Lecture Notes on Cut Elimination

15-317: Constructive Logic
Frank Pfenning

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1 Introduction

The identity rule of the sequent calculus exhibits one connection between the judgments $A\ ante$ and $A\ succ$: If we assume A as an antecedent we can prove A as a succedent. Because we designed the sequent calculus to model verifications, this rule applies to propositional variables P (also called *atomic propositions*) only, because these are the propositions allowed for the $\downarrow\uparrow$ rule. The *identity theorem* then says we can safely generalize the identity rule to all propositions A. As we will see in Section 2, this amounts to a global version of the local completeness of the eliminations.

The *cut theorem* of the sequent calculus expresses the opposite: if we have a proof of $A\ succ$ we are licensed to assume $A\ ante$. This can be interpreted as saying the left rules are not too strong: whatever we can do with the antecedent A can also be deduced without that, if we know A is true. Because $A\ succ$ occurs only as a succedent, and $A\ ante$ only as an antecedent, we must formulate this in a somewhat roundabout manner: If $\Gamma \Longrightarrow A\ succ$ and $\Gamma, A\ ante \Longrightarrow C\ succ$ then $\Gamma \Longrightarrow C\ succ$.

Because it is very easy to go back and forth between sequent calculus deductions of $A\ succ$ and verifications of $A\uparrow$, we can use the cut theorem to show that every true proposition has a verification, which establishes a fundamental, global connection between truth and verifications. While the sequent calculus is a convenient intermediary (and was conceived as such by Gentzen [1935]), this theorem can also be established directly using verifications.

2 Identity

In today's lecture we squarely put ourselves outside the philosophical foundations and study the structure of sequent proofs with mathematical means. In particular, we will carry about proofs about the structure and existence of sequent calculus derivations, using forms of mathematical induction. Nevertheless, our proofs will remain constructive even at the metalevel and we will briefly discuss their computational contents for the theorems we prove.

We restrict the identity rule in our sequent calculus to atomic propositions P.

$$\dfrac{}{\Gamma, P \Longrightarrow P}\ \mathsf{id}$$

Are our rules strong enough so we can prove $\Gamma, A \Longrightarrow A$ for an arbitrary proposition A? Yes!

Theorem 1 (Identity) *For any proposition A, we have $\Gamma, A \Longrightarrow A$.*

How might we prove this? Many of our metatheoretic proofs go by induction over the structure of a derivation because that's what establishes a judgment. But here, the only thing we are given is a proposition A. So we are led instead to consider an *induction over the structure of A*. This gives use three base cases (atomic propositions P, $\top$, and $\bot$) and three inductive cases $A_1 \wedge A_2$, $A_1 \supset A_2$, and $A_1 \vee A_2$.

Proof: By induction on the structure of A. We show several representative cases and leave the remaining ones to the reader.

Case: $A = P$ for an atomic proposition P. Then

$$\dfrac{}{\Gamma, P \Longrightarrow P}\ \mathsf{id}$$

Case: $A = \bot$. Then

$$\dfrac{}{\Gamma, \bot \Longrightarrow \bot}\ {\bot}L$$

Case: $A = A_1 \wedge A_2$. Then

$$\dfrac{\dfrac{\begin{array}{c}\text{By i.h. on } A_1\\ \Gamma, A_1 \wedge A_2, A_1 \Longrightarrow A_1\end{array}}{\Gamma, A_1 \wedge A_2 \Longrightarrow A_1}\ {\wedge}L_1 \qquad \dfrac{\begin{array}{c}\text{By i.h. on } A_2\\ \Gamma, A_1 \wedge A_2, A_2 \Longrightarrow A_2\end{array}}{A_1 \wedge A_2 \Longrightarrow A_2}\ {\wedge}L_2}{\Gamma, A_1 \wedge A_2 \Longrightarrow A_1 \wedge A_2}\ {\wedge}R$$

