Lecture Notes on Inversion

15-317: Constructive Logic
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Lecture 15
Thursday, March 16, 2023

1 Introduction

The contraction-free sequent calculus can be seen as describing a decision procedure for intuitionistic propositional logic. Great! But as soon as we have a sequent with many antecedents, there are many choices of rules to apply. Unless we somehow "optimize" we would have to try them all for each sequent we are trying to prove. This turns out not to be feasible except for very small examples.

Fortunately, some rules have the property that we can *always* apply them without having to consider alternatives. Loosely speaking, this is because whenever the conclusion is provable, so are all the premises. Because rules can have many different formulations, we do not think of this so much a property of a rule, but of a connective. We say a connective is *invertible on the right* if we can always apply the right rule on the connective without having to consider alternatives and remain complete in our search. Symmetrically, a connective is *invertible on the left* if we can always apply its left rule and remove the principal formula from the sequent.

The strategy then is to decompose all invertible connectives on the right and left of a sequent in a deterministic fashion until we reach one where we finally have to make a choice. When there are multiple choices, we may then need to try them in some order and backtrack if necessary.

As we do most of the time in this course, we capture this strategy by a restrictive set of rules. When attempting to construct a derivation bottom-up with these rules, we should be forced to follow this strategy.

We develop this step by step so we can see where the various components of the judgments come from. For simplicity, we don't use the contraction-free sequent calculus but the restrictive sequent calculus without the refinements of implication.

2 Restrictive Sequent Calculus

For reference, our starting point is shown in Figure 1.

$$\frac{}{\Gamma, P \longrightarrow P}\ \mathsf{id}^*$$

$$\frac{\Gamma \longrightarrow A \quad \Gamma \longrightarrow B}{\Gamma \longrightarrow A \wedge B}\ {\wedge}R \qquad \frac{\Gamma, A, B \longrightarrow C}{\Gamma, A \wedge B \longrightarrow C}\ {\wedge}L$$

$$\frac{}{\Gamma \longrightarrow \top}\ {\top}R \qquad \frac{\Gamma \longrightarrow C}{\Gamma, \top \longrightarrow C}\ {\top}L$$

$$\frac{\Gamma \longrightarrow A}{\Gamma \longrightarrow A \vee B}\ {\vee}R_1 \qquad \frac{\Gamma \longrightarrow B}{\Gamma \longrightarrow A \vee B}\ {\vee}R_2 \qquad \frac{\Gamma, A \longrightarrow C \quad \Gamma, B \longrightarrow C}{\Gamma, A \vee B \longrightarrow C}\ {\vee}L$$

$$\text{no } {\perp}R \text{ rule} \qquad \frac{}{\Gamma, \perp \longrightarrow C}\ {\perp}L$$

$$\frac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \supset B}\ {\supset}R \qquad \frac{\Gamma, A \supset B \longrightarrow A \quad \Gamma, B \longrightarrow C}{\Gamma, A \supset B \longrightarrow C}\ {\supset}L$$

Figure 1: Restrictive Sequent Calculus

3 Right Invertible Connectives

The restrictive sequent calculus in the previous section is a big improvement over the original one, but if we use it directly to implement a search procedure it is hopelessly inefficient. The problem is that for any goal sequent, any left or right rule might be applicable. But the application of a rule changes the sequent just a little—most formulas are preserved and we are faced with the same choices at the next step. Eliminating this kind of inefficiency is crucial for a practical theorem proving procedure.

The first observation, to be refined later, is that certain rules are *invertible*, that is, the premises hold iff the conclusion holds. This is powerful, because we can apply the rule and never look back and consider any other choice.

Conjunction. As an example of an invertible rule, consider ${\wedge}R$ again:

$$\frac{\Gamma \longrightarrow A \quad \Gamma \longrightarrow B}{\Gamma \longrightarrow A \wedge B}\ {\wedge}R$$

The premises already imply the conclusion since the rule is sound. So for the *rule* ${\wedge}R$ to be invertible means that if the conclusion holds then both premises hold as well.

