Lecture Notes on Linear Logic

15-317: Constructive Logic
Frank Pfenning

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1 Introduction

In previous lectures we saw a computational interpretation of constructive proofs as functional programming as well as a computational interpretation of proof search as logical programming. While state-change can be understood indirectly in both paradigms as well, today's lecture develops a direct handle on a logical account of state change.

In order to allow this we need to generalize the logic to handle state intrinsically, something provided by *linear logic* [Girard, 1987]. We provide an introduction to linear logic as a *sequent calculus*, which generalizes our previous way of specifying truth. In the next lecture we will see that this sequent calculus can be given a computational interpretation in terms of message-passing concurrency.

2 Linear Sequent Calculus

Linear logic has been described as a logic of state or a resource-aware logic. Formally, it arises from complementing the usual notion of logical assumption with so-called *linear assumptions* or *linear hypotheses*. Unlike traditional assumptions which may be used many times in a proof, linear assumptions must be used *exactly once* during a proof. Linear assumptions then become (consumable) *resources* in the course of a proof. Because linear assumptions are consumable, they can represent ephemeral truth about the current state, e.g., of a computation, because what is no longer true can be consumed and is then gone. Facts that become true can be made available as resources.

This generalization of the usual mathematical standpoint may seem slight, but as we will see it is quite expressive. We write

$$A_1\ \mathit{res}, \ldots, A_n\ \mathit{res} \vdash C\ \mathit{true}$$

for a linear hypothetical judgment with resources $A_1, \ldots, A_n$ and goal C. If we can prove this, it means that we can achieve that C is true, given resources A_1 through A_n. Here, all A_i and C are propositions. As usual, we expect a cut rule to be admissible and we consider this at various points in our development today.

The version of linear logic defined by this judgment is called *intuitionistic linear logic* [Girard and Lafont, 1987, Chang et al., 2003], sometimes contrasted with *classical linear logic* in which the sequent calculus has multiple conclusions [Girard, 1987]. It is not quite so straightforward to combine classical linear logic with functional programming based on intuitionstic logic, so we stick with the intuitionistic approach.

Hidden in the judgment are other assumptions, usually abbreviated as Γ, which can be used arbitrarily often (including not at all), and are therefore called the *unrestricted assumptions*. If we need to make them explicit in a rule we will write

$$\Gamma \,;\, \Delta \vdash C \ \mathit{true}$$

where Δ abbreviates the resources. As in our development so far, unrestricted assumption are fixed and are carried through from every conclusion to all premisses.

In the development of the remainder of this lecture we will mostlyomit *res* and *true* because, as usual in the sequent calculus, the position in the sequent determines the judgment uniquely.

The first rule of linear logic is that if we have a resource P we can achieve goal P, where P is an atomic proposition. It will be a consequence of our definitions that this will be true for arbitrary propositions A, but we need it as a rule only for the atomic case, where the structure of the propositions can not be broken down further.

$$\frac{}{P \vdash P}\ \mathrm{id}$$

The fact that linear resources *must be used exactly once* means, for example, that we cannot prove

$$P, Q \vdash P$$

because Q is unused.

3 Connectives of Linear Logic

One of the curious phenomena of linear logic is that the ordinary connectives multiply. This is because the presence of linear assumptions allows us to make distinctions we ordinarily could not. The first example of this kind is conjunction. It turns out that linear logic possesses two forms of conjunction.

Simultaneous Conjunction ($A \otimes B$). A simultaneous (or *multiplicative*) conjunction $A \otimes B$ is true if we can achieve both A and B in the same state. This means we have to subdivide our resources, devoting some of them to achieve A and the others to achieve B.

$$\frac{\Delta_1 \vdash A \quad \Delta_2 \vdash B}{\Delta_1, \Delta_2 \vdash A \otimes B}\ \otimes R$$

The order of linear assumptions is irrelevant, so in Δ_1, Δ_2 the comma denotes the multi-set union. In other words, every occurrence of a proposition in Δ will end up in exactly one of Δ_1 and Δ_2, if we read the rule bottom-up.

As a trivial example, consider the books "A is for Alibi" (Al) and "B is for Burglar" (Bu) and each costs \$10. Then

$$\dfrac{\begin{matrix}\vdots & \vdots \\ \$10 \vdash \mathsf{Al} & \$10 \vdash \mathsf{Bu}\end{matrix}}{\$10, \$10 \vdash \mathsf{Al} \otimes \mathsf{Bu}}\ \otimes R$$

where we are purposely vague at this point about how we express that each of these books costs \$10.