Case: $A = A_1 \vee A_2$. Then

$$\dfrac{\dfrac{\begin{array}{c}\text{By i.h. on } A_1\\ \Gamma, A_1 \vee A_2, A_1 \Longrightarrow A_1\end{array}}{\Gamma, A_1 \vee A_2, A_1 \Longrightarrow A_1 \vee A_2}\ {\vee}R_1 \qquad \dfrac{\begin{array}{c}\text{By i.h. on } A_2\\ \Gamma, A_1 \vee A_2, A_2 \Longrightarrow A_2\end{array}}{A_1 \vee A_2, A_2 \Longrightarrow A_1 \vee A_2}\ {\vee}R_2}{\Gamma, A_1 \vee A_2 \Longrightarrow A_1 \vee A_2}\ {\vee}L$$

$\square$

An interesting point in the last two case is that we apply the induction hypothesis with a larger collection of antecedents, for example, $\Gamma, A_1 \wedge A_2$ in the case of conjunction. Alternatively, we could apply the induction hypothesis with just Γ and obtain $\Gamma, A_1 \Longrightarrow A_1$ and then apply *weakening* by adding the antecedent $A_1 \wedge A_2$. Let's separate out this observation as another important property of the sequent calculus.

Theorem 2 (Weakening) *If* $\Gamma \Longrightarrow C$ *then* $\Gamma, A \Longrightarrow C$.

Proof: We add the unused antecedent to every sequent in the given derivation of $\Gamma \Longrightarrow C$. Formally, this would be an induction over the structure of the given derivation. An important observation here is that the structure of the derivation does not change. □

With this theorem, an alternative case in the proof of conjunction could be

Case: $A = A_1 \wedge A_2$. Then

$$\dfrac{\dfrac{\dfrac{}{\Gamma, A_1 \Longrightarrow A_1}\ \text{i.h.}(A_1)}{\Gamma, A_1 \wedge A_2, A_1 \Longrightarrow A_1}\ \text{weaken} \qquad \dfrac{\dfrac{\dfrac{}{\Gamma, A_2 \Longrightarrow A_2}\ \text{i.h.}(A_2)}{\Gamma, A_1 \wedge A_2, A_2 \Longrightarrow A_2}\ \text{weaken}}{\Gamma, A_1 \wedge A_2 \Longrightarrow A_2}\ \wedge L_2}{\dfrac{\Gamma, A_1 \wedge A_2 \Longrightarrow A_1}{\Gamma, A_1 \wedge A_2 \Longrightarrow A_1 \wedge A_2}\ \wedge R}\ \wedge L_1$$

The identity theorem is the global version of the local completeness property for each individual connective. Local completeness shows that a connective can be re-verified from a proof that gives us license to use it, which directly corresponds to $A \Longrightarrow A$. One can recognize the local expansion as embodied in each case of the inductive proof of identity.

3 Derivability vs. Admissibility

We call a rule *derivable* if it has a closed form derivation within the calculus. For example, the rule

$$\dfrac{\Gamma \Longrightarrow A \quad \Gamma \Longrightarrow B}{\Gamma, \neg(A \wedge B) \Longrightarrow C}$$

is derivable, because it has the following derivation

$$\dfrac{\dfrac{\Gamma \Longrightarrow A \quad \Gamma \Longrightarrow B}{\Gamma \Longrightarrow A \wedge B}\ \wedge R \qquad \dfrac{}{\Gamma, \bot \Longrightarrow C}\ \bot L}{\Gamma, \neg(A \wedge B) \Longrightarrow C}\ \supset L$$

where we have omitted some antecedents we don't plan on using. The great property of a derivable rule that it will also derivable in any extension of our language or logic as long as the meaning of preexisting connectives (and therefore their rules) do not change.

> *Derivable rules: once valid, always valid.*

On the other hand, the identity theorem embodies and *admissible rules* (with no premises)

$$\frac{}{\Gamma, A \vdash A}\ \mathsf{id}$$

The difference is that every instance of an admissible rule has a derivation, but that this derivation may not be the same for different instances. An admissible rule is established by metatheoretic reasoning, which means we have to reconsider every time we extend our language.

> *Admissible rules: prove time and time again.*

In addition to weakening (which expresses that we don't have to use an antecedent) there is also *contraction*, which expresses that we may use antecedent multiple times. This means the one copy of an antecedent is sufficient.

Theorem 3 (Contraction) *If* $\Gamma, A, A \Longrightarrow C$ *then* $\Gamma, A \Longrightarrow C$*. Stated succinctly:*

$$\frac{\Gamma, A, A \Longrightarrow C}{\Gamma, A \Longrightarrow C}\ \mathsf{contr}$$

is admissible.