That is, we have to show that the following rules are *admissible*:

$$\frac{\Gamma \longrightarrow A \wedge B}{\Gamma \longrightarrow A} \qquad \frac{\Gamma \longrightarrow A \wedge B}{\Gamma \longrightarrow B}$$

Fortunately, this follows easily by cut and identity, since $\Gamma, A \wedge B \longrightarrow A$ and $\Gamma, A \wedge B \longrightarrow B$.

$$\dfrac{\Gamma \longrightarrow A \wedge B \qquad \dfrac{\dfrac{}{\Gamma, A, B \longrightarrow A}\,\mathsf{id}}{\Gamma, A \wedge B \longrightarrow A}\,{\wedge}L}{\Gamma \longrightarrow A}\,\mathsf{cut}$$

In order to formalize the strategy of applying inversions eagerly, without backtracking over the choices of which invertible rules to try, we refine the restricted sequent calculus further into three, mutually dependent forms of sequents. We start by trying

$$\Gamma \xrightarrow{\mathsf{R}} A$$

and write the rules for this judgment such that *we can only apply right rules to right invertible connectives*. In the process, we'll have to discover which these are. To start with, we get:

$$\dfrac{\Gamma \xrightarrow{\mathsf{R}} A \qquad \Gamma \xrightarrow{\mathsf{R}} B}{\Gamma \xrightarrow{\mathsf{R}} A \wedge B}\,{\wedge}R$$

Since the rule mimics the usual right rule for conjunction, we keep the same name.

Implication What about implication? In order to see if implication is invertible on the right, let's look at the rule:

$$\dfrac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \supset B}\,{\supset}R$$

Not losing provability means that the rule

$$\dfrac{\Gamma \longrightarrow A \supset B}{\Gamma, A \longrightarrow B}$$

would have to be admissible. It is, again using cut and identity:

$$\dfrac{\dfrac{\Gamma \longrightarrow A \supset B}{\Gamma, A \longrightarrow A \supset B}\,\mathsf{weaken} \qquad \dfrac{\dfrac{}{\Gamma, A, A \supset B \Longrightarrow A}\,\mathsf{id} \qquad \dfrac{}{\Gamma, A, B \longrightarrow B}\,\mathsf{id}}{\Gamma, A, A \supset B \longrightarrow B}\,{\supset}L}{\Gamma, A \longrightarrow B}\,\mathsf{cut}$$

We have shown here the weakening step, although as we have grown used to we will apply this silently from now on. For implication, we get the rule

$$\dfrac{\Gamma, A \xrightarrow{\mathsf{R}} B}{\Gamma \xrightarrow{\mathsf{R}} A \supset B}\,{\supset}R$$

Disjunction. It turns out that neither of the two right rules for disjunction is invertible. Also, it would involve a *choice* which of the two rules apply, so it doesn't have a place in the right inversion judgment. In fact, we may have to apply a left rule before we apply a right rule. For example, when proving

$$P \vee Q \longrightarrow Q \vee P$$

we have to first apply $\vee L$: starting with either of the two right rules will immediately run into trouble.

Falsehood. There is no right rule for $\bot$, so it certainly cannot be part of the right inversion judgment.

Truth. We can always apply $\top R$ because it completes the proof and therefore never requires considering other choices.

$$\frac{}{\Gamma \xrightarrow{\mathsf{R}} \top}\ \top R$$

Atomic propositions. In a sequent $\Gamma \longrightarrow P$ we can not necessarily apply the identity because P may not be in Γ. Consider, for example, $Q, Q \supset P \longrightarrow P$. So it would not be in the right inversion judgment.

Summary. We found:

Right invertible	$A \wedge B, A \supset B, \top$
Not right invertible	$A \vee B, \bot, P$

4 Left Inversion

Once we have broken down the right-hand side of a sequent until we have reached a proposition that is not right invertible, we have to switch to a new judgment for left inversion. We'll have to change this judgment shortly, but let's start as we did for right inversion.

$$\frac{\Gamma, A, B \xrightarrow{\mathsf{L}} C}{\Gamma, A \wedge B \xrightarrow{\mathsf{L}} C}\ \wedge L$$

This looks okay, because we can prove

$$\dfrac{\dfrac{\dfrac{}{A \longrightarrow A}\ \mathsf{id} \quad \dfrac{}{B \longrightarrow B}\ \mathsf{id}}{A, B \longrightarrow A \wedge B}\ \wedge R \qquad \Gamma, A \wedge B \longrightarrow C}{\Gamma, A, B \longrightarrow C}\ \mathsf{cut}$$

However, what happens if we have two conjunctions among the antecedents?

$$\Gamma, A \wedge B, D \wedge E \xrightarrow{\mathsf{L}} C$$

We could apply $\wedge L$ to $A \wedge B$ or we could apply $\wedge L$ to $D \wedge E$. But we wanted to design our calculus so there is no choice during inversion!

