Lecture Notes on The Inverse Method

15-317: Constructive Logic
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1 Introduction

The standard way we have mostly been interpreting inference rules is bottom-up, reading them from the conclusion to the premises. This is natural for goal-directed search and decision procedures for the propositional case like loop checking or the contraction-free sequent calculus. An important example of goal-directed search with a fixed strategy is logic programming (a la Prolog) with Horn clauses. Forward chaining works in the opposite direction, but can we take it as a basis of a general proof search or decision procedure? And can we use it as a foundation of a programming language?

In today's lecture we look at the first question: if we are interested in general theorem proving (and maybe a decision procedure for the propositional case), can we use forward inference and saturation in a practical way? The answer is "yes", which is an insight originally due to Maslov [1964]. We develop this today for the propositional case, it also applies equally well to the predicate calculus. Maslov called his approach the "inverse method" because it works inversely to the way we usually approach proof search. It has nothing to do with with inversion as a general method for reducing nondeterminism in proof search.

2 The Problem with Sequent Calculus

When compared to resolution [Robinson, 1965], a great advantage of the inverse method is that it is a general technique that applies to (cut-free) sequent calculi, while resolution in its original conception was quite specialized to classical logic. But when looking at the rules of the sequent calculus, they are *fundamentally* constructed so as to decompose propositions reading the rules from the conclusion to the premises. This seems entirely unsuitable for bottom-up proof construction since we could indefinitely apply a rule such as

$$\frac{\Gamma \Longrightarrow A \quad \Gamma \Longrightarrow B}{\Gamma \Longrightarrow A \wedge B} \wedge R$$

to infer bigger and bigger conjunctions. Proof search would *never* saturate.

Maslov's exploited Gentzen's insight that in a derivation in the sequent calculus, any proposition appearing in the derivation is a *subformula* of our overall goal sequent. Then we apply a rule such as $\wedge R$ only if the $A \wedge B$ is a subformula of our goal! If any sequent is only allowed to consist of subformulas, then in the propositional case (without quantification) there are only finitely many sequents if we treat antecedents as sets (that is, either implicitly or explicitly apply *contraction*). There, forward search must saturate in the propositional case. In the case of predicate logic, if we have a proposition such as $\forall x.\, P(x)$ then instances such as $P(0)$, $P(\mathsf{s}(0))$, $P(\mathsf{s}(\mathsf{s}(0)))$, etc. may appear in backward search, so the space is still infinite and saturation is not guaranteed. Nevertheless, we can base a semi-decision procedure on forward inference.

3 Specializing Inference Rules

Actually, Maslov's insight is a bit more specific than just that a derivation consists only of subformulas. Each subformula occurrence is destined to only appear either on the left or right of a sequent, further cutting down on the number of rules we can apply. Sometimes, this may be called *signed subformulas*. Since we reserved the positive/negative terminology for intrinsic properties of the connectives, we just call them *left* and *right subformulas*.

In the following example we have marked every subformula with either L or R, depending on whether it might appear on the left or right of the sequent arrow in its derivation. This process is straightforward, we just have to remember that the implication flips the sidedness of it first argument. All other connectives just propagate it to the subformulas unchanged.

$$(A \supset (B \supset C)) \supset ((A \wedge B) \supset C)$$

$$\begin{array}{ccccccccccccccccc} & \mathsf{R} & \mathsf{L} & & \mathsf{R} & \mathsf{L} & \mathsf{L} & & \mathsf{R} & & \mathsf{L} & \mathsf{L} & \mathsf{L} & & \mathsf{R} & \mathsf{R} & \\ (& A & \supset & (& B & \supset & C &)) & \supset & ((& A & \wedge & B &) & \supset & C &) \end{array}$$

In this (and the remaining examples) we write A, B, and C for atomic formulas instead of P, Q and R.

Next, based on these subformula occurrences, we can generate possible instances of the inference rules that might occur in a derivation of this sequent.