Proof: Whenever one of the two copies of A is used in the given derivation, we just use the other. Formally, this is an induction over the structure of the given derivation. As for weakening, we note that the structure of the derivation does not change in this process. □

4 Admissibility of Cut

The cut theorem is one of the most fundamental properties of logic. Because of its central role, we will spend some time on its proof. In lecture we developed the proof and the required induction principle incrementally; here we present the final result as is customary in mathematics. The proof is amenable to formalization in a logical framework; details can be found in a paper by the instructor Pfenning [2000].

Theorem 4 (Admissibility of Cut) *If* $\Gamma \Longrightarrow A$ *and* $\Gamma, A \Longrightarrow C$ *then* $\Gamma \Longrightarrow C$*. Alternatively:*

$$\frac{\Gamma \Longrightarrow A \quad \Gamma, A \Longrightarrow C}{\Gamma \Longrightarrow C}\ \mathsf{cut}$$

is admissible

Proof: By nested inductions on the structure of A, the derivation $\mathcal{D}$ of $\Gamma \Longrightarrow A$ and $\mathcal{E}$ of $\Gamma, A \Longrightarrow C$. More precisely, we appeal to the induction hypothesis either with a strictly smaller cut formula, or with an identical cut formula and two derivations, one of which is strictly smaller while the other stays the same. The proof is constructive, which means we show how to transform

$$\begin{array}{c}\mathcal{D}\\ \Gamma \Longrightarrow A\end{array} \quad \text{and} \quad \begin{array}{c}\mathcal{E}\\ \Gamma, A \Longrightarrow C\end{array} \quad \text{into} \quad \begin{array}{c}\mathcal{F}\\ \Gamma \Longrightarrow C\end{array}$$

The proof is divided into several classes of cases. More than one case may be applicable, which means that the algorithm for constructing the derivation of $\Gamma \Longrightarrow C$ from the two given derivations is naturally non-deterministic.

Case: $\mathcal{D}$ is an initial sequent, $\mathcal{E}$ is arbitrary.

$$\mathcal{D} = \frac{}{\Gamma', A \Longrightarrow A}\ \mathsf{id} \quad \text{and} \quad \begin{array}{c}\mathcal{E}\\ \Gamma', A, A \Longrightarrow C\end{array}$$

$\Gamma = (\Gamma', A)$	this case
$\Gamma', A, A \Longrightarrow C$	deduction $\mathcal{E}$
$\Gamma', A \Longrightarrow C$	by contraction (Theorem 3)
$\Gamma \Longrightarrow C$	since $\Gamma = (\Gamma', A)$

Case: $\mathcal{D}$ is arbitrary and $\mathcal{E}$ is an initial sequent using the cut formula.

$$\begin{array}{c}\mathcal{D}\\ \Gamma \Longrightarrow A\end{array} \quad \text{and} \quad \mathcal{E} = \frac{}{\Gamma, A \Longrightarrow A}\ \mathsf{id}$$

$A = C$	this case
$\Gamma \Longrightarrow A$	deduction $\mathcal{D}$

Case: $\mathcal{E}$ is an initial sequent *not* using the cut formula.

$$\mathcal{E} = \frac{}{\Gamma', C, A \Longrightarrow C}\ \mathsf{id}$$

$\Gamma = (\Gamma', C)$	this case
$\Gamma', C \Longrightarrow C$	by rule id
$\Gamma \Longrightarrow C$	since $\Gamma = (\Gamma', C)$

In the next set of cases, the cut formula is the principal formula of the final inference in both $\mathcal{D}$ and $\mathcal{E}$. We only show two of these cases.