In a linear sequent calculus, the right rules show when we can conclude a proposition. The left rules show how we can use a resource. In this case, the resource $A \otimes B$ means that we have A and B simultaneously, so the left rule reads

$$\dfrac{\Delta, A, B \vdash C}{\Delta, A \otimes B \vdash C}\ \otimes L$$

Essentially, this rule "unbundles" the resources A and B so we have them separately and can use them in separate branches of a proof. For example:

$$\dfrac{\dfrac{\begin{matrix}\vdots & \vdots \\ \$10 \vdash \mathsf{Al} & \$10 \vdash \mathsf{Bu}\end{matrix}}{\$10, \$10 \vdash \mathsf{Al} \otimes \mathsf{Bu}}\ \otimes R}{\$10 \otimes \$10 \vdash \mathsf{Al} \otimes \mathsf{Bu}}\ \otimes L$$

Here we need to apply $\otimes L$ first, because otherwise we couldn't split the resources at the point of of the $\otimes R$ rule.

The sequent calculus analogue of the local reduction from natural deduction is the principal case in the proof of the admissibility of cut. We want to reduce a cut on a proposition A to (zero or more) cuts on strict subformulas of A.

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1 & \mathcal{D}_2 \\ \Delta_1 \vdash A_1 & \Delta_2 \vdash A_2\end{matrix}}{\Delta_1, \Delta_2 \vdash A_1 \otimes A_2}\ \otimes R \quad \dfrac{\begin{matrix}\mathcal{E}' \\ \Delta_3, A_1, A_2 \vdash C\end{matrix}}{\Delta_3, A_1 \otimes A_2 \vdash C}\ \otimes L}{\Delta_1, \Delta_2, \Delta_3 \vdash C}\ \mathsf{cut}_{A_1 \otimes A_2}$$

$$\longrightarrow_R \quad \dfrac{\begin{matrix}\mathcal{D}_2 \\ \Delta_2 \vdash A_2\end{matrix} \quad \dfrac{\begin{matrix}\mathcal{D}_1 & \mathcal{E}' \\ \Delta_1 \vdash A_1 & \Delta_3, A_1, A_2 \vdash C\end{matrix}}{\Delta_1, \Delta_3, A_2 \vdash C}\ \mathsf{cut}_{A_1}}{\Delta_1, \Delta_2, \Delta_3 \vdash C}\ \mathsf{cut}_{A_2}$$

Here we have annotated the cut with the cut formula to make it visually clear that the complexity of the cut goes down.

For the ultimate proof of the admissibility of cut we also need to check the commuting cases, where the cut formula is a side formula of the last inference. This is what is need to extend the local check to a global theorem.

We should also check the analogue of the local expansion, which reduces the identity at a proposition to the identity at subformulas.

$$\dfrac{}{A_1 \otimes A_2 \vdash A_1 \otimes A_2}\,\mathsf{id}_{A_1 \otimes A_2} \quad \longrightarrow_E \quad \dfrac{\dfrac{\dfrac{}{A_1 \vdash A_1}\,\mathsf{id}_{A_1} \quad \dfrac{}{A_2 \vdash A_2}\,\mathsf{id}_{A_2}}{A_1, A_2 \vdash A_1 \otimes A_2}\,{\otimes}R}{A_1 \otimes A_2 \vdash A_1 \otimes A_2}\,{\otimes}L$$

Alternative Conjunction ($A \mathbin{\&} B$). An alternative (or *additive*) conjunction is true if we can achieve both conjuncts, separately, with the current resources. This means if we have a linear assumption $A \mathbin{\&} B$ we have to make a choice: either we use A or we use B, but we cannot use them both since A and B are formed from the same resources in $\& R$.

$$\dfrac{\Delta \vdash A \quad \Delta \vdash B}{\Delta \vdash A \mathbin{\&} B}\,\& R$$

$$\dfrac{\Delta, A \vdash C}{\Delta, A \mathbin{\&} B \vdash C}\,\& L_1 \qquad \dfrac{\Delta, B \vdash C}{\Delta, A \mathbin{\&} B \vdash C}\,\& L_2$$

It looks like the right rule duplicates the assumptions, but this does not violate linearity because in a use of the assumption $A \mathbin{\&} B$ *res* we have to commit to one or the other.

Returning to our example, with \$10 I can buy "A is for Alibi" and I can also by \$10 "B is for Burglar", but I cannot buy them both. I have to make a choice. Therefore, the alternative conjunction is also *external choice* because the provider of the resource has to be prepared for both. So we have (for example)

$$\dfrac{\begin{matrix}\vdots \\ \$10 \vdash \mathsf{Al}\end{matrix} \quad \begin{matrix}\vdots \\ \$10 \vdash \mathsf{Bu}\end{matrix}}{\$10 \vdash \mathsf{Al} \mathbin{\&} \mathsf{Bu}}\,\& R$$

To check for (sequent-style) harmony we can consider the principal cases of the cut.