First, we see that each of A, B, and C occur marked as both left and right subformulas. As such it is possible they might meet in an initial sequent and we generate:

$$\dfrac{}{A \longrightarrow A}\ R_0(=\mathsf{id}) \qquad \dfrac{}{B \longrightarrow B}\ R_1(=\mathsf{id}) \qquad \dfrac{}{C \longrightarrow C}\ R_2(=\mathsf{id})$$

We write $\Gamma \longrightarrow A$ for the rules in the *forward sequent calculus* and refer to the original calculus as the *backward sequent calculus*. We also introduce new names for the specialized rules. Note that A, B, and C do *not* stand here for arbitrary propositions, but the specific A, B, and C in our goal $(A \supset (B \supset C)) \supset ((A \wedge B) \supset C)$. We have named the new rules but also indicated which more general rules they are specific instances of.

One difference we can already notice here is that in the (backward) sequent calculus we write $\Gamma, A \Longrightarrow A$ and here we write $A \longrightarrow A$. That's because we want to have just the three axioms rather than possibly exponentially many ones for different version of Γ. More generally, in backward sequent calculus we read the sequent

$\Gamma \Longrightarrow A$	We may use Γ in the derivation of A
$\Gamma \longrightarrow A$	We have used Γ in the derivation of A

Then a forward rule for conjunction would be

$$\frac{\Gamma_1 \longrightarrow A \quad \Gamma_2 \longrightarrow B}{\Gamma_1, \Gamma_2 \longrightarrow A \wedge B} \wedge R$$

where we may also immediately contract propositions that occur in both Γ_1 and Γ_2.

Now back to our example.

$$\begin{array}{ccccccccccccccccc} & \mathsf{R} & \mathsf{L} & & \mathsf{R} & \mathsf{L} & \mathsf{L} & & \mathsf{R} & & \mathsf{L} & \mathsf{L} & \mathsf{L} & & \mathsf{R} & \mathsf{R} & \\ (& A & \supset & (& B & \supset & C &)) & \supset & ((& A & \wedge & B &) & \supset & C &) \end{array}$$

Working from the inside outward, we see a left occurrence of $B \supset C$, so we generate:

$$\frac{? \longrightarrow B \quad C \longrightarrow ?}{?, B \supset C \longrightarrow ?} R_3(= \supset L)?$$

We have left question marks where in the original sequent calculus we had Γ and a succedent. So as to avoid a clash of notation, we write γ for an arbitrary succedent and fill in the question marks as indicated above: if the derivation of each premise requires some antecedents we have to combine them in the derivation of the conclusion.

$$\frac{\Gamma_1 \longrightarrow B \quad \Gamma_2, C \longrightarrow \gamma}{\Gamma_1, \Gamma_2, B \supset C \longrightarrow \gamma} R_3(= \supset L)$$

Next, we move on to $A \supset (B \supset C)$, which is also a left subformula.

$$\frac{\Gamma_1 \longrightarrow A \quad \Gamma_2, B \supset C \longrightarrow \gamma}{\Gamma_1, \Gamma_2, A \supset (B \supset C) \longrightarrow \gamma} R_4(= \supset L)$$

This leaves $A \wedge B$, which is once again a left subformula. It gives us two new rules:

$$\frac{\Gamma, A \longrightarrow \gamma}{\Gamma, A \wedge B \longrightarrow \gamma} R_5(= \wedge L_1) \qquad \frac{\Gamma, B \longrightarrow \gamma}{\Gamma, A \wedge B \longrightarrow \gamma} R_6(= \wedge L_2)$$

Next is the right subformula $(A \wedge B) \supset C$. At first this looks straightforward:

$$\frac{\Gamma, A \wedge B \longrightarrow C}{\Gamma \longrightarrow (A \wedge B) \supset C} R_7(= \supset R)$$

and the same for the whole goal proposition, which is also a right (sub)formula.

$$\frac{\Gamma, A \supset (B \supset C) \longrightarrow (A \wedge B) \supset C}{\Gamma \longrightarrow (A \supset (B \supset C)) \supset ((A \wedge B) \supset C)} R_8(= \supset R)$$

It turns out that in general these last two rules are not sufficient, but they are sufficient in this example so let's return to this question in the next section after we see forward reasoning in action.