In order to achieve this we replace the antecedents Γ (which have always been unordered) with the ordered antecedents Ω.

$$\text{Ordered antecedents} \quad \Omega \quad ::= \quad \epsilon \mid A \cdot \Omega$$

Using ordered antecedents we can force a rule to apply only to its first element.

$$\frac{A \cdot B \cdot \Omega \xrightarrow{\mathsf{L}} C}{(A \wedge B) \cdot \Omega \xrightarrow{\mathsf{L}} C} \wedge L$$

That seems fine, but what if we encounter a proposition whose principal connective is *not* left invertible? Somehow, we have to be able to "skip past it" and find other left invertible propositions in Ω. The way we capture this is by having *two different collections of antecedents*. These are separated by a semicolon ";" which is a notation frequently used for separating different kinds of antecedents.

$$\Gamma \,;\, \Omega \xrightarrow{\mathsf{L}} C$$

Here, Γ is unordered as usual and Ω is ordered. When a noninvertible proposition is first in Ω we shuffle it into Γ. For example (anticipating that implication is not left invertible):

$$\frac{\Gamma, A \supset B \,;\, \Omega \xrightarrow{\mathsf{L}} C}{\Gamma \,;\, (A \supset B) \cdot \Omega \xrightarrow{\mathsf{L}} C}$$

This is a so-called *structural rule* because it changes the structure of the sequent without changing its components.

We can now go through the connectives and write out the left invertible rules, to be applied deterministically.

Conjunction. We have already seen this.

$$\frac{\Gamma \,;\, A \cdot B \cdot \Omega \xrightarrow{\mathsf{L}} C}{\Gamma \,;\, (A \wedge B) \cdot \Omega \xrightarrow{\mathsf{L}} C} \wedge L$$

Implication. Implication is not left invertible. We already saw what works as a counterexample in the last lecture, namely the proof of $\neg\neg(A \vee \neg A)$. We may need the implication again, so we cannot eliminate it from the left premise. Therefore, we just obtain the structural rule already shown above:

$$\frac{\Gamma, A \supset B \,;\, \Omega \xrightarrow{\mathsf{L}} C}{\Gamma \,;\, (A \supset B) \cdot \Omega \xrightarrow{\mathsf{L}} C} \mathsf{LL}$$

We label this structural rules as LL because both premise and conclusion are left inversion judgments.

Disjunction. Disjunction *is* left invertible. In a proof where we know $A \vee B$, we can always proceed by considering the two cases separately. This may do redundant work (if this assumption isn't actually needed), but such potential redundancy doesn't affect completeness.

$$\frac{\Gamma\ ;\ A \cdot \Omega \xrightarrow{\mathsf{L}} C \quad \Gamma\ ;\ B \cdot \Omega \xrightarrow{\mathsf{L}} C}{\Gamma\ ;\ (A \vee B) \cdot \Omega \xrightarrow{\mathsf{L}} C}\ \vee L$$

Quick check: we need the admissibility of

$$\frac{\Gamma, A \vee B \longrightarrow C}{\Gamma, A \longrightarrow C} \qquad \frac{\Gamma, A \vee B \longrightarrow C}{\Gamma, B \longrightarrow C}$$

which are easy to prove with the techniques we have seen and left as an exercise.

Falsehood. Falsehood is also left invertible, trivially.

$$\frac{}{\Gamma\ ;\ \bot \cdot \Omega \xrightarrow{\mathsf{L}} C}\ \bot L$$

Truth. We can always just remove $\top$ from the antecedents since it contains no information and therefore isn't useful in a proof.

$$\frac{\Gamma\ ;\ \Omega \xrightarrow{\mathsf{L}} C}{\Gamma\ ;\ \top \cdot \Omega \xrightarrow{\mathsf{L}} C}\ \top L$$

This is the nullary case of conjunction on the left. We can also check the admissibility of the corresponding inverse rule, which is easy to verify.

Atomic propositions. Since the succedent may not be exactly P, the identity (the only rule that could apply to P) is not always applicable and we have to move P to Γ with a structural rule.

$$\frac{\Gamma, P\ ;\ \Omega \xrightarrow{\mathsf{L}} C}{\Gamma\ ;\ P \cdot \Omega \xrightarrow{\mathsf{L}} C}\ \mathsf{LL}$$

We reuse the rule name, because the action is the same: a noninvertible proposition is moved from the ordered to the unordered antecedents. We'll see below what happens when Ω becomes empty.

