Lecture Notes on Propositional Theorem Proving

15-317: Constructive Logic
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Lecture 14
Tuesday, March 14, 2023

1 Introduction

The sequent calculus we have introduced so far maintains a close correspondence to natural deductions or, more specifically, to verifications. One consequence is *persistence of antecedents*: once an assumption has been introduced in the course of a deduction, it will remain available in any sequent above this point. While this is appropriate in a foundational calculus, it is not ideal for proof search since rules can be applied over and over again without necessarily making progress. We therefore develop a second sequent calculus and then a third in order to make the process of bottom-up search for a proof more efficient by reducing unnecessary choices in proof search. By way of the previous link of the sequent calculus with verification-style natural deductions, this lecture will, thus, give rise to a more efficient way of coming up with natural deduction proofs.

This lecture marks the begin of a departure from the course of the lectures so far, which, broadly construed, focused on understanding what a constructive proof is and what can be read off or done once one has such a proof. Now we begin to move toward the question of how to find such a proof in the first place.

More ambitiously, we are looking for a *decision procedure* for intuitionistic propositional logic. Specifically, we would like to prove that for every proposition A, either $\Longrightarrow A$ or not $\Longrightarrow A$. Based on experience, we suspect this could be proved by induction on A, but this will fail for various reasons. We somehow need to generalize it to prove that for *every sequent*, either $\Gamma \Longrightarrow A$ or not. That, however, has its own problems because the premises of the rules can be larger than the conclusion so it is not clear how one might apply an induction hypothesis.

There several possible ways forward. One is to construct a derivation bottom up and fail if a proof goal (that is, a sequent) recurs. In such a case there is no point continuing this particular attempt and we can backtrack. This technique is called *loop-checking*. There is more to be considered, and we will do so in the next lecture.

Another approach is to write a new set of rules such that (a) everything we can derive can also be derived in the sequent calculus (it is *sound*), (b) everything that can be derived in the sequent calculus can be derived in the new calculus (it is *complete*), and (c)

the premises of all rules are smaller than the conclusion so the search for a proof will terminate. "Smaller" here means according to some *well-founded measure*, by which we mean a measure where any strictly decreasing chain of elements eventually has to arrive at a minimal element.

This approach, developed for the propositional calculus by **?**, is the subject of this lecture. We will do this in two steps: in the first we rewrite almost all rules so the premises are smaller than the conclusion, isolating the reason proof-search may not terminate. In the second step we further refine the rules to avoid this issue. The result is a beautiful calculus which Dyckhoff calls *contraction-free* because there is no rule of contraction, and, furthermore, the principal formula of each left rule is consumed as part of the rule application rather than copied to any premise, so we never duplicate reasoning (which we could if there were a contraction rule).

In this process we have to accept that we restrict the set of derivations we may find. This means in turn that if we use theorem proving to synthesize programs of a given type, we will miss some potential programs. This is traded off against being able to decide whether a formula is true (intuitionistically, of course).

2 A More Restrictive Sequent Calculus

Ideally, once we have applied an inference rule during proof search (that is, bottom-up), we should not have to apply the same rule again to the same proposition. Since all rules decompose formulas, if we had such a sequent calculus, we would have a simple and clean decision procedure. As it turns out, there is a fly in the ointment, but let us try to derive such a system.

We write $\Gamma \longrightarrow A$ for a sequent whose deductions try to eliminate principal formulas as much as possible. We keep the names of the rules, since they are largely parallel to the rules of the original sequent calculus, $\Gamma \Longrightarrow A$.

Conjunction. The right rule works as before; the left rule extracts *both* conjuncts so that the conjunction itself is no longer needed.

$$\frac{\Gamma \longrightarrow A \quad \Gamma \longrightarrow B}{\Gamma \longrightarrow A \wedge B}\ \wedge R \qquad \frac{\Gamma, A, B \longrightarrow C}{\Gamma, A \wedge B \longrightarrow C}\ \wedge L$$

Observe that for both rules, all premises have smaller sequents than the conclusion if one counts the number of connectives in a sequent. So applying either rule obviously made progress toward simplifying the sequent.