We now take rules R_0 through R_8, discarding all other rules. Essentially, we have specialized the rules of the sequent calculus to possible left and right subformulas that could appear in a derivation of our proposed theorem. Here is a summary of the rules:

$$\frac{}{A \longrightarrow A}\ R_0(=\mathsf{id}) \qquad \frac{}{B \longrightarrow B}\ R_1(=\mathsf{id}) \qquad \frac{}{C \longrightarrow C}\ R_2(=\mathsf{id})$$

$$\frac{\Gamma_1 \longrightarrow B \quad \Gamma_2, C \longrightarrow \gamma}{\Gamma_1, \Gamma_2, B \supset C \longrightarrow \gamma}\ R_3(= {\supset}L) \qquad \frac{\Gamma_1 \longrightarrow A \quad \Gamma_2, B \supset C \longrightarrow \gamma}{\Gamma_1, \Gamma_2, A \supset (B \supset C) \longrightarrow \gamma}\ R_4(= {\supset}L)$$

$$\frac{\Gamma, A \longrightarrow \gamma}{\Gamma, A \wedge B \longrightarrow \gamma}\ R_5(= \wedge L_1) \qquad \frac{\Gamma, B \longrightarrow \gamma}{\Gamma, A \wedge B \longrightarrow \gamma}\ R_6(= \wedge L_2)$$

$$\frac{\Gamma, A \wedge B \longrightarrow C}{\Gamma \longrightarrow (A \wedge B) \supset C}\ R_7(= {\supset}R)$$

$$\frac{\Gamma, A \supset (B \supset C) \longrightarrow (A \wedge B) \supset C}{\Gamma \longrightarrow (A \supset (B \supset C)) \supset ((A \wedge B) \supset C)}\ R_8(= {\supset}R)$$

Now we can apply the rules in the forward direction, starting with the identity sequents, until we have reached saturation. Here is a first round of inferences, applying all applicable R_3 through R_8 to the initial sequents.

$$\begin{array}{lll}
(1) & A \longrightarrow A & (R_0) \\
(2) & B \longrightarrow B & (R_1) \\
(3) & C \longrightarrow C & (R_2) \\ \hline
(4) & B, B \supset C \longrightarrow C & (R_3\, 2\, 3) \\
(5) & A \wedge B \longrightarrow A & (R_5\, 1) \\
(6) & A \wedge B \longrightarrow B & (R_6\, 2)
\end{array}$$

At this point we can make a second round of inferences. Of course, the first round of inferences still apply, but they give us sequents we already know.

$$\begin{array}{rll}
(7) & A, B, A \supset (B \supset C) \longrightarrow C & (R_4\, 1\, 4) \\ \hline
(8) & A \wedge B, B, A \supset (B \supset C) \longrightarrow C & (R_5\, 7) \\
(9) & A, A \wedge B, A \supset (B \supset C) \longrightarrow C & (R_6\, 7) \\ \hline
(10) & B, A \supset (B \supset C) \longrightarrow (A \wedge B) \supset C & (R_7\, 8) \\
(11) & A, A \supset (B \supset C) \longrightarrow (A \wedge B) \supset C & (R_7\, 9) \\
(12) & A \wedge B, A \wedge B, A \supset (B \supset C) \longrightarrow C & (R_6\, 9) \text{ or } (R_5\, 10)
\end{array}$$

We see that (10) and (11) are essentially dead ends and won't help us, but (12) (which can be deduced two ways) is interesting. We usually implicitly contract that two copies of any formula (here $A \wedge B$), which gives us

$$(12') \quad A \wedge B, A \supset (B \supset C) \longrightarrow C \quad \text{(contract (12))}$$

And this point R_7 followed by R_8 can complete the derivation.

$$\frac{(13) \quad A \supset (B \supset C) \longrightarrow (A \wedge B) \supset C \qquad\qquad (R_7\, 12')}{(14) \quad \longrightarrow (A \supset (B \supset C)) \supset ((A \wedge B) \supset C) \quad (R_8\, 13)}$$

This may also be the fully saturated database if we always treat the antecedents as sets, that is, fully contract them.