Case:

$$\mathcal{D} = \frac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \Longrightarrow A_1\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \Gamma \Longrightarrow A_2\end{array}}{\Gamma \Longrightarrow A_1 \wedge A_2}\ \wedge R$$

$$\text{and} \qquad \mathcal{E} = \cfrac{\begin{array}{c}\mathcal{E}_1 \\ \Gamma, A_1 \wedge A_2, A_1 \Longrightarrow C\end{array}}{\Gamma, A_1 \wedge A_2 \Longrightarrow C} \wedge L_1$$

$A = A_1 \wedge A_2$	this case
$\Gamma, A_1 \Longrightarrow C$	by i.h. on $A_1 \wedge A_2$, $\mathcal{D}$ and $\mathcal{E}_1$
$\Gamma \Longrightarrow C$	by i.h. on A_1, $\mathcal{D}_1$, and previous line

Actually we have ignored a detail: in the first appeal to the induction hypothesis, $\mathcal{E}_1$ has an additional hypothesis, A_1, and therefore does not match the statement of the theorem precisely. However, we can always weaken $\mathcal{D}$ to include this additional hypothesis without changing the structure of $\mathcal{D}$ (see the Theorem 2) and then appeal to the induction hypothesis. We will not be explicit about these trivial weakening steps in the remaining cases.

It is crucial for a well-founded induction that $\mathcal{E}_1$ is smaller than $\mathcal{E}$, so even if the same cut formula and same $\mathcal{D}$ is used, $\mathcal{E}_1$ got smaller. Note that we cannot directly appeal to induction hypothesis on A_1,$\mathcal{D}_1$ and $\mathcal{E}_1$ because the additional formula $A_1 \wedge A_2$ might still be used in $\mathcal{E}_1$, e.g., by a subsequent use of $\wedge L_2$.

Case:

$$\mathcal{D} = \cfrac{\begin{array}{c}\mathcal{D}_2 \\ \Gamma, A_1 \Longrightarrow A_2\end{array}}{\Gamma \Longrightarrow A_1 \supset A_2} \supset R$$

$$\text{and} \quad \mathcal{E} = \cfrac{\begin{array}{c}\mathcal{E}_1 \\ \Gamma, A_1 \supset A_2 \Longrightarrow A_1\end{array} \quad \begin{array}{c}\mathcal{E}_2 \\ \Gamma, A_1 \supset A_2, A_2 \Longrightarrow C\end{array}}{\Gamma, A_1 \supset A_2 \Longrightarrow C} \supset L$$

$A = A_1 \supset A_2$	this case
$\Gamma \Longrightarrow A_1$	by i.h. on $A_1 \supset A_2$, $\mathcal{D}$ and $\mathcal{E}_1$
$\Gamma \Longrightarrow A_2$	by i.h. on A_1 from above and $\mathcal{D}_2$
$\Gamma, A_2 \Longrightarrow C$	by i.h. on $A_1 \supset A_2$, $\mathcal{D}$ and $\mathcal{E}_2$
$\Gamma \Longrightarrow C$	by i.h. on A_2 from above

Note that the proof constituents of the last step $\Gamma \Longrightarrow C$ may be longer than the original deductions $\mathcal{D}, \mathcal{E}$. Hence, it is crucial for a well-founded induction that the cut formula A_2 is smaller than $A_1 \supset A_2$.

Finally note the resemblance of these principal cases to the local soundness reductions in harmony arguments for natural deduction.

In the next set of cases, the principal formula in the last inference in $\mathcal{D}$ is *not* the cut formula. We sometimes call such formulas *side formulas* of the cut.

Case: If $\mathcal{D}$ ended with an $\wedge L_1$:

$$\mathcal{D} = \cfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma', B_1 \wedge B_2, B_1 \Longrightarrow A\end{array}}{\Gamma', B_1 \wedge B_2, \Longrightarrow A} \wedge L_1 \quad \text{and} \quad \begin{array}{c}\mathcal{E}\\ \Gamma', B_1 \wedge B_2, A \Longrightarrow C\end{array}$$

$\Gamma = (\Gamma', B_1 \wedge B_2)$	this case
$\Gamma', B_1 \wedge B_2, B_1 \Longrightarrow C$	by i.h. on A, $\mathcal{D}_1$ and $\mathcal{E}$
$\Gamma', B_1 \wedge B_2 \Longrightarrow C$	by rule $\wedge L_1$
$\Gamma \Longrightarrow C$	Since $\Gamma = (\Gamma', B_1 \wedge B_2)$

Case:

$$\mathcal{D} = \cfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma', B_1 \supset B_2 \Longrightarrow B_1\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \Gamma', B_1 \supset B_2, B_2 \Longrightarrow A\end{array}}{\Gamma', B_1 \supset B_2 \Longrightarrow A} \supset L$$