It is easy to see that these rules are sound with respect to the ordinary sequent calculus rules. Soundness here is the property that if $\Gamma \longrightarrow C$ then $\Gamma \Longrightarrow C$. This is straightforward since $\wedge R$ is the same rule and $\wedge L$ is the same as $\wedge L_1$ followed by $\wedge L_2$ followed by weakening the original $A \wedge B$ away. Completeness if generally more difficult. What we want to show is that if $\Gamma \Longrightarrow C$ then also $\Gamma \longrightarrow C$, where the rules for the latter sequents are more restrictive, by design. The proof of this will eventually proceed by induction on the structure of the given deduction $\mathcal{D}$ and appeal to lemmas on the restrictive sequent calculus. For example:

Case: (of completeness proof)

$$\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}_1 \\ \Gamma, A \wedge B, A \Longrightarrow C\end{array}}{\Gamma, A \wedge B \Longrightarrow C}\ \wedge L_1$$

$\Gamma, A \wedge B, A \longrightarrow C$	By i.h. on $\mathcal{D}_1$
$\Gamma, A, B \longrightarrow A$	By identity for $\longrightarrow$
$\Gamma, A \wedge B \longrightarrow A$	By $\wedge L$
$\Gamma, A \wedge B \longrightarrow C$	By cut for $\longrightarrow$

The induction hypothesis is applicable to $\mathcal{D}_1$ because, even if it is a longer sequent, $\mathcal{D}_1$ is a shorter proof than $\mathcal{D}$. We see that identity and cut for the restricted sequent calculus are needed to show completeness in the sense described above. Fortunately, they hold (see further notes at the end of this section). We will not formally justify many of the rules, but give informal justifications or counterexamples.

Truth. There is a small surprise here, in that, unlike in natural deduction which had no elimination rule for $\top$, we can have a left rule for $\top$, which eliminates it from the antecedents to make progress (cleanup). It is analogous to the nullary case of conjunction.

$$\dfrac{}{\Gamma \longrightarrow \top}\ \top R \qquad \dfrac{\Gamma \longrightarrow C}{\Gamma, \top \longrightarrow C}\ \top L$$

Atomic propositions. They are straightforward, since the initial sequents do not change.

$$\dfrac{}{\Gamma, P \longrightarrow P}\ \mathsf{id}$$

Disjunction. The right rules to do not change; in the left rule we can eliminate the principal formula.

$$\dfrac{\Gamma \longrightarrow A}{\Gamma \longrightarrow A \vee B}\ \vee R_1 \qquad \dfrac{\Gamma \longrightarrow B}{\Gamma \longrightarrow A \vee B}\ \vee R_2 \qquad \dfrac{\Gamma, A \longrightarrow C \quad \Gamma, B \longrightarrow C}{\Gamma, A \vee B \longrightarrow C}\ \vee L$$

Intuitively, the assumption $A \vee B$ can be eliminated from both premises of the $\vee L$ rule, because the new assumptions A and B are stronger. More formally:

Case: (of completeness proof)

$$\mathcal{D} = \dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2 \\ \Gamma, A \vee B, A \Longrightarrow C & \Gamma, A \vee B, B \Longrightarrow C\end{array}}{\Gamma, A \vee B \Longrightarrow C}\ \vee L$$

$\Gamma, A \vee B, A \longrightarrow C$	By i.h. on $\mathcal{D}_1$
$\Gamma, A \longrightarrow A$	By identity for $\longrightarrow$
$\Gamma, A \longrightarrow A \vee B$	By $\vee R_1$
$\Gamma, A \longrightarrow C$	By cut for $\longrightarrow$
$\Gamma, A \vee B, B \longrightarrow C$	By i.h. on $\mathcal{D}_2$
$\Gamma, B \longrightarrow B$	By identity for $\longrightarrow$
$\Gamma, B \longrightarrow A \vee B$	By $\vee R_2$
$\Gamma, B \longrightarrow C$	By cut for $\longrightarrow$
$\Gamma, A \vee B \longrightarrow C$	By rule $\vee L$

Falsehood. There is no right rule, and the left rule has no premise, which means it transfers directly.

$$\text{no } \bot R \text{ rule} \qquad \frac{}{\Gamma, \bot \longrightarrow C}\ \bot L$$

Implication. In all the rules so far, all premises have fewer connectives than the conclusion. For implication, we will not be able to maintain this property.