4 Soundness and Completeness of the Forward Sequent Calculus

As mentioned above, our forward sequent calculus isn't quite correct. In order to debug it, we state its soundness and completness theorem and see where a problem might arise. For these theorems, we do not yet restrict the rules to left/right subformulas, which we take as a second step. The rules for the forward sequent calculus are collected in Figure 1. For now, we consider propositions without falsehood, so that $\bot L$ drops out and the succedents γ are always a singleton.

Theorem 1 (Soundness of Forward Sequent Calculus) *If* $\Gamma \longrightarrow A$ *then* $\Gamma \Longrightarrow A$*.*

Proof: By rule induction on the structure of the given derivation. There are no surprises. To make the backwards rule applicable we have to routinely apply weakening. □

The corresponding conjecture, namely that $\Gamma \longrightarrow A$ if $\Gamma \Longrightarrow A$ fails. Already the identity rule shows the problem: we have $B, A \Longrightarrow A$ but not $B, A \longrightarrow A$ (only $A \longrightarrow A$). To repair this we remember that $\Gamma \longrightarrow A$ only records the antecedents that were needed in a particular proof of A, while there may be unused antecedents in $\Gamma \Longrightarrow A$. To bridge this gap we generalize our theorem to:

Theorem 2 (Completeness of Forward Sequent Calculus) *If* $\Gamma \Longrightarrow A$ *then* $\Gamma' \longrightarrow A$ *for some* $\Gamma' \subseteq \Gamma$*.*

Proof: By rule induction on the structure of the given derivation $\mathcal{D}$. We show a couple of cases.

Case:

$$\mathcal{D} = \frac{}{\Gamma, P \Longrightarrow P}\ \mathsf{id}$$

Then

$$\frac{}{P \longrightarrow P}\ \mathsf{id}$$

and $\{P\} \subseteq (\Gamma, P)$

Case:

$$\mathcal{D} = \frac{\begin{array}{c}\mathcal{D}_1 \\ \Gamma, B \supset C \Longrightarrow B\end{array} \quad \begin{array}{c}\mathcal{D}_2 \\ \Gamma, B \supset C, C \Longrightarrow A\end{array}}{\Gamma, B \supset C \Longrightarrow A}\ {\supset}L$$

Then

$$\begin{array}{c} \mathsf{IH}(\mathcal{D}_2) \\ \Gamma_2 \longrightarrow A \end{array}$$

for some $\Gamma_2 \subseteq (\Gamma, B \supset C, C)$. We distinguish two subcases.

Subcase: $C \notin \Gamma_2$. Then $\Gamma_2 \subseteq (\Gamma, B \supset C)$ and $\mathsf{IH}(\mathcal{D}_2)$ satisfies the desired property.

Subcase: $C \in \Gamma_2$ so $\Gamma_2 = (\Gamma_2', C)$. Then we construct

$$\dfrac{\begin{array}{c} \mathsf{IH}(\mathcal{D}_1) \\ \Gamma_1 \longrightarrow B \end{array} \quad \begin{array}{c} \mathsf{IH}(\mathcal{D}_2) \\ \Gamma_2', C \longrightarrow A \end{array}}{\Gamma_1, \Gamma_2', B \supset C \longrightarrow A} \supset\! L$$

where $\Gamma_1 \subseteq (\Gamma, B \supset C)$ and $\Gamma_2' \subseteq (\Gamma, B \supset C)$. Therefore $(\Gamma_1, \Gamma_2', B \supset C) \subseteq (\Gamma, B \supset C)$ and the constructed derivation satisfies the desired property. Note that by reasoning about sets we implicitly apply contraction.

Case:

$$\mathcal{D} = \dfrac{\begin{array}{c} \mathcal{D}' \\ \Gamma, B \Longrightarrow C \end{array}}{\Gamma \Longrightarrow B \supset C} \supset\! R$$

By the induction hypothesis we get

$$\begin{array}{c} \mathsf{IH}(\mathcal{D}') \\ \Gamma' \longrightarrow C \end{array}$$

for some $\Gamma' \subseteq \Gamma$. Again, we distinguish two subcases.

