{id}(0)}{A \supset B, A \Longrightarrow B}\,{\supset}L(1)}{A \supset B, A \Longrightarrow B \vee C}\,{\vee}R_1 \qquad \begin{array}{c}\vdots\\ A \supset C, A \Longrightarrow B \vee C\end{array}}{\dfrac{(A \supset B) \vee (A \supset C), A \Longrightarrow B \vee C}{\dfrac{(A \supset B) \vee (A \supset C) \Longrightarrow A \supset (B \vee C)}{\Longrightarrow ((A \supset B) \vee (A \supset C)) \supset (A \supset (B \vee C))}\,{\supset}R}\,{\supset}R}\,{\vee}L(1)$$

This time, we use the rightmost antecedent, so the index of rule application is 0. The remaining subderivation is symmetric to the first, so we just fill it in.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \Longrightarrow A}\,\mathsf{id}(0) \quad \dfrac{}{B \Longrightarrow B}\,\mathsf{id}(0)}{A \supset B, A \Longrightarrow B}\,{\supset}L(1)}{A \supset B, A \Longrightarrow B \vee C}\,{\vee}R_1 \qquad \dfrac{\dfrac{\dfrac{}{A \Longrightarrow A}\,\mathsf{id}(0) \quad \dfrac{}{C \Longrightarrow C}\,\mathsf{id}(0)}{A \supset C, A \Longrightarrow C}\,{\supset}L(1)}{A \supset C, A \Longrightarrow B \vee C}\,{\vee}R_2}{\dfrac{(A \supset B) \vee (A \supset C), A \Longrightarrow B \vee C}{\dfrac{(A \supset B) \vee (A \supset C) \Longrightarrow A \supset (B \vee C)}{\Longrightarrow ((A \supset B) \vee (A \supset C)) \supset (A \supset (B \vee C))}\,{\supset}R}\,{\supset}R}\,{\vee}L(1)$$

Now let's look at the other direction.

$$\begin{array}{c}\vdots\\ \Longrightarrow (A \supset (B \vee C)) \supset ((A \supset B) \vee (A \supset C))\end{array}$$

Before we try to prove it, we can apply a quick test: is this *classically* true? Since classical logic has just one additional axiom (the law of excluded middle), every intuitionistically true proposition is also classically true. Therefore, if the proposition is classically false, it cannot be intuitionistically false. Here, if either B or C is true, then the whole proposition is (emphasizing again) *classically* true. If they are both false, then the proposition becomes $\neg A \vee \neg A \supset \neg A$, which is certainly true.

But is intuitionistically true? Well, the first step is forced, because there is only one rule that could possibly apply: ${\supset}R$.

$$\dfrac{\begin{array}{c}\vdots\\ A \supset (B \vee C) \Longrightarrow (A \supset B) \vee (A \supset C)\end{array}}{\Longrightarrow (A \supset (B \vee C)) \supset ((A \supset B) \vee (A \supset C))}\,{\supset}R$$

Now there are three possibilities: ${\vee}R_1$, ${\vee}R_2$, and ${\supset}L$. Trying ${\vee}R_1$ yields the following incomplete proof:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ A \supset (B \vee C) \Longrightarrow A \supset B\end{array}}{A \supset (B \vee C) \Longrightarrow (A \supset B) \vee (A \supset C)}\,{\vee}R_1}{\Longrightarrow (A \supset (B \vee C)) \supset ((A \supset B) \vee (A \supset C))}\,{\supset}R$$

But this subgoal isn't even classically true: pick A and C to be true and B to be false and then the antecedent is true but the succedent is false.

$\vee R_2$ cannot lead to a proof, either, for a symmetric reason. If we try $\supset L$ we obtain

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ A \supset (B \vee C) \Longrightarrow A\end{array} \quad \begin{array}{c}\vdots\\ B \vee C \Longrightarrow (A \supset B) \vee (A \supset C)\end{array}}{A \supset (B \vee C) \Longrightarrow (A \supset B) \vee (A \supset C)} \vee R_1}{\Longrightarrow (A \supset (B \vee C)) \supset ((A \supset B) \vee (A \supset C))} \supset R$$

We can actually prove the rightmost subgoal by cases on $B \vee C$, but the leftmost subgoal isn't even classically true (pick A to be false).

Since we have exhausted all the possibilities, we know there is no intuitionistic proof of this implication. In other words, implication does not distribute over disjunction. You might try constructing a function of the corresponding type to see why such an attempt also fails. It is difficult to prove that it is in fact impossible, except perhaps via knowing it is not intuitionistically provable.

5 Optimizations

In future lectures, we will devote more time to "optimizations" of the sequent calculus where we try to eliminate redundancies while preserving the same set of theorems. One form of redundancy arises if an antecedent of the premise of the rules are in general not needed to prove the success. For example, in the rule

$$\dfrac{\Gamma, A \vee B, A \Longrightarrow C \quad \Gamma, A \vee B, B \Longrightarrow C}{\Gamma, A \vee B \Longrightarrow C} \vee L$$

the antecedent $A \vee B$ is redundant in both premises. In the first premise, for example, we also have A and A is stronger than $A \vee B$ (in the sense that $A \supset (A \vee B)$). In the second premise, it is B which is stronger than $A \vee B$.

In the following summary of the sequent calculus we put [brackets] around the antecedents that could be considered redundant *after optimization*.

$$\frac{}{\Gamma, A \Longrightarrow A}\ \mathsf{id}$$

$$\frac{\Gamma \Longrightarrow A \quad \Gamma \Longrightarrow B}{\Gamma \Longrightarrow A \wedge B}\ {\wedge}R \qquad \frac{\Gamma, A \wedge B, A \Longrightarrow C}{\Gamma, A \wedge B \Longrightarrow C}\ {\wedge}L_1 \quad \frac{\Gamma, A \wedge B, B \Longrightarrow C}{\Gamma, A \wedge B \Longrightarrow C}\ {\wedge}L_2$$

$$\frac{}{\Gamma \Longrightarrow \top}\ {\top}R \qquad \text{no rule } {\top}L$$

$$\frac{\Gamma, A \Longrightarrow B}{\Gamma \Longrightarrow A \supset B}\ {\supset}R \qquad \frac{\Gamma, A \supset B \Longrightarrow A \quad \Gamma, [A \supset B], B \Longrightarrow C}{\Gamma, A \supset B \Longrightarrow C}\ {\supset}L$$

$$\frac{\Gamma \Longrightarrow A}{\Gamma \Longrightarrow A \vee B}\ {\vee}R_1 \quad \frac{\Gamma \Longrightarrow B}{\Gamma \Longrightarrow A \vee B}\ {\vee}R_2 \qquad \frac{\Gamma, [A \vee B], A \Longrightarrow C \quad \Gamma, [A \vee B], B \Longrightarrow C}{\Gamma, A \vee B \Longrightarrow C}\ {\vee}L$$

$$\text{no rule } {\bot}R \qquad \frac{}{\Gamma, \bot \Longrightarrow C}\ {\bot}L$$

We could also replace the two left rules for conjunction with

$$\frac{\Gamma, [A \wedge B], A, B \Longrightarrow C}{\Gamma, A \wedge B \Longrightarrow C}\ {\wedge}L$$

Such a rule is also sound, but it does not directly correspond to the rules for conjunction elimination we have in natural deduction. Can you spot which elimination rule from the homework assignment it corresponds to?

One point here is that the elimination rules for the connectives are not uniquely determined by the introduction rules. Consequently, the left rules in the sequent calculus also have some leeway.

References

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