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1 \\ \Delta' \vdash A_1\end{matrix} \quad \begin{matrix}\mathcal{D}_2 \\ \Delta' \vdash A_2\end{matrix}}{\Delta' \vdash A_1 \mathbin{\&} A_2}\,\& R \quad \dfrac{\begin{matrix}\mathcal{E}_1 \\ \Delta_1, A_1 \vdash C\end{matrix}}{\Delta_1, A_1 \mathbin{\&} A_2 \vdash C}\,\& L_1}{\Delta', \Delta_1 \vdash C}\,\mathsf{cut}_{A_1 \& A_2}$$

$$\longrightarrow_R \quad \dfrac{\begin{matrix}\mathcal{D}_1 \\ \Delta' \vdash A_1\end{matrix} \quad \begin{matrix}\mathcal{E}_1 \\ \Delta_1, A_1 \vdash C\end{matrix}}{\Delta', \Delta_1 \vdash C}\,\mathsf{cut}_{A_1}$$

The other principal case is entirely symmetric. For the identity expansion we have

$$\dfrac{}{A_1 \mathbin{\&} A_2 \vdash A_1 \mathbin{\&} A_2}\,\mathsf{id}_{A_1 \& A_2} \quad \longrightarrow_E \quad \dfrac{\dfrac{\dfrac{}{A_1 \vdash A_1}\,\mathsf{id}_{A_1}}{A_1 \mathbin{\&} A_2 \vdash A_1}\,\& L_1 \quad \dfrac{\dfrac{}{A_2 \vdash A_2}\,\mathsf{id}_{A_2}}{A_1 \mathbin{\&} A_2 \vdash A_2}\,\& L_2}{A_1 \mathbin{\&} A_2 \vdash A_1 \mathbin{\&} A_2}\,\& R$$

Additive Truth ($\top$). We have seen two forms of conjunction, which are distinguished because of their resource behavior. There are also two truth constants, which correspond to nullary conjunctions. The first is *additive truth* $\top$. A proof of it consumes all current resources. As such we can extract no information from its presence as an assumption.

$$\frac{}{\Delta \vdash \top}\ \top R \qquad\qquad \text{(no } \top L \text{ rule)}$$

Consumptive truth is important in applications where there is an aspect of the state we do not care about, because of the stipulation of linear logic that every linear assumption must be used *exactly once*.

For example, if we don't care about burning \$20, we can still by "A is for Alibi" even if we have \$30.

$$\frac{\begin{array}{c}\vdots\\ \$10 \vdash \mathsf{AI}\end{array} \quad \dfrac{}{\$20 \vdash \top}\ \top R}{\$10, \$20 \vdash \mathsf{AI} \otimes \top}\ \otimes R$$

Consumptive truth is the unit of alternative conjunction in that $A \,\&\, \top$ is equivalent to A.

Empty Truth (1). The other form of truth holds only if there are no resources. If we have this as a linear hypothesis we can transform it into the empty set of resources.

$$\frac{}{\cdot \vdash 1}\ 1R \qquad\qquad \frac{\Delta \vdash C}{\Delta, 1 \vdash C}\ 1L$$

Empty truth can be useful to dispose explicitly of specific resources. As an example, consider *affine logic*. In affine logic, we can use all resources *at most once*. We can encode this by making their use *optional* and reason in linear logic. Recasting the example from consumptive truth, if both bills we have may be "burned", we can formula this as

$$\frac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \$10 \vdash \mathsf{AI}\end{array}}{\$10, 1 \vdash \mathsf{AI}}\ 1L}{\$10, \$20 \,\&\, 1 \vdash \mathsf{AI}}\ \&L_2}{\$10 \,\&\, 1, \$20 \,\&\, 1 \vdash \mathsf{AI}}\ \&L_1 \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\cdot \vdash 1}\ 1R}{1 \vdash 1}\ 1L}{1, 1 \vdash 1}\ 1L}{1, \$20 \,\&\, 1 \vdash 1}\ \&L_2}{\$10 \,\&\, 1, \$20 \,\&\, 1 \vdash 1}\ \&L_2}{\$10 \,\&\, 1, \$20 \,\&\, 1 \vdash \mathsf{AI} \,\&\, 1}\ \&R$$

We can easily check the sequent-style harmony of the left and right rule (but don't write it out here)

Linear Implication ($A \multimap B$). A linear implication $A \multimap B$ is true if we can achieve B given resource A.

$$\frac{\Delta, A \vdash B}{\Delta \vdash A \multimap B}\ \multimap R$$

Conversely, if we have $A \multimap B$ as a resource, it means that we could transform the resource A into the resource B. We capture this in the following left rule:

$$\frac{\Delta_1 \vdash A \quad \Delta_2, B \vdash C}{\Delta_1, \Delta_2, A \multimap B \vdash C} \multimap L$$

An assumption $A \multimap B$ therefore represents a means to transition from a state with A to a state with B.

The check for sequent-style harmony is quite similar to $A_1 \otimes A_2$, except the side at which some subformulas appear is different.

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}'\\ \Delta, A_1 \vdash A_2\end{array}}{\Delta' \vdash A_1 \multimap A_2} \multimap R \quad \dfrac{\begin