5 Right Inversion Revisited

When writing our right inversion rules, we did not antipate the need for the ordered antecedents Ω, so we now have to rewrite these rules. The unordered context Γ should only consist of propositions that are not left invertible. So we obtain the following invariants:

$$\begin{array}{llcl}\text{Antecedents, not left invertible, unordered} & \Gamma & ::= & A \supset B \mid P \mid \cdot \mid \Gamma_1, \Gamma_2 \\ \text{Antecedents, ordered} & \Omega & ::= & A \cdot \Omega \mid \epsilon \\ \text{Succedent, not right invertible} & C & ::= & A \vee B \mid \bot \mid P\end{array}$$

and judgments

$$\begin{array}{ll}\text{Right inversion} & \Gamma \,;\, \Omega \xrightarrow{\mathsf{R}} A \\ \text{Left inversion} & \Gamma \,;\, \Omega \xrightarrow{\mathsf{L}} C\end{array}$$

Since Γ consists only of propositions that are not left invertible, the right rule for implication $A \supset B$ has to add A to Ω. The other rules are straightforward, so we just summarize them.

$$\frac{\Gamma \,;\, \Omega \xrightarrow{\mathsf{R}} A \quad \Gamma \,;\, \Omega \xrightarrow{\mathsf{R}} B}{\Gamma \,;\, \Omega \xrightarrow{\mathsf{R}} A \wedge B} \wedge R \qquad \frac{\Gamma \,;\, A \cdot \Omega \xrightarrow{\mathsf{R}} B}{\Gamma \,;\, \Omega \xrightarrow{\mathsf{R}} A \supset B} \supset R \qquad \frac{}{\Gamma \,;\, \Omega \xrightarrow{\mathsf{R}} \top} \top R$$

$$\frac{\Gamma \,;\, \Omega \xrightarrow{\mathsf{L}} A \vee B}{\Gamma \,;\, \Omega \xrightarrow{\mathsf{R}} A \vee B} \mathsf{LR} \qquad \frac{\Gamma \,;\, \Omega \xrightarrow{\mathsf{L}} \bot}{\Gamma \,;\, \Omega \xrightarrow{\mathsf{R}} \bot} \mathsf{LR} \qquad \frac{\Gamma \,;\, \Omega \xrightarrow{\mathsf{L}} P}{\Gamma \,;\, \Omega \xrightarrow{\mathsf{R}} P} \mathsf{LR}$$

The last three rules are the structural rules that transition from right inversion to left inversion. Next, the rules for left inversion, summarized from above.

$$\frac{\Gamma \,;\, A \cdot B \cdot \Omega \xrightarrow{\mathsf{L}} C}{\Gamma \,;\, (A \wedge B) \cdot \Omega \xrightarrow{\mathsf{L}} C} \wedge L \qquad \frac{\Gamma \,;\, A \cdot \Omega \xrightarrow{\mathsf{L}} C \quad \Gamma \,;\, B \cdot \Omega \xrightarrow{\mathsf{L}} C}{\Gamma \,;\, (A \vee B) \cdot \Omega \xrightarrow{\mathsf{L}} C} \vee L$$

$$\frac{}{\Gamma \,;\, \bot \cdot \Omega \xrightarrow{\mathsf{L}} C} \bot L \qquad \frac{\Gamma \,;\, \Omega \xrightarrow{\mathsf{L}} C}{\Gamma \,;\, \top \cdot \Omega \xrightarrow{\mathsf{L}} C} \top L$$

$$\frac{\Gamma, A \supset B \,;\, \Omega \xrightarrow{\mathsf{L}} C}{\Gamma \,;\, (A \supset B) \cdot \Omega \xrightarrow{\mathsf{L}} C} \mathsf{LL} \qquad \frac{\Gamma, P \,;\, \Omega \xrightarrow{\mathsf{L}} C}{\Gamma \,;\, P \cdot \Omega \xrightarrow{\mathsf{L}} C} \mathsf{LL}$$

The last two rules and the structural rules for propositions that are not left invertible.

6 Choice

With the rules so far, we can deterministically break down a sequent until we reach

$$\Gamma \,;\, \epsilon \xrightarrow{\mathsf{L}} C$$

where Γ is not left invertible and C is not right invertible.

