Lecture Notes on
Focusing

15-317: Constructive Logic
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Lecture 17
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1 Introduction

Inversion removes the major source of nondeterminism in proof search. However, there is still more improvements to make while remaining at a high level of abstraction, and these are about making choices. Consider the goal to prove

$$A_1 \supset A_2, B_1 \supset B_2, C_1 \supset C_2 \longrightarrow D_1 \vee D_2 \vee D_3 \vee D_4$$

Let's assume all the D_i are atomic propositions. At an intuitive level, there should be 7 ways to proceed: 3 possibilities for $\supset L$ and 4 possibilities to prove one of the disjuncts on the right.

In the current system, it doesn't work out this way. We can choose among 3 $\supset L$ rules, and we can choose either D_1 or $D_2 \vee D_3 \vee D_4$ which is only 5 choices. Unfortunately, if D_1 fails we then again have the choice between 3 $\supset L$ rules and now between D_2 and $D_3 \vee D_4$, again 5 choices. If D_2 fails, we once again have to choose between 5 alternatives, and if D_3 fails then 3. So, overall, we make $5 + 5 + 5 + 3 = 18$ choices when it should only be 7. This explodes quickly on even slightly larger examples.

The key insight here is that we *chain* together the choices on the succedent without having to reconsider choices among the antecedents. This is not an obvious property, and its proof is nontrivial [Simmons, 2014]. And we can in fact carry this forward to choice among the antecedents. For example, if we are trying to prove

$$(A_1 \vee A_2) \supset A_3 \longrightarrow D_1 \vee D_2 \vee D_3 \vee D_4$$

we can either focus on the succedent (giving us 4 choices) or focus on the antecedent (giving us 2 choices, namely between A_1 and A_2). This improves on the "small-step choice" in the inversion calculus similar to the reasoning above.

The property that we can chain together choices by focusing on a particular antecedent or the succedent was first observed by Andreoli [1992] for *linear logic*. It turns out to be fundamental to just about any logic and has many interesting consequences, one of which we will examine in the next lecture.

There is one further refinement (also due to Andreoli) which we discuss in Section 5.

2 Inversion Recalled

As before, we imagine we carry out all invertible rules until we arrive at a choice. We recall material from Lecture 15 on inversion. We streamline the choice judgment by removing the ordered context ϵ which is always empty and therefore doesn't contribute to its meaning.

Judgments.

Right inversion	$\Gamma\,;\Omega \xrightarrow{\mathsf{R}} A$
Left inversion	$\Gamma\,;\Omega \xrightarrow{\mathsf{L}} C$
Choice	$\Gamma \xrightarrow{\mathsf{C}} C$

Rules.

Right Inversion.

$$\frac{\Gamma\,;\Omega \xrightarrow{\mathsf{R}} A \quad \Gamma\,;\Omega \xrightarrow{\mathsf{R}} B}{\Gamma\,;\Omega \xrightarrow{\mathsf{R}} A \wedge B}\,\wedge R \qquad \frac{\Gamma\,;A\cdot\Omega \xrightarrow{\mathsf{R}} B}{\Gamma\,;\Omega \xrightarrow{\mathsf{R}} A \supset B}\,\supset R \qquad \frac{}{\Gamma\,;\Omega \xrightarrow{\mathsf{R}} \top}\,\top R$$

$$\frac{\Gamma\,;\Omega \xrightarrow{\mathsf{L}} A \vee B}{\Gamma\,;\Omega \xrightarrow{\mathsf{R}} A \vee B}\,\mathsf{LR} \qquad \frac{\Gamma\,;\Omega \xrightarrow{\mathsf{L}} \bot}{\Gamma\,;\Omega \xrightarrow{\mathsf{R}} \bot}\,\mathsf{LR} \qquad \frac{\Gamma\,;\Omega \xrightarrow{\mathsf{L}} P}{\Gamma\,;\Omega \xrightarrow{\mathsf{R}} P}\,\mathsf{LR}$$

Left Inversion.