$$\frac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \supset B}\ \supset R \qquad \frac{\Gamma, A \supset B \longrightarrow A \quad \Gamma, B \longrightarrow C}{\Gamma, A \supset B \longrightarrow C}\ \supset L$$

Here, the assumption $A \supset B$ persists in the first premise but not in the second. While the assumption B is more informative than $A \supset B$, so only B is kept in the second premise, this is not the case in the first premise. Unfortunately, $A \supset B$ may be needed again in that branch of the proof. An example which requires the implication more than once is $\longrightarrow \neg\neg(A \vee \neg A)$, where $\neg A = A \supset \bot$ as usual. Without that additional assumption (marked in red below), the proof would not work:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg(A \vee \neg A), A \longrightarrow A}\,\mathsf{id}}{\neg(A \vee \neg A), A \longrightarrow A \vee \neg A}\,\vee R_1 \quad \dfrac{}{A, \bot \longrightarrow \bot}\,\bot L}{\neg(A \vee \neg A), A \longrightarrow \bot}\,\supset L}{\neg(A \vee \neg A) \longrightarrow \neg A}\,\supset R}{\neg(A \vee \neg A) \longrightarrow A \vee \neg A}\,\vee R_2 \qquad \dfrac{}{\bot \longrightarrow \bot}\,\bot L}{\neg(A \vee \neg A) \longrightarrow \bot}\,\supset L}{\longrightarrow \neg\neg(A \vee \neg A)}\,\supset R$$

Now all rules have smaller premises (if one counts the number of logical constants and connectives in them) except for the $\supset L$ rule. We will address the issue with $\supset L$ in Section **??**.

3 Metatheory of the Restricted Sequent Calculus[1]

We only enumerate the basic properties.

Theorem 1 (Weakening) *If* $\Gamma \longrightarrow C$ *then* $\Gamma, A \longrightarrow C$ *with a structurally identical deduction.*

Theorem 2 (Atomic contraction) *If* $\Gamma, P, P \longrightarrow C$ *then* $\Gamma, P \longrightarrow C$ *with a structurally identical deduction.*

Theorem 3 (Identity) $\Gamma, A \longrightarrow A$ *for any proposition* A*.*

Proof: By induction on the structure of A. □

Theorem 4 (Cut) *If* $\Gamma \longrightarrow A$ *and* $\Gamma, A \longrightarrow C$ *then* $\Gamma \longrightarrow C$

Proof: Analogous to the proof for the ordinary sequent calculus in Lecture 8. In the case where the first deduction is initial, we use atomic contraction. □

Theorem 5 (Contraction) *If* $\Gamma, A, A \longrightarrow C$ *then* $\Gamma, A \longrightarrow C$*.*

Proof: $\Gamma, A \longrightarrow A$ by identity and weakening. Therefore $\Gamma, A \longrightarrow C$ by cut. □

Theorem 6 (Soundness wrt. $\Longrightarrow$) *If* $\Gamma \longrightarrow A$ *then* $\Gamma \Longrightarrow A$*.*

Proof: By induction on the structure of the given deduction. □

Theorem 7 (Completeness wrt. $\Longrightarrow$) *If* $\Gamma \Longrightarrow A$ *then* $\Gamma \longrightarrow A$*.*

Proof: By induction on the structure of the given deduction, appealing to identity and cut in many cases. See the cases for $\wedge L_1$ and $\vee L$ in the previous section. □

We repeat the rules of the restrictive sequent calculus here for reference.

$$\frac{}{\Gamma, P \longrightarrow P}\ \mathsf{id}^*$$

$$\frac{\Gamma \longrightarrow A \quad \Gamma \longrightarrow B}{\Gamma \longrightarrow A \wedge B}\ \wedge R \qquad \frac{\Gamma, A, B \longrightarrow C}{\Gamma, A \wedge B \longrightarrow C}\ \wedge L$$

$$\frac{}{\Gamma \longrightarrow \top}\ \top R \qquad \frac{\Gamma \longrightarrow C}{\Gamma, \top \longrightarrow C}\ \top L$$

$$\frac{\Gamma \longrightarrow A}{\Gamma \longrightarrow A \vee B}\ \vee R_1 \qquad \frac{\Gamma \longrightarrow B}{\Gamma \longrightarrow A \vee B}\ \vee R_2 \qquad \frac{\Gamma, A \longrightarrow C \quad \Gamma, B \longrightarrow C}{\Gamma, A \vee B \longrightarrow C}\ \vee L$$

$$\text{no } \bot R \text{ rule} \qquad \frac{}{\Gamma, \bot \longrightarrow C}\ \bot L$$

$$\frac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \supset B}\ \supset R \qquad \frac{\Gamma, A \supset B \longrightarrow A \quad \Gamma, B \longrightarrow C}{\Gamma, A \supset B \longrightarrow C}\ \supset L$$