At this point we should make a choice. While not strictly necessary, it makes sense to introduce a new judgment

$$\Gamma\,;\,\epsilon \xrightarrow{\mathsf{C}} C$$

where the "C" stands for Choice. Note that there are never any ordered antecedents because we have applied all left invertible rules already. Examining the possibilities for the propositions in this judgment gives use the following rules.

$$\frac{\Gamma\,;\,\epsilon \xrightarrow{\mathsf{C}} C}{\Gamma\,;\,\epsilon \xrightarrow{\mathsf{L}} C}\ \mathsf{CL}$$

$$\frac{\Gamma\,;\,\epsilon \xrightarrow{\mathsf{R}} A}{\Gamma\,;\,\epsilon \xrightarrow{\mathsf{C}} A \vee B}\ {\vee}R_1 \qquad \frac{\Gamma\,;\,\epsilon \xrightarrow{\mathsf{R}} B}{\Gamma\,;\,\epsilon \xrightarrow{\mathsf{C}} A \vee B}\ {\vee}R_1$$

$$\frac{}{\Gamma, P\,;\,\epsilon \xrightarrow{\mathsf{C}} P}\ \mathsf{id}^* \qquad \frac{\Gamma, A \supset B\,;\,\epsilon \xrightarrow{\mathsf{R}} A \quad \Gamma, [A \supset B]\,;\,B \xrightarrow{\mathsf{R}} C}{\Gamma, A \supset B\,;\,\epsilon \xrightarrow{\mathsf{C}} C}\ {\supset}L$$

In the cases of ${\vee}R_i$ and ${\supset}L$ we circle back to the $\xrightarrow{\mathsf{R}}$ judgment since the component types may be invertible, so we have to go through a phase of inversion. As a small "optimization" the second premise of ${\supset}L$ could be $\Gamma\,;\,B \xrightarrow{\mathsf{L}} C$, because we already know that C is not right invertible and that $A \supset B$ is redundant. But we need to preserve $A \supset B$ in the first premise, as the example of $\neg\neg(A \vee \neg A)$ from Lecture 14.4 shows.

7 Soundness and Completeness of Inversion

We define a translation between ordered an unordered contexts via

$$\begin{array}{rcl} \overline{\epsilon} & = & \cdot \\ \overline{\Omega \cdot A} & = & \overline{\Omega}, A \end{array}$$

Then the soundness theorem states

1. If $\Gamma\,;\,\Omega \xrightarrow{\mathsf{R}} A$ then $\Gamma, \overline{\Omega} \longrightarrow A$
2. If $\Gamma\,;\,\Omega \xrightarrow{\mathsf{L}} C$ then $\Gamma, \overline{\Omega} \longrightarrow C$
3. If $\Gamma\,;\,\epsilon \xrightarrow{\mathsf{C}} C$ then $\Gamma \longrightarrow C$

The proof is straightforward by simultaneous induction over the structure of the given derivations. The logical rules map to rules of the same name, and the structural rules have identical premises and conclusion, so they collapse.

Completeness is a much more difficult theorem. What we want is

1. If $\Gamma, \overline{\Omega} \longrightarrow A$ then $\Gamma \mathrel{;} \Omega \xrightarrow{\mathsf{R}} A$
2. If $\Gamma, \overline{\Omega} \longrightarrow C$ then $\Gamma \mathrel{;} \Omega \xrightarrow{\mathsf{L}} C$
3. If $\Gamma \longrightarrow C$ then $\Gamma \mathrel{;} \epsilon \xrightarrow{\mathsf{C}} C$

The key to this property, as for many completeness theorems, is the admissibility of cut and identity in the more restricted system. Both of these are significantly more complicated than for ordinary sequent calculus. Simple properties, such as weakening, no longer hold in the strong form we had earlier. For example, we might have

$$\frac{}{\Gamma \mathrel{;} \bot \cdot \Omega \xrightarrow{\mathsf{L}} C}\ \bot L$$

but if we weaken, for example, as

$$\Gamma^{-} \mathrel{;} ((A \vee B) \wedge (C \vee D)) \cdot \bot \cdot \Omega \xrightarrow{\mathsf{L}} C$$

we are now *forced* to break down $(A \vee B) \wedge (C \vee D)$ entirely before we can apply $\bot L$ in each branch.

We do not replicate the proof here, but the interested reader is referred to the elegant solution by Simmons [2014] elegant solution for an even more restricted system, into which the inversion calculus can be easily embedded.

8 Loop Checking

The inversion calculus we have developed in this lecture so far is not yet a decision procedure. The problem from last lecture did not go away: in the rule

$$\frac{\Gamma, A \supset B \mathrel{;} \epsilon \xrightarrow{\mathsf{R}} A \quad \Gamma, [A \supset B] \mathrel{;} B \xrightarrow{\mathsf{R}} C}{\Gamma, A \supset B \mathrel{;} \epsilon \xrightarrow{\mathsf{C}} C}\ {\supset}L$$

the first premise is not necessarily smaller than the conclusion.