$$\frac{\Gamma\,;A\cdot B\cdot\Omega \xrightarrow{\mathsf{L}} C}{\Gamma\,;(A\wedge B)\cdot\Omega \xrightarrow{\mathsf{L}} C}\,\wedge L \qquad \frac{\Gamma\,;A\cdot\Omega \xrightarrow{\mathsf{L}} C \quad \Gamma\,;B\cdot\Omega \xrightarrow{\mathsf{L}} C}{\Gamma\,;(A\vee B)\cdot\Omega \xrightarrow{\mathsf{L}} C}\,\vee L$$

$$\frac{}{\Gamma\,;\bot\cdot\Omega \xrightarrow{\mathsf{L}} C}\,\bot L \qquad \frac{\Gamma\,;\Omega \xrightarrow{\mathsf{L}} C}{\Gamma\,;\top\cdot\Omega \xrightarrow{\mathsf{L}} C}\,\top L$$

$$\frac{\Gamma, A\supset B\,;\Omega \xrightarrow{\mathsf{L}} C}{\Gamma\,;(A\supset B)\cdot\Omega \xrightarrow{\mathsf{L}} C}\,\mathsf{LL} \qquad \frac{\Gamma, P\,;\Omega \xrightarrow{\mathsf{L}} C}{\Gamma\,;P\cdot\Omega \xrightarrow{\mathsf{L}} C}\,\mathsf{LL}$$

$$\frac{\Gamma \xrightarrow{\mathsf{C}} C}{\Gamma \xrightarrow{\mathsf{L}} C}\,\mathsf{CL}$$

3 Focusing on the Succedent

When we use the inversion rules in bottom-up search we reach the choice sequent $\Gamma \xrightarrow{\mathsf{C}} C$ where Γ consists of implications and atoms and C is a disjunction, falsehood, or an atom. At this point we can *focus* on the succedent or any of the antecedents. We indicate this by [bracketing] the proposition we are focused on. If the bracket is on the right, we are focusing on the succedent, if it is on the left, we are focusing on this particular antecedent. In Section 5 we will further refine these two rules.

$$\frac{\Gamma \xrightarrow{\mathsf{FR}} [C]}{\Gamma \xrightarrow{\mathsf{C}} C}\ \mathsf{FRC} \qquad \frac{A \in \Gamma \quad \Gamma, [A] \xrightarrow{\mathsf{FL}} C}{\Gamma \xrightarrow{\mathsf{C}} C}\ \mathsf{FLC}$$

Note that in the second rule we focus on a copy of A, because in general it may be needed again.

We know the noninvertible connectives on the right are $A \vee B$, $\perp$ and atoms P.

$$\frac{\Gamma \xrightarrow{\mathsf{FR}} [A]}{\Gamma \xrightarrow{\mathsf{FR}} [A \vee B]}\ {\vee}R_1 \qquad \frac{\Gamma \xrightarrow{\mathsf{FR}} [B]}{\Gamma \xrightarrow{\mathsf{FR}} [A \vee B]}\ {\vee}R_2$$

When the succedent is $\perp$, we fail:

$$\text{(no rule for } {\perp}R\text{)}$$

We consider the case for atoms P later.

We expect that for connectives that are invertible, we switch back to the inversion right judgment. This is correct for implication.

$$\frac{\Gamma \xrightarrow{\mathsf{R}} A \supset B}{\Gamma \xrightarrow{\mathsf{FR}} [A \supset B]}\ \mathsf{RFR}$$

On the other hand, we have already seen that conjunction (which is really two different connectives that are not distinguished) can be treated as invertible on the right as well as the left. Symmetrically, we can also treat it as *noninvertible* on both the right and the left because we want to chain together as many choices as possible without losing focus. Similarly, for $\top$.

$$\frac{\Gamma \xrightarrow{\mathsf{FR}} [A] \quad \Gamma \xrightarrow{\mathsf{FR}} [B]}{\Gamma \xrightarrow{\mathsf{FR}} [A \wedge B]}\ {\wedge}R \qquad \frac{}{\Gamma \xrightarrow{\mathsf{FR}} [\top]}\ {\top}R$$

4 Focusing on an Antecedent

Focus is inherited by all subformulas in the premises (whether in the antecedent or succedent). It starts with an implication (postponing the case for atoms):

$$\frac{\Gamma \xrightarrow{\mathsf{FR}} [A] \quad \Gamma, [B] \xrightarrow{\mathsf{FL}} C}{\Gamma, [A \supset B] \xrightarrow{\mathsf{FL}} C}\ {\supset}L$$

As already mentioned above, conjunction can also be considered noninvertible on the left, so it can keep the chaining phase alive. We just choose between the two conjuncts.