[1] not covered in lecture

4 Refining the Left Rule for Implication

In order to find a more efficient form of the problematic rule $\supset L$, we consider each possibility for the antecedent of the implication in turn. We will start with more obvious cases to find out the principles behind the design of the rules.

Truth. Consider a sequent

$$\Gamma, \top \supset B \longrightarrow C$$

Can we find a simpler proposition expressing the same as $\top \supset B$? Yes, namely just B, since $(\top \supset B) \equiv B$. So we can propose the following specialized rule:

$$\frac{\Gamma, B \longrightarrow C}{\Gamma, \top \supset B \longrightarrow C}\ \top\supset L$$

This rule derives from $\supset L$ and $\top R$, which are both sound. Note that we expect the resulting calculus to remain complete because we replace a proposition with an equivalent one, preserving provability.

Falsehood. Consider a sequent

$$\Gamma, \bot \supset B \longrightarrow C$$

Can we find a simpler proposition expressing the same contents? Yes, namely $\top$, since $(\bot \supset B) \equiv \top$. But $\top$ on the left-hand side can be eliminated by $\top L$, so we can specialize the general rule as follows:

$$\frac{\Gamma \longrightarrow C}{\Gamma, \bot \supset B \longrightarrow C}\ \bot\supset L$$

Soundness of this rule also follows from weakening. Are we losing information compared to applying $\supset L$ here? No because that would require a proof of $\Gamma, \bot \supset B \longrightarrow \bot$ which will succeed if $\bot$ can be proved from Γ, but then there also is a direct proof without using $\bot \supset B$.

Disjunction. Now we consider a sequent

$$\Gamma, (A_1 \vee A_2) \supset B \longrightarrow C$$

Again, we have to ask if there is a simpler equivalent formula we can use instead of $(A_1 \vee A_2) \supset B$. If we consider the $\vee L$ rule, we might consider $(A_1 \supset B) \wedge (A_2 \supset B)$. A little side calculation confirms that, indeed,

$$((A_1 \vee A_2) \supset B) \equiv ((A_1 \supset B) \wedge (A_2 \supset B))$$

The computational intuition is that getting a B out of having either a A_1 or an A_2 is equivalent to separate ways of getting a B out of a A_1 as well as a way of getting a B out of an

A_2. We can exploit this, playing through the rules as follows

$$\dfrac{\dfrac{\Gamma, A_1 \supset B, A_2 \supset B \longrightarrow C}{\Gamma, (A_1 \supset B) \wedge (A_2 \supset B) \longrightarrow C}\ \wedge L}{\Gamma, (A_1 \vee A_2) \supset B \longrightarrow C}\ \mathsf{equiv}$$

This suggests the specialized rule

$$\dfrac{\Gamma, A_1 \supset B, A_2 \supset B \longrightarrow C}{\Gamma, (A_1 \vee A_2) \supset B \longrightarrow C}\ \vee\supset L$$

The question is whether the premise is really smaller than the conclusion in some well-founded measure. We note that both $A_1 \supset B$ and $A_2 \supset B$ are smaller than the original formula $(A_1 \vee A_2) \supset B$. Replacing one element in a multiset by several, each of which is strictly smaller according to some well-founded ordering, induces another well-founded ordering on multisets (**?**). So, the premises are indeed smaller in the multiset ordering. A few more remarks on this in **??**. Operationally, the effect of $\vee\supset L$ is to separately consider the smaller implications $A_1 \supset B$ and $A_2 \supset B$.