We can try to merge the ideas behind the contraction-free calculus and the inversion calculus. In this lecture we pursue a different path: we can cut off search along a branch in the partial proof tree if we encounter a sequent which is the same as an earlier sequent. Given the overall goal of proving A, there are only finitely many subformulas of A, and therefore there are also only finitely many sequents *with distinct antecedents* that can occur in a proof. Owing to the contraction theorem, we can always achieve that every antecedent is unique. Alternatively, assume we have a situation such as

$$\begin{array}{c} \Gamma' \mathrel{;} \epsilon \xrightarrow{\mathsf{C}} C' \\ \vdots \\ \Gamma \mathrel{;} \epsilon \xrightarrow{\mathsf{C}} C \end{array}$$

When can we give up on this branch? We can fail this branch when

$$\Gamma' \subseteq \Gamma \quad \text{and} \quad C = C'$$

That's because where there is a derivation of $\Gamma' \;;\; \epsilon \xrightarrow{\mathsf{C}} C$ then the same derivation also shows $\Gamma \;;\; \epsilon \xrightarrow{\mathsf{C}} C$. So why make the detour? This can be formalized as an induction, which we do not make precise here.

Note that we used the choice sequents only for this argument, even though it holds for sequents in general. That's because inversion steps are uniquely determined, so we might as well carry them out and save ourselves a lot of redundant loop checks.

As an example of the system, including loop checking, we verify that $\neg\neg P \supset P$ is not provable. We start the proof attempt (which will fail) in the right inversion phase.

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ (1)\ \neg\neg P \,;\, \epsilon \xrightarrow{\mathsf{C}} P\end{array}}{\neg\neg P \,;\, \epsilon \xrightarrow{\mathsf{L}} P}\,\mathsf{CL}}{\cdot \,;\, \neg\neg P \xrightarrow{\mathsf{L}} P}\,\mathsf{LL}}{\cdot \,;\, \neg\neg P \xrightarrow{\mathsf{R}} P}\,\mathsf{LR}}{\cdot \,;\, \epsilon \xrightarrow{\mathsf{R}} \neg\neg P \supset P}\,{\supset}R$$

Among the choice rules, only ${\supset}L$ is applicable. We label the choice sequent in case we later want to indicate a loop, that is, the same sequent recurs later.

$$\dfrac{\begin{array}{c}\vdots\\ \neg\neg P \,;\, \epsilon \xrightarrow{\mathsf{R}} \neg P\end{array} \qquad \begin{array}{c}\vdots\\ \cdot \,;\, \bot \xrightarrow{\mathsf{R}} P\end{array}}{(1)\ \neg\neg P \,;\, \epsilon \xrightarrow{\mathsf{C}} P}\,{\supset}L$$

Let's first look at the second premise. This is actually easy to prove:

$$\dfrac{\begin{array}{c}\vdots\\ \neg\neg P \,;\, \epsilon \xrightarrow{\mathsf{R}} \neg P\end{array} \qquad \dfrac{\dfrac{}{\cdot \,;\, \bot \xrightarrow{\mathsf{L}} P}\,{\bot}L}{\cdot \,;\, \bot \xrightarrow{\mathsf{R}} P}\,\mathsf{LR}}{(1)\ \neg\neg P \,;\, \epsilon \xrightarrow{\mathsf{C}} P}\,{\supset}L$$

In the first premise, we can apply ${\supset}R$, then switch to left inversion, and finally arrive at another choice sequent. Again, we label it.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ (2)\ \neg\neg P, P \,;\, \epsilon \xrightarrow{\mathsf{C}} \bot\end{array}}{\neg\neg P, P \,;\, \epsilon \xrightarrow{\mathsf{L}} \bot}\,\mathsf{CL}}{\neg\neg P \,;\, P \xrightarrow{\mathsf{L}} \bot}\,\mathsf{LL}}{\neg\neg P \,;\, P \xrightarrow{\mathsf{R}} \bot}\,\mathsf{LR}}{\neg\neg P \,;\, \epsilon \xrightarrow{\mathsf{R}} \neg P}\,{\supset}R \qquad \dfrac{\dfrac{}{\cdot \,;\, \bot \xrightarrow{\mathsf{L}} P}\,{\bot}L}{\cdot \,;\, \bot \xrightarrow{\mathsf{R}} P}\,\mathsf{LR}}{(1)\ \neg\neg P \,;\, \epsilon \xrightarrow{\mathsf{C}} P}\,{\supset}L$$

Let's look back on the partial proof tree. The previous choice sequent was $\neg\neg P \,;\, \epsilon \xrightarrow{\mathsf{C}} P$, so we cannot form a loop because we now have P as an additional assumption.