$$\frac{\Gamma, [A] \xrightarrow{\mathsf{FL}} C}{\Gamma, [A \wedge B] \xrightarrow{\mathsf{FL}} C} \wedge L_1 \qquad \frac{\Gamma, [B] \xrightarrow{\mathsf{FL}} C}{\Gamma, [A \wedge B] \xrightarrow{\mathsf{FL}} C} \wedge L_2 \qquad \text{(no rule for } \top L\text{)}$$

In the case of disjunction and falsehood, we circle back to the left inversion judgment. We can do that, because we know that C is not right invertible.

$$\frac{\Gamma \,;\, A \vee B \xrightarrow{\mathsf{L}} C}{\Gamma, [A \vee B] \xrightarrow{\mathsf{FL}} C} \mathsf{LFL} \qquad \frac{\Gamma \,;\, \bot \xrightarrow{\mathsf{L}} C}{\Gamma, [\bot] \xrightarrow{\mathsf{FL}} C} \mathsf{LFL}$$

In the case $\bot$ we actually immediately succeed with the $\bot L$ rule in the next step, but it still seems stylistically cleaner to circle back to left inversion for one extra step.

This takes care of all focusing rules except those for atoms.

5 Focusing on Atoms

Atoms, as often, seem to be a special case. We can think of them as *negative* (like implication), in which case we should be able to make progress if we are focused on them on the left. Or we can think of them as *positive* (like disjunction), in which case we should be able to make progress if we are focused on them on the right.

Interestingly, we can also mix them: we can declare some atoms as negative and some atoms as positive, as long as all occurrence of an atom are treated consistently. We write P^+ for positive atoms and P^- for negative atoms.

Negative atoms. We can use the identity rule exactly when we are left focused on a negative atom and it matches the succedent.

$$\frac{}{\Gamma, [P^-] \xrightarrow{\mathsf{FL}} P^-} \mathsf{id}^- \qquad \begin{array}{c}\text{(no rule for } P^- \neq Q^-\text{)} \\ \Gamma, [P^-] \xrightarrow{\mathsf{FL}} Q^-\end{array}$$

We see that negative atoms have to be allowed to appear in the succedent, so we lose focus if we see it. Because P^- is not invertible, we circle back immediately to a choice sequent.

$$\frac{\Gamma \xrightarrow{\mathsf{C}} P^-}{\Gamma \xrightarrow{\mathsf{FR}} [P^-]} \mathsf{CFR}$$

In order to avoid an obvious loop, we can focus on the right only if the succedent is positive, that is, a disjunction, falsehood, or positive atom but not a negative atom.

$$\frac{\Gamma \xrightarrow{\mathsf{FR}} [C^+]}{\Gamma \xrightarrow{\mathsf{C}} C^+} \mathsf{FRC}$$

where

$$\text{Positive propositions} \quad C^+ \quad ::= \quad A \vee B \mid \bot \mid P^+ \;[\;\mid A \wedge B\;]$$

Because conjunction can be both right and left invertible, it may be permitted as positive.

Positive atoms. We can use the identity exactly when we are right focused on a positive atom and it matches an antecedent.

$$\frac{}{\Gamma, P^+ \xrightarrow{\mathsf{FR}} [P^+]}\ \mathsf{id}^+ \qquad \begin{array}{c}\text{(no rule if } P^+ \notin \Gamma) \\ \Gamma \xrightarrow{\mathsf{FR}} [P^+]\end{array}$$

We see that positive atoms have to be allowed among the antecedents, so we lose focus if we see it. Because P^+ is not left invertible, we will immediately circle back to a choice sequent.

$$\frac{\Gamma, P^+ \xrightarrow{\mathsf{C}} C}{\Gamma, [P^+] \xrightarrow{\mathsf{FL}} C}\ \mathsf{CFL}$$

Here, too, we should avoid the obvious cycle and can left focus on a proposition only if it is negative:

$$\frac{A^- \in \Gamma \quad \Gamma, [A^-] \xrightarrow{\mathsf{FL}} C}{\Gamma \xrightarrow{\mathsf{C}} C}\ \mathsf{FLC}$$

where

$$\text{Negative propositions} \quad A^+ \quad ::= \quad A \supset B \mid P^- \;[\;\mid A \wedge B\;]$$

Because conjunction con be considered both left and right invertible, it may be permitted as negative.