Conjunction. Next we consider

$$\Gamma, (A_1 \wedge A_2) \supset B \longrightarrow C$$

In this case we can create an equivalent formula by currying using that $(A_1 \wedge A_2) \supset B \equiv A_1 \supset (A_2 \supset B)$.

$$\dfrac{\Gamma, A_1 \supset (A_2 \supset B) \longrightarrow C}{\Gamma, (A_1 \wedge A_2) \supset B \longrightarrow C}\ \wedge\supset L$$

This formula is not strictly smaller, but we can make it so by giving conjunction a weight of 2 while counting implications as 1. Fortunately, this weighting does not conflict with any of the other rules we have. Operationally, the effect of $\wedge\supset L$ is to first consider what to make of the first assumed conjunct A_1 by the other rules and then subsequently consider the second conjunct A_2.

Atomic propositions. How do we use an assumption $P \supset B$? We can conclude if we also know P, so we restrict the rule to the case where P is already among the assumption.

$$\dfrac{P \in \Gamma \quad \Gamma, B \longrightarrow C}{\Gamma, P \supset B \longrightarrow C}\ P\supset L$$

Clearly, the premise is smaller than the conclusion. If we were to use $\supset L$ instead, $P \supset B$ would remain in the first premise. The intuitive reason why we do not have to keep it is because the only way to make use of $P \supset B$ is to produce a P. But if we have such an atomic P, the above rule already establishes B. Note that, unlike a premise $\Gamma \longrightarrow P$, the premise $P \in \Gamma$ will obviously never search for possible proof rule applications within Γ. Indeed, those would not be useful, because we might as well apply them first before splitting into two premises.

Implication. Last, but not least, we consider the case

$$\Gamma, (A_1 \supset A_2) \supset B \longrightarrow C$$

We start by playing through the left rule $\supset L$ for this particular case because, as we have already seen, an implication on the left does not directly simplify when interacting with another implication.

$$\frac{\dfrac{\Gamma, (A_1 \supset A_2) \supset B, A_1 \longrightarrow A_2}{\Gamma, (A_1 \supset A_2) \supset B \longrightarrow A_1 \supset A_2}\supset R \qquad \Gamma, B \longrightarrow C}{\Gamma, (A_1 \supset A_2) \supset B \longrightarrow C}\supset L$$

The second premise is smaller and does not require any further attention. For the first premise, we need to find a smaller formula that is equivalent to $((A_1 \supset A_2) \supset B) \wedge A_1$. The conjunction here is a representation of two distinguished formulas in the context. Fortunately, we find

$$((A_1 \supset A_2) \supset B) \wedge A_1 \equiv (A_2 \supset B) \wedge A_1$$

which can be checked easily since $A_1 \supset A_2$ is equivalent to A_2 if we already have A_1. This leads to the specialized rule

$$\frac{\Gamma, A_2 \supset B, A_1 \longrightarrow A_2 \quad \Gamma, B \longrightarrow C}{\Gamma, (A_1 \supset A_2) \supset B \longrightarrow C}\supset\supset L$$

Indeed, all premises of $\supset\supset L$ are simpler now, because $A_2 \supset B$ has strictly less operators than $(A_1 \supset A_2) \supset B$ and its operators are of the same weight.

There is a minor variation of this rule, which is also both sound and complete, and the premises are all smaller (by the multiset ordering) than the conclusion.

$$\frac{\Gamma, A_2 \supset B \longrightarrow A_1 \supset A_2 \quad \Gamma, B \longrightarrow C}{\Gamma, (A_1 \supset A_2) \supset B \longrightarrow C}\supset\supset L$$

They are equivalent because, in general, $\Gamma \longrightarrow A_1 \supset A_2$ iff $\Gamma, A_1 \longrightarrow A_2$.

This concludes the presentation of the specialized rules so that the only rule that kept its principal formula around, $\supset L$, is no longer needed since all forms of implications are covered. The complete set of rule is summarized in Figure **??**.

The proof that these set of rules are *sound* is quite straightforward, but the fact that they are *complete* is nontrivial. We refer the reader to the paper by **?**. Termination of the bottom-up construction is again not difficult once one is familiar with the multiset ordering on sequents.