Even though we are at a choice sequent, there is once again just a single left rule applicable. We fill in the second premise because it is easy to prove.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \neg\neg P, P \,;\, \epsilon \xrightarrow{\mathsf{R}} \neg P\end{array} \quad \dfrac{\dfrac{}{P \,;\, \bot \xrightarrow{\mathsf{L}} \bot}\,{\bot}L}{P \,;\, \bot \xrightarrow{\mathsf{R}} \bot}\,\mathsf{LR}}{(2)\ \neg\neg P, P \,;\, \epsilon \xrightarrow{\mathsf{C}} \bot}\,{\supset}L}{\neg\neg P, P \,;\, \epsilon \xrightarrow{\mathsf{L}} \bot}}{\neg\neg P \,;\, P \xrightarrow{\mathsf{L}} \bot}\,\mathsf{LL}}{\neg\neg P \,;\, P \xrightarrow{\mathsf{R}} \bot}\,\mathsf{LR}}{\neg\neg P \,;\, \epsilon \xrightarrow{\mathsf{R}} \neg P}\,{\supset}R \qquad \dfrac{\dfrac{}{\cdot \,;\, \bot \xrightarrow{\mathsf{L}} P}\,{\bot}L}{\cdot \,;\, \bot \xrightarrow{\mathsf{R}} P}\,\mathsf{LR}}{(1)\ \neg\neg P \,;\, \epsilon \xrightarrow{\mathsf{C}} P}\,{\supset}L$$

In the first premise, we go through a similar sequence as before, first $\supset R$ and then some structural rules.

$$
\begin{prooftree}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\neg\neg P, P, P \,;\, \epsilon \xrightarrow{\mathsf{C}} \bot$}
\RightLabel{CL}
\UnaryInfC{$\neg\neg P, P, P \,;\, \epsilon \xrightarrow{\mathsf{L}} \bot$}
\RightLabel{LL}
\UnaryInfC{$\neg\neg P, P \,;\, P \xrightarrow{\mathsf{L}} \bot$}
\RightLabel{LR}
\UnaryInfC{$\neg\neg P, P \,;\, P \xrightarrow{\mathsf{R}} \bot$}
\RightLabel{${\supset}R$}
\UnaryInfC{$\neg\neg P, P \,;\, \epsilon \xrightarrow{\mathsf{R}} \neg P$}
\AxiomC{}
\RightLabel{$\bot L$}
\UnaryInfC{$P \,;\, \bot \xrightarrow{\mathsf{L}} \bot$}
\RightLabel{LR}
\UnaryInfC{$P \,;\, \bot \xrightarrow{\mathsf{R}} \bot$}
\RightLabel{${\supset}L$}
\BinaryInfC{$(2)\ \neg\neg P, P \,;\, \epsilon \xrightarrow{\mathsf{C}} \bot$}
\RightLabel{CL}
\UnaryInfC{$\neg\neg P, P \,;\, \epsilon \xrightarrow{\mathsf{L}} \bot$}
\RightLabel{LL}
\UnaryInfC{$\neg\neg P \,;\, P \xrightarrow{\mathsf{L}} \bot$}
\RightLabel{LR}
\UnaryInfC{$\neg\neg P \,;\, P \xrightarrow{\mathsf{R}} \bot$}
\RightLabel{${\supset}R$}
\UnaryInfC{$\neg\neg P \,;\, \epsilon \xrightarrow{\mathsf{R}} \neg P$}
\AxiomC{}
\RightLabel{$\bot L$}
\UnaryInfC{$\cdot \,;\, \bot \xrightarrow{\mathsf{L}} P$}
\RightLabel{LR}
\UnaryInfC{$\cdot \,;\, \bot \xrightarrow{\mathsf{R}} P$}
\RightLabel{${\supset}L$}
\BinaryInfC{$(1)\ \neg\neg P \,;\, \epsilon \xrightarrow{\mathsf{C}} P$}
\end{prooftree}
$$

This time we can cut off the search, because $\{\neg\neg P, P, P\} \subseteq \{\neg\neg P, P\}$ (viewing the antecedents as sets).

$$
\begin{prooftree}
\AxiomC{(loop with (2))}
\noLine
\UnaryInfC{$(3)\ \neg\neg P, P, P \,;\, \epsilon \xrightarrow{\mathsf{C}} \bot$}
\RightLabel{CL}
\UnaryInfC{$\neg\neg P, P, P \,;\, \epsilon \xrightarrow{\mathsf{L}} \bot$}
\RightLabel{LL}
\UnaryInfC{$\neg\neg P, P \,;\, P \xrightarrow{\mathsf{L}} \bot$}
\RightLabel{LR}
\UnaryInfC{$\neg\neg P, P \,;\, P \xrightarrow{\mathsf{R}} \bot$}
\RightLabel{${\supset}R$}
\UnaryInfC{$\neg\neg P, P \,;\, \epsilon \xrightarrow{\mathsf{R}} \neg P$}
\AxiomC{}
\RightLabel{$\bot L$}
\UnaryInfC{$P \,;\, \bot \xrightarrow{\mathsf{L}} \bot$}
\RightLabel{LR}
\UnaryInfC{$P \,;\, \bot \xrightarrow{\mathsf{R}} \bot$}
\RightLabel{${\supset}L$}
\BinaryInfC{$(2)\ \neg\neg P, P \,;\, \epsilon \xrightarrow{\mathsf{C}} \bot$}
\RightLabel{CL}
\UnaryInfC{$\neg\neg P, P \,;\, \epsilon \xrightarrow{\mathsf{L}} \bot$}
\RightLabel{LL}
\UnaryInfC{$\neg\neg P \,;\, P \xrightarrow{\mathsf{L}} \bot$}
\RightLabel{LR}
\UnaryInfC{$\neg\neg P \,;\, P \xrightarrow{\mathsf{R}} \bot$}
\RightLabel{${\supset}R$}
\UnaryInfC{$\neg\neg P \,;\, \epsilon \xrightarrow{\mathsf{R}} \neg P$}
\AxiomC{}
\RightLabel{$\bot L$}
\UnaryInfC{$\cdot \,;\, \bot \xrightarrow{\mathsf{L}} P$}
\RightLabel{LR}
\UnaryInfC{$\cdot \,;\, \bot \xrightarrow{\mathsf{R}} P$}
\RightLabel{${\supset}L$}
\BinaryInfC{$(1)\ \neg\neg P \,;\, \epsilon \xrightarrow{\mathsf{C}} P$}
\end{prooftree}
$$

What we show above is of course not a proof. Instead, it represents a *refutation* of the conclusion because (a) at every choice sequent we have explored all choices (in each case, there was just one), and (b) the only possible choice fails because of a loop, that is, a repeating sequent.

9 Conclusion

We have introduced *inversion*, an important concept in proof search because it allows us to proceed deterministically in many cases. Eventually, though, some choices still have to be made, but hopefully many fewer.

We have taken the restrictive sequent calculus and engineered a formal system with three kinds of sequents (right inversion, left inversion, choice), utilizing *ordered antecedents* to eliminate some nondeterminism in the application of rules. In the end, we found the following:

Right invertible	$A \wedge B, A \supset B, \top$
Not right invertible	$A \vee B, \bot, P$
Left invertible	$A \wedge B, \top, A \vee B, \bot$
Not left invertible	$A \supset B, P$

Conjunction seems to have a special status, because it is invertible both on the left and on the right. This is a reflection that there actually two forms of conjunction: a lazy and an eager one. The one that is invertible on the right is lazy, and the one that is invertible left is eager. Since in intuitionistic logic the two forms of conjunction are logically equivalent we don't need to make a formal distinction unless we care about the dynamic behavior of the proofs interpreted as programs. An analogous remark applies about $\top$, for a similar reason.

In the end, we obtained a decision procedure from the inversion calculus by doing loop checking between choice sequents.

We also observed that the inversion calculus is tedious to use by hand because many structural rules have to be applied. Nevertheless, it still serves two important purposes: (a) it clarifies which connectives *are* invertible on the right and left and (b) embodies a particular strategy (right inversion, left inversion, choice, right inversion, left inversion, choice, etc.). The latter is what makes this calculus simple to implement, which we will explore in the next lecture.

References

Robert J. Simmons. Structural focalization. *ACM Transactions on Computational Logic*, 15(3): 21:1–21:33, 2014.