Subcase: $B \in \Gamma'$ so $\Gamma' = (\Gamma'', B)$. Then we construct

$$\dfrac{\begin{array}{c} \mathsf{IH}(\mathcal{D}') \\ \Gamma'', B \longrightarrow C \end{array}}{\Gamma'' \longrightarrow B \supset C} \supset\! R$$

Subcase: $B \notin \Gamma'$. At this point we are stuck, because we cannot deduce $\Gamma' \longrightarrow B \supset C$. So we need another rule in the forward sequent calculus that does *not* require the proposition B to be used in the derivation of C. With such a rule we can then construct:

$$\dfrac{\begin{array}{c} \mathsf{IH}(\mathcal{D}') \\ \Gamma' \longrightarrow C \end{array}}{\Gamma' \longrightarrow B \supset C} \supset\! R'$$

$\square$

So the conclusion is that we should have two forms of the $\supset R$ rule in the forward sequent calculus. There was some concern during lecture that we might generate an inordinate number of rules, but it turns out there are at most two rules for any left/right subformula. So there are $O(n)$ rules altogether. The bottleneck overall, then, is not the number of rules, but the number of sequents that may be derived before either saturation is reached or the goal sequent is deduced.

5 Subsumption

Let's try the rules from the previous section in the example

$$\begin{array}{ccccccc} \mathsf{L} & \mathsf{R} & & \mathsf{L} & \mathsf{R} & \mathsf{R} & \\ A & \supset & (& B & \supset & A &) \end{array}$$

Only A occurs as both a left and right subformula, so we only have one specialized identity rule. In addition, we have four rules for implication.

$$\frac{}{A \longrightarrow A}\ R_0$$

$$\frac{\Gamma, B \longrightarrow A}{\Gamma \longrightarrow B \supset A}\ R_1 \qquad \frac{\Gamma \longrightarrow A}{\Gamma \longrightarrow B \supset A}\ R_2$$

$$\frac{\Gamma, A \longrightarrow B \supset A}{\Gamma \longrightarrow A \supset (B \supset A)}\ R_3 \qquad \frac{\Gamma \longrightarrow B \supset A}{\Gamma \longrightarrow A \supset (B \supset A)}\ R_4$$

We deduce

$$\begin{array}{lll} (1) & A \longrightarrow A & (R_0) \\ \hline (2) & A \longrightarrow (B \supset A) & (R_2\ 1) \\ \hline (3) & \cdot \longrightarrow A \supset (B \supset A) & (R_3\ 2) \\ (4) & A \longrightarrow A \supset (B \supset A) & (R_3\ 2) \end{array}$$

We see that at saturation, (3) is what we wanted to prove. We also see that (3) is strictly stronger than (4) because we can obtain (4) from (3) by weakening. We say that (3) *subsumes* (4). We can always delete a subsumed sequent from our database of sequents because any inference that could be done with the subsumed sequent can also be done with the subsuming one, leading to equal or stronger results.

6 Inversion

We can use inversion as well as general focusing to streamline the forward inference system and improve the process of generating specialized rules. This is the basis of the Imogen theorem prover for intuitionistic propositional calculus McLaughlin and Pfenning [2008, 2009]. We show here only a small example that doesn't require any additional metatheoretic proof for its correctness. When given a goal sequent we first apply inversion until we reach a choice sequent. Then we proceed as before.

Let's try this on

$$\Longrightarrow ((A \vee B) \supset C) \supset (A \supset C) \wedge (B \supset C)$$

Applying inversion multiple times gives us the goal sequents

$$(A \vee B) \supset C, A \Longrightarrow C$$
$$(A \vee B) \supset C, B \Longrightarrow C$$

We now analyze the first one; the second one is symmetric.

$$\begin{array}{ccccccccccc} & \mathsf{R} & & \mathsf{R} & & \mathsf{L} & \mathsf{L} & & \mathsf{L} & & \mathsf{R} \\ (& A & \vee & B &) \supset & & C & , & A & \Longrightarrow & C \end{array}$$

We see that only A and C occur as both left and right subformulas, so an identity sequent $B \Longrightarrow B$ could not occur in a backwards derivation. So we generate only

$$\frac{}{A \longrightarrow A}\,R_0 \qquad \frac{}{C \longrightarrow C}\,R_1$$

Working again from the insider out, we have $A \vee B$ as a right subformula so we generate

$$\frac{\Gamma \longrightarrow A}{\Gamma \longrightarrow A \vee B}\,R_2 \qquad \frac{\Gamma \longrightarrow B}{\Gamma \longrightarrow A \vee B}\,R_3$$