Even though these rules can be interpreted as defining a decision procedure, such a procedure would not be practical except for small examples because there is too much nondeterminism in choosing which rule to apply when. We will discuss this in the next lecture.

$$\frac{}{\Gamma, P \longrightarrow P}\ \mathsf{id}^*$$

$$\frac{\Gamma \longrightarrow A \quad \Gamma \longrightarrow B}{\Gamma \longrightarrow A \wedge B}\ \wedge R \qquad \frac{\Gamma, A, B \longrightarrow C}{\Gamma, A \wedge B \longrightarrow C}\ \wedge L$$

$$\frac{}{\Gamma \longrightarrow \top}\ \top R \qquad \frac{\Gamma \longrightarrow C}{\Gamma, \top \longrightarrow C}\ \top L$$

$$\frac{\Gamma \longrightarrow A}{\Gamma \longrightarrow A \vee B}\ \vee R_1 \qquad \frac{\Gamma \longrightarrow B}{\Gamma \longrightarrow A \vee B}\ \vee R_2 \qquad \frac{\Gamma, A \longrightarrow C \quad \Gamma, B \longrightarrow C}{\Gamma, A \vee B \longrightarrow C}\ \vee L$$

$$\text{no } \bot R \text{ rule} \qquad \frac{}{\Gamma, \bot \longrightarrow C}\ \bot L$$

$$\frac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \supset B}\ \supset R$$

$$\frac{\Gamma, P, B \longrightarrow C}{\Gamma, P, P \supset B \longrightarrow C}\ P \supset L$$

$$\frac{\Gamma, A_1 \supset (A_2 \supset B) \longrightarrow C}{\Gamma, (A_1 \wedge A_2) \supset B \longrightarrow C}\ \wedge \supset L \qquad \frac{\Gamma, B \longrightarrow C}{\Gamma, \top \supset B \longrightarrow C}\ \top \supset L$$

$$\frac{\Gamma, A_1 \supset B, A_2 \supset B \longrightarrow C}{\Gamma, (A_1 \vee A_2) \supset B \longrightarrow C}\ \vee \supset L \qquad \frac{\Gamma \longrightarrow C}{\Gamma, \bot \supset B \longrightarrow C}\ \bot \supset L$$

$$\frac{\Gamma, A_2 \supset B, A_1 \longrightarrow A_2 \quad \Gamma, B \longrightarrow C}{\Gamma, (A_1 \supset A_2) \supset B \longrightarrow C}\ \supset \supset L$$

Figure 1: Contraction-free sequent calculus

5 Multiset Ordering

An ordering is *well-founded* if any strictly decreasing chain of elements has a least element, that is, one that does not have any strictly smaller ones. Well-founded orderings support *well-founded induction*: to prove that a property $A(x)$ holds for an element x, we may assume that it holds for all strictly smaller elements $y < x$. The base case arises for minimal elements: we have to prove $A(x)$ outright since there is no $y < x$ when x is minimal.

If we have a well-founded ordering on elements from some set X we can extend it to a well-founded ordering on *finite sets of elements*. We start by defining that $Y < X$ if we can take an element $x \in X$ and replace it by an arbitrary finite set $\{y_1, \ldots, y_n\}$ such that each $y_i < x$. Because $n = 0$ is allowed, this means we can also just delete $x \in X$ and the only minimal finite set is just the empty set. The actual multiset ordering is then the transitive closure.

As explained, for example by **?**, if the ordering on elements is well-founded, so is the multiset ordering based on it.

Here is an example of the beginning of a chain on multisets of natural numbers:

$$
\begin{array}{ll}
 & \{0, 2, \quad 5\} \\
> & \{0, 1, 1, 1, 5\} \\
> & \{0, 0, 1, 1, 5\} \\
> & \{0, 0, 1, 1, 2, 3, 3, 3, 4\} \\
> & \{\ \ 0, 1, 1, 2, 3, 3, 3, 4\} \\
> & \{\ \ 0, 1, 1, 2, 3, \ \ 3, 4\} \\
> & \cdots
\end{array}
$$

One can observe that even though we eventually have to reach the empty set, there isn't a fixed bound on the number of steps because any element can be replaced by *any finite number of strictly smaller elements*. If we have a bound on *how many* smaller elements can replace a given one, then we can calculate some bound on the overall length of the strictly decreasing chain of finite sets (assuming we also have bounds for the order between the elements).

In the particular application here we consider a sequent $\Gamma \Longrightarrow A$ as a multiset $\Gamma \cup \{A\}$ and use our ordering on formulas as a basis. This ordering assigns 1 to every logical constant, atomic proposition, and connective except for conjunction, which is assigned the value 2.

References

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