$\Gamma = (\Gamma', B_1 \supset B_2)$	this case
$\Gamma', B_1 \supset B_2, B_2 \Longrightarrow C$	by i.h. on A, $\mathcal{D}_2$ and $\mathcal{E}$
$\Gamma', B_1 \supset B_2 \Longrightarrow C$	by rule $\supset L$ on $\mathcal{D}_1$ and above
$\Gamma \Longrightarrow C$	Since $\Gamma = (\Gamma', B_1 \supset B_2)$

In the final set of cases, A is not the principal formula of the last inference in $\mathcal{E}$. This overlaps with the previous cases since A may not be principal on either side. In this case, we appeal to the induction hypothesis on the subderivations of $\mathcal{E}$ and directly infer the conclusion from the results.

Case:

$$\begin{array}{c}\mathcal{D}\\ \Gamma \Longrightarrow A\end{array} \quad \text{and} \quad \mathcal{E} = \cfrac{\begin{array}{c}\mathcal{E}_1\\ \Gamma, A \Longrightarrow C_1\end{array} \quad \begin{array}{c}\mathcal{E}_2\\ \Gamma, A \Longrightarrow C_2\end{array}}{\Gamma, A \Longrightarrow C_1 \wedge C_2} \wedge R$$

$C = C_1 \wedge C_2$	this case
$\Gamma \Longrightarrow C_1$	by i.h. on A, $\mathcal{D}$ and $\mathcal{E}_1$
$\Gamma \Longrightarrow C_2$	by i.h. on A, $\mathcal{D}$ and $\mathcal{E}_2$
$\Gamma \Longrightarrow C_1 \wedge C_2$	by rule $\wedge R$ on above

Case:

$$\begin{array}{c}\mathcal{D}\\ \Gamma \Longrightarrow A\end{array} \quad \text{and} \quad \mathcal{E} = \cfrac{\begin{array}{c}\mathcal{E}_1\\ \Gamma', B_1 \wedge B_2, B_1, A \Longrightarrow C\end{array}}{\Gamma', B_1 \wedge B_2, A \Longrightarrow C} \wedge L_1$$

$\Gamma = (\Gamma', B_1 \wedge B_2)$	this case
$\Gamma', B_1 \wedge B_2, B_1 \Longrightarrow C$	by i.h. on A, $\mathcal{D}$ and $\mathcal{E}_1$
$\Gamma', B_1 \wedge B_2 \Longrightarrow C$	By rule $\wedge L_1$ from above

□

5 Cut Elimination[1]

Gentzen's original presentation of the sequent calculus included an inference rule for cut, and identity was allowed for arbitrary propositions. We write $\Gamma \stackrel{\mathsf{cut}}{\Longrightarrow} A$ for this system, which is just like $\Gamma \Longrightarrow A$, with the additional rule

$$\frac{\Gamma \stackrel{\mathsf{cut}}{\Longrightarrow} A \quad \Gamma, A \stackrel{\mathsf{cut}}{\Longrightarrow} C}{\Gamma \stackrel{\mathsf{cut}}{\Longrightarrow} C}\ \mathsf{cut}$$

The advantage of this calculus is that it more directly corresponds to natural deduction in its full generality, rather than verifications, because just like in natural deduction, the cut rule makes it possible to prove an arbitrary other A from the available assumptions Γ (left premise) and then use that A as an additional assumption in the rest of the proof (right premise). The disadvantage is that it cannot easily be seen as capturing the meaning of the connectives by inference rules, because with the rule of cut the meaning of C might depend on the meaning of any other proposition A (possibly even including C as a subformula).

In order to clearly distinguish between the two kinds of calculi, the one we presented is sometimes called the *cut-free sequent calculus*, while Gentzen's calculus would be a *sequent calculus with cut*. The theorem connecting the two is called *cut elimination*: for any deduction in the sequent calculus with cut, there exists a cut-free deduction of the same sequent. The proof is a straightforward induction on the structure of the deduction, appealing to the cut theorem in one crucial place.

Theorem 5 (Cut Elimination) *If $\mathcal{D}$ is a deduction of $\Gamma \stackrel{\mathsf{cut}}{\Longrightarrow} C$ possibly using the* cut *rule, then there exists a cut-free deduction $\mathcal{D}'$ of $\Gamma \stackrel{\mathsf{cut}}{\Longrightarrow} C$.*

Proof: By induction on the structure of $\mathcal{D}$. In each case, we appeal to the induction hypothesis on all premises and then apply the same rule to the result. The only interesting case is when a cut rule is encountered.