Finally, the implication yields

$$\frac{\Gamma_1 \longrightarrow A \vee B \quad \Gamma_2, C \longrightarrow \gamma}{\Gamma_1, \Gamma_2, (A \vee B) \supset C \longrightarrow \gamma}\,R_4$$

Now we can start forward inference. Since we may actually derive a stronger sequent that the goal sequent, according to Theorem 4 we have to derive

$$\Gamma \longrightarrow C \quad \text{such that} \quad \Gamma \subseteq ((A \vee B) \supset C, A)$$

We saturate rather quickly:

$$\begin{array}{lll} (1) & A \longrightarrow A & (R_0) \\ (2) & C \longrightarrow C & (R_1) \\ \hline (3) & A \longrightarrow A \vee B & (R_2\,1) \\ \hline (4) & A, (A \vee B) \supset C \longrightarrow C & (R_3\,3) \end{array}$$

Here, (4) is just what we were trying to derive and we succeed.

Inversion was significant in cutting the overhead of building and applying the rules.

7 Naming Subformulas

The way we have written the rules so far requires a considerable amount of checking equality between propositions. In order to avoid this significant overhead we can assign unique names to left/right subformulas and generate rules using these names.

We illustrate this using the reverse of the previous example:

$$((A \supset C) \wedge (B \supset C)) \supset ((A \vee B) \supset C)$$

First, we apply inversion to arrive at

$$A \supset C, B \supset C, A \vee B \Longrightarrow C$$

then we name subformulas (that don't already have names) and label them.

$$\begin{array}{rcccc} & & \mathsf{R} & \mathsf{L} & \mathsf{L} \\ L_1 & = & A & \supset & C \\ & & \mathsf{R} & \mathsf{L} & \mathsf{L} \\ L_2 & = & B & \supset & C \\ & & \mathsf{L} & \mathsf{L} & \mathsf{L} \\ L_3 & = & A & \vee & B \end{array}$$

All of A, B, and C (as the succedent in the original sequent) appear as left and right subformulas, so we start with

$$\frac{}{A \longrightarrow A}\ R_0 \qquad \frac{}{B \longrightarrow B}\ R_1 \qquad \frac{}{C \longrightarrow C}\ R_2$$

For L_1, L_2, and L_3 we generate the following rules:

$$\frac{\Gamma_1 \longrightarrow A \quad \Gamma_2, C \longrightarrow \gamma}{\Gamma_1, \Gamma_2, L_1 \longrightarrow \gamma}\ R_3 \qquad \frac{\Gamma_1 \longrightarrow B \quad \Gamma_2, C \longrightarrow \gamma}{\Gamma_1, \Gamma_2, L_2 \longrightarrow \gamma}\ R_4$$

$$\frac{\Gamma_1, A \longrightarrow \gamma \quad \Gamma_2, B \longrightarrow \gamma}{\Gamma_1, \Gamma_2, L_3 \longrightarrow \gamma}\ R_5$$

Now rule applications are more easily checked. Our goal sequent is

$$L_1, L_2, L_3 \longrightarrow C$$

We saturate as follows.

$$\begin{array}{lll} (1) & A \longrightarrow A & (R_0) \\ (2) & B \longrightarrow B & (R_1) \\ (3) & C \longrightarrow C & (R_2) \\ \hline (4) & A, L_1 \longrightarrow C & (R_3\ 1\ 3) \\ (5) & B, L_2 \longrightarrow C & (R_4\ 2\ 3) \\ \hline (6) & L_1, L_2, L_3 \longrightarrow C & (R_5\ 4\ 5) \end{array}$$

8 Adding Falsehood[1]

[1] not covered in lecture

The $\bot L$ rule raises a new question. We might think it should be

$$\frac{}{\bot \longrightarrow \gamma}\ \bot L$$

The question is what should γ be? We certainly do not want to allow all right subformulas. Instead, we allow the succedent to be *empty*, leading to the syntax

$$\text{Succedent} \quad \gamma \quad ::= \quad A \mid \cdot$$

Then we have

$$\frac{}{\bot \longrightarrow \cdot}\ \bot L$$

We further extend our notion of subsumption.

$$(\Gamma \longrightarrow \gamma) \leq (\Gamma' \longrightarrow \gamma') \quad \text{if} \quad \Gamma \subseteq \Gamma' \text{ and } \gamma \subseteq \gamma'$$

The soundness and completeness theorems are updated as below, with analogous proofs.