6 Backward Chaining

What are the consequences of the polarity assignment to the atoms? In general a choice sequent has the form

$$\Gamma \xrightarrow{\mathsf{C}} C$$

where

$$\begin{array}{lcl} \Gamma & ::= & A \supset B \mid P \\ C & ::= & A \vee B \mid \bot \mid P \\ P & ::= & P^+ \mid P^- \end{array}$$

To examine the consequence of polarity, let's consider the following sequent:

$$P, P \supset Q, Q \supset R \longrightarrow R$$

First, let's make all atoms negative. Then it becomes the following choice sequent

$$P^-, P^- \supset Q^-, Q^- \supset R^- \xrightarrow{\mathsf{C}} R^-$$

In this case we cannot focus on the right, because R^- is not positive. We can try to focus on P^-, or $P^- \supset Q^-$, but both of these will fail very quickly. For example:

$$
\dfrac{
\dfrac{
\dfrac{\vdots}{P^-, P^- \supset Q^-, Q^- \supset R^- \xrightarrow{\mathsf{C}} P^-}
}{P^-, P^- \supset Q^-, Q^- \supset R^- \xrightarrow{\mathsf{FR}} [P^-]}\ \mathsf{CFR}
\qquad
\dfrac{\text{fails, since } Q^- \neq R^-}{P^-, P^- \supset Q^-, Q^- \supset R^-, [Q^-] \xrightarrow{\mathsf{FL}} R^-}
}{
\dfrac{P^-, P^- \supset Q^-, Q^- \supset R^-, [P^- \supset Q^-] \xrightarrow{\mathsf{FL}} R^-}{P^-, P^- \supset Q^-, Q^- \supset R^- \xrightarrow{\mathsf{C}} R^-}\ \mathsf{FLC}
}\ \supset L
$$

So we can only successfully focus on $Q^- \supset R^-$.

$$
\dfrac{
\dfrac{
\dfrac{\vdots}{P^-, P^- \supset Q^-, Q^- \supset R^- \xrightarrow{\mathsf{C}} Q^-}
}{P^-, P^- \supset Q^-, Q^- \supset R^- \xrightarrow{\mathsf{FR}} [Q^-]}\ \mathsf{CFR}
\qquad
\dfrac{}{P^-, P^- \supset Q^-, Q^- \supset R^-, [R^-] \xrightarrow{\mathsf{FL}} R^-}\ \mathsf{id}^-
}{
\dfrac{P^-, P^- \supset Q^-, Q^- \supset R^-, [Q^- \supset R^-] \xrightarrow{\mathsf{FL}} R^-}{P^-, P^- \supset Q^-, Q^- \supset R^- \xrightarrow{\mathsf{C}} R^-}\ \mathsf{FLC}
}\ \supset L
$$

In the open subgoal we can only focus on $P^- \supset Q^-$, leading to the subgoal of proving P^-, and for that we can only focus on P^- on the left. This means that in the focusing calculus there is only a single derivation of

$$P^-, P^- \supset Q^-, Q^- \supset R^- \xrightarrow{\mathsf{C}} R^-$$

We call this search behavior *backward chaining* because we go from proving R^- to proving Q^- and then P^-.

7 Forward Chaining

Now if we make all of the atoms positive, as in

$$P^+, P^+ \supset Q^+, Q^+ \supset R^+ \xrightarrow{\mathsf{C}} R^+$$

then focusing on $Q^+ \supset R^+$ will fail because Q^+ is not already among the antecedents. You are encouraged to play through the rules to confirm that. Similarly, we cannot focus on R^+ on the right, because R^+ is not already among the antecedents and we fail immediately.

The only possibility is to focus on $P^+ \supset Q^+$:

$$
\dfrac{\dfrac{}{P^+, P^+ \supset Q^+, Q^+ \supset R^+ \xrightarrow{\mathsf{FR}} [P^+]}\,\mathsf{id}^+ \qquad \dfrac{\begin{array}{c}\vdots \\ P^+, P^+ \supset Q^+, Q^+ \supset R^+, Q^+ \xrightarrow{\mathsf{C}} R^+\end{array}}{P^+, P^+ \supset Q^+, Q^+ \supset R^+, [Q^+] \xrightarrow{\mathsf{FL}} R^+}\,\mathsf{CFL}}{\dfrac{P^+, P^+ \supset Q^+, Q^+ \supset R^+, [P^+ \supset Q^+] \xrightarrow{\mathsf{FL}} R^+}{P^+, P^+ \supset Q^+, Q^+ \supset R^+ \xrightarrow{\mathsf{C}} R^+}\,\mathsf{FLC}}\,{\supset}L
$$

We see we have added Q^+ to our antecedents. Now we can focus on $Q^+ \supset R^+$, adding R^+ to the antecedents and then we can focus on R^+ on the right and succeed. We could also again focus in $P^+ \supset Q^+$, but it would only add more copy of Q^+ to our antecedents so we could fail this branch due to loop checking. If we do, then there is also just a single derivation of the given sequent.

We call this *forward chaining* because we go from P^+ to Q^+ to R^+, using the implications in the forward direction.

As we will see in the next lecture, forward and backward chaining are crucial building blocks for designing programming languages where computation is proof construction.

8 Summary

We summarize the rules and invariants of the focusing calculus.

Propositions.

$$
\begin{array}{lcl}
\Gamma & ::= & A_1 \supset A_2 \mid P \mid \cdot \mid \Gamma_1, \Gamma_2 \\
\Omega & ::= & \epsilon \mid A \cdot \Omega \\
C & ::= & A \vee B \mid \bot \mid P \\
A^+ & ::= & A \vee B \mid \bot \mid P^+ \; [\; \mid A \wedge B \;] \\
A^- & ::= & A \supset B \mid P^- \; [\; \mid A \wedge B \;] \\
P & ::= & P^- \mid P^+
\end{array}
$$

Judgments.

Right inversion	$\Gamma \,;\, \Omega \xrightarrow{\mathsf{R}} A$
Left inversion	$\Gamma \,;\, \Omega \xrightarrow{\mathsf{L}} C$
Choice	$\Gamma \xrightarrow{\mathsf{C}} C$
Right focus	$\Gamma \xrightarrow{\mathsf{FR}} [C]$
Left focus	$\Gamma, [A] \xrightarrow{\mathsf{FL}} C$

Rules.

Right Inversion.

$$\frac{\Gamma\,;\Omega \xrightarrow{\mathsf{R}} A \quad \Gamma\,;\Omega \xrightarrow{\mathsf{R}} B}{\Gamma\,;\Omega \xrightarrow{\mathsf{R}} A \wedge B}\,{\wedge}R \qquad \frac{\Gamma\,;A\cdot\Omega \xrightarrow{\mathsf{R}} B}{\Gamma\,;\Omega \xrightarrow{\mathsf{R}} A \supset B}\,{\supset}R \qquad \frac{}{\Gamma\,;\Omega \xrightarrow{\mathsf{R}} \top}\,{\top}R$$

$$\frac{\Gamma\,;\Omega \xrightarrow{\mathsf{L}} A \vee B}{\Gamma\,;\Omega \xrightarrow{\mathsf{R}} A \vee B}\,\mathsf{LR} \qquad \frac{\Gamma\,;\Omega \xrightarrow{\mathsf{L}} \bot}{\Gamma\,;\Omega \xrightarrow{\mathsf{R}} \bot}\,\mathsf{LR} \qquad \frac{\Gamma\,;\Omega \xrightarrow{\mathsf{L}} P}{\Gamma\,;\Omega \xrightarrow{\mathsf{R}} P}\,\mathsf{LR}$$

Left Inversion.

$$\frac{\Gamma\,;A\cdot B\cdot\Omega \xrightarrow{\mathsf{L}} C}{\Gamma\,;(A\wedge B)\cdot\Omega \xrightarrow{\mathsf{L}} C}\,{\wedge}L \qquad \frac{\Gamma\,;A\cdot\Omega \xrightarrow{\mathsf{L}} C \quad \Gamma\,;B\cdot\Omega \xrightarrow{\mathsf{L}} C}{\Gamma\,;(A\vee B)\cdot\Omega \xrightarrow{\mathsf{L}} C}\,{\vee}L$$

$$\frac{}{\Gamma\,;\bot\cdot\Omega \xrightarrow{\mathsf{L}} C}\,{\bot}L \qquad \frac{\Gamma\,;\Omega \xrightarrow{\mathsf{L}} C}{\Gamma\,;\top\cdot\Omega \xrightarrow{\mathsf{L}} C}\,{\top}L$$

$$\frac{\Gamma, A\supset B\,;\Omega \xrightarrow{\mathsf{L}} C}{\Gamma\,;(A\supset B)\cdot\Omega \xrightarrow{\mathsf{L}} C}\,\mathsf{LL} \qquad \frac{\Gamma, P\,;\Omega \xrightarrow{\mathsf{L}} C}{\Gamma\,;P\cdot\Omega \xrightarrow{\mathsf{L}} C}\,\mathsf{LL}$$

$$\frac{\Gamma \xrightarrow{\mathsf{C}} C}{\Gamma \xrightarrow{\mathsf{L}} C}\,\mathsf{CL}$$

Choice.

$$\frac{\Gamma \xrightarrow{\mathsf{FR}} [C^+]}{\Gamma \xrightarrow{\mathsf{C}} C^+}\,\mathsf{FRC} \qquad \frac{A^- \in \Gamma \quad \Gamma, [A^-] \xrightarrow{\mathsf{FL}} C}{\Gamma \xrightarrow{\mathsf{C}} C}\,\mathsf{FLC}$$

Right Focus.

$$\frac{}{\Gamma, P^+ \xrightarrow{\mathsf{FR}} [P^+]}\ \mathsf{id}^+ \qquad \begin{array}{c}(\text{no rule if } P^+ \notin \Gamma)\\ \Gamma \xrightarrow{\mathsf{FR}} [P^+]\end{array}$$

$$\frac{\Gamma \xrightarrow{\mathsf{FR}} [A]}{\Gamma \xrightarrow{\mathsf{FR}} [A \vee B]}\ \vee R_1 \qquad \frac{\Gamma \xrightarrow{\mathsf{FR}} [B]}{\Gamma \xrightarrow{\mathsf{FR}} [A \vee B]}\ \vee R_2$$

$$\begin{array}{c}(\text{no rule})\\ \Gamma \xrightarrow{\mathsf{FR}} [\bot]\end{array}$$

$$\frac{\Gamma \xrightarrow{\mathsf{FR}} [A] \quad \Gamma \xrightarrow{\mathsf{FR}} [B]}{\Gamma \xrightarrow{\mathsf{FR}} [A \wedge B]}\ \wedge R \qquad \frac{}{\Gamma \xrightarrow{\mathsf{FR}} [\top]}\ \top R$$

$$\frac{\Gamma \xrightarrow{\mathsf{R}} A \supset B}{\Gamma \xrightarrow{\mathsf{FR}} [A \supset B]}\ \mathsf{RFR}$$

$$\frac{\Gamma \xrightarrow{\mathsf{C}} P^-}{\Gamma \xrightarrow{\mathsf{FR}} [P^-]}\ \mathsf{CFR}$$

Left Focus.

$$\frac{}{\Gamma, [P^-] \xrightarrow{\mathsf{FL}} P^-}\ \mathsf{id}^- \qquad \begin{array}{c}(\text{no rule for } P^- \neq Q^-)\\ \Gamma, [P^-] \xrightarrow{\mathsf{FL}} Q^-\end{array}$$

$$\frac{\Gamma \xrightarrow{\mathsf{FR}} [A] \quad \Gamma, [B] \xrightarrow{\mathsf{FL}} C}{\Gamma, [A \supset B] \xrightarrow{\mathsf{FL}} C}\ \supset L$$

$$\frac{\Gamma, [A] \xrightarrow{\mathsf{FL}} C}{\Gamma, [A \wedge B] \xrightarrow{\mathsf{FL}} C}\ \wedge L_1 \qquad \frac{\Gamma, [B] \xrightarrow{\mathsf{FL}} C}{\Gamma, [A \wedge B] \xrightarrow{\mathsf{FL}} C}\ \wedge L_2 \qquad \begin{array}{c}(\text{no rule})\\ \Gamma, [\top] \xrightarrow{\mathsf{FL}} C\end{array}$$

$$\frac{\Gamma\ ;\ A \vee B \xrightarrow{\mathsf{L}} C}{\Gamma, [A \vee B] \xrightarrow{\mathsf{FL}} C}\ \mathsf{LFL} \qquad \frac{\Gamma\ ;\ \bot \xrightarrow{\mathsf{L}} C}{\Gamma, [\bot] \xrightarrow{\mathsf{FL}} C}\ \mathsf{LFL}$$

$$\frac{\Gamma, P^+ \xrightarrow{\mathsf{C}} C}{\Gamma, [P^+] \xrightarrow{\mathsf{FL}} C}\ \mathsf{CFL}$$

References

Jean-Marc Andreoli. Logic programming with focusing proofs in linear logic. *Journal of Logic and Computation*, 2(3):197–347, 1992.

Robert J. Simmons. Structural focalization. *ACM Transactions on Computational Logic*, 15(3): 21:1–21:33, 2014.