Case:

$$\mathcal{D} = \frac{\begin{array}{c}\mathcal{D}_1 \\ \Gamma \stackrel{\mathsf{cut}}{\Longrightarrow} A\end{array} \quad \begin{array}{c}\mathcal{D}_2 \\ \Gamma, A \stackrel{\mathsf{cut}}{\Longrightarrow} C\end{array}}{\Gamma \stackrel{\mathsf{cut}}{\Longrightarrow} C}\ \mathsf{cut}$$

$\Gamma \Longrightarrow A$ without cut	By i.h. on $\mathcal{D}_1$
$\Gamma, A \Longrightarrow C$ without cut	By i.h. on $\mathcal{D}_2$
$\Gamma \Longrightarrow C$	By the admissibility of cut (Theorem 4)

□

[1] not covered in lecture

6 Summary

We summarize the rules for the sequent calculus, with the admissible rules emphasized with dashed lines. In some versions of the calculus, like Gentzen's [1935] original, these rules were primitive. The fact that they are admissible in the calculus without these rules is an example showing that *you can have your cake and eat it, too*.

Having your cake: If you want to derive a theorem either in or about the sequent calculus, you can freely use cut and identity. That's because the rules are *admissible*: a derivation for the conclusion exists, even if it may be much longer than the one that uses cut.

Eating your cake: If you want to derive a theorem either in or about the sequent calculus, you do not *need* to consider the rules of cut or general identity. That's because every derivable sequent can be derived without either of them. This is particularly useful if you want to design or even implement systematic search procedures, since cut, seen as a rule, introduces an arbitrary proposition A and is therefore extremely prolific. As mentioned before, with this additional rule, all other rules satisfy the *subformula property* and just break down the propositions in the sequent you are trying to prove.

$$\frac{\Gamma \Longrightarrow A \quad \Gamma, A \Longrightarrow C}{\Gamma \Longrightarrow C}\ \text{cut} \qquad \frac{}{\Gamma, A \Longrightarrow A}\ \text{id}$$

$$\frac{\Gamma \Longrightarrow C}{\Gamma, A \Longrightarrow C}\ \text{weaken} \qquad \frac{\Gamma, A, A \Longrightarrow C}{\Gamma, A \Longrightarrow C}\ \text{contract}$$

$$\frac{}{\Gamma, P \Longrightarrow P}\ \text{id}^*$$

$$\frac{\Gamma \Longrightarrow A \quad \Gamma \Longrightarrow B}{\Gamma \Longrightarrow A \wedge B}\ \wedge R \qquad \frac{\Gamma, A \wedge B, A \Longrightarrow C}{\Gamma, A \wedge B \Longrightarrow C}\ \wedge L_1 \qquad \frac{\Gamma, A \wedge B, B \Longrightarrow C}{\Gamma, A \wedge B \Longrightarrow C}\ \wedge L_2$$

$$\frac{\Gamma, A \Longrightarrow B}{\Gamma \Longrightarrow A \supset B}\ \supset R \qquad \frac{\Gamma, A \supset B \Longrightarrow A \quad \Gamma, [A \supset B], B \Longrightarrow C}{\Gamma, A \supset B \Longrightarrow C}\ \supset L$$

$$\frac{\Gamma \Longrightarrow A}{\Gamma \Longrightarrow A \vee B}\ \vee R_1 \qquad \frac{\Gamma \Longrightarrow B}{\Gamma \Longrightarrow A \vee B}\ \vee R_2 \qquad \frac{\Gamma, [A \vee B], A \Longrightarrow C \quad \Gamma, [A \vee B], B \Longrightarrow C}{\Gamma, A \vee B \Longrightarrow C}\ \vee L$$

$$\frac{}{\Gamma \Longrightarrow \top}\ \top R \qquad \text{no rule } \top L$$

$$\text{no rule } \bot R \qquad \frac{}{\Gamma, \bot \Longrightarrow C}\ \bot L$$

As in the last lecture, we have [bracketed] some antecedents that are redundant, but removing them would break the close relationship to verifications.

References

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