Theorem 3 (Soundness of Forward Sequent Calculus with Falsehhood)

(i) If $\Gamma \longrightarrow A$ then $\Gamma \Longrightarrow A$

(ii) If $\Gamma \longrightarrow \cdot$ then $\Gamma \Longrightarrow C$ for any C

Theorem 4 (Completeness of Forward Sequent Calculus with Falsehoo)
If $\Gamma \Longrightarrow A$ then $\Gamma' \longrightarrow \gamma'$ for some Γ' and γ' with $(\Gamma' \longrightarrow \gamma') \leq (\Gamma \longrightarrow A)$.

You can find the complete set of rules for the forward sequent calculus in Figure 1. The $\vee L$ rule is a bit unusual because either of the two premises could have an empty succedent. So we take the union and make sure it is either empty (both premises have empty succedent) or a singleton (the succedents of the premises agree, or one of them is empty).

9 Summary

We have developed the *inverse method* that works by forward inference starting from identity sequents. This is feasible because we can specialize all inference rules to those on left and right subformulas of our goal sequent.

There are many optimizations and other considerations for predicate calculus, but the inverse method is remarkably robust [Voronkov, 1992, Degtyarev and Voronkov, 2001]. This is because, fundamentally it is based on the subformula property of the (cut-free) sequent calculus.

Backward and forward proof search each have their own strengths and weaknesses. Backward search has to backtrack, and it is difficult to learn from the failure of a given attempt. When loop-checking, then loops can only be formed on a single branch, which limits reuse. Forward search avoids some of these problems, but it might generate many sequents that could not be reached by backward search. So the size of the generated database of sequence and the time to process it all is a limiting factor.

Both approaches benefit from important proof-theoretic properties such as inversion and focusing.

$$\frac{}{P \longrightarrow P}\ \mathsf{id}$$

$$\frac{\Gamma_1 \longrightarrow A \quad \Gamma_2 \longrightarrow B}{\Gamma_1, \Gamma_2 \longrightarrow A \wedge B}\ \wedge R \qquad \frac{\Gamma, A \longrightarrow \gamma}{\Gamma, A \wedge B \longrightarrow \gamma}\ \wedge L_1 \quad \frac{\Gamma, B \longrightarrow \gamma}{\Gamma, A \wedge B \longrightarrow \gamma}\ \wedge L_2$$

$$\frac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \supset B}\ \supset R_1 \quad \frac{\Gamma \longrightarrow B}{\Gamma \longrightarrow A \supset B}\ \supset R_2 \quad \frac{\Gamma, A \longrightarrow \cdot}{\Gamma \longrightarrow A \supset B}\ \supset R_3 \quad \frac{\Gamma_1 \longrightarrow A \quad \Gamma_2, B \longrightarrow \gamma}{\Gamma_1, \Gamma_2, A \supset B \longrightarrow \gamma}\ \supset L$$

$$\frac{\Gamma \longrightarrow A}{\Gamma \longrightarrow A \vee B}\ \vee R_1 \quad \frac{\Gamma \longrightarrow B}{\Gamma \longrightarrow A \vee B}\ \vee R_2 \qquad \frac{\Gamma_1, A \longrightarrow \gamma_1 \quad \Gamma_2, B \longrightarrow \gamma_2}{\Gamma_1, \Gamma_2, A \vee B \longrightarrow \gamma_1, \gamma_2}\ \vee L^*$$

$$\frac{}{\cdot \longrightarrow \top}\ \top R \qquad \frac{}{\bot \longrightarrow \cdot}\ \bot L$$

Figure 1: Forward sequent calculus (antecedents sets, succedents singletons or empty)
(*) γ_1, γ_2 either empty or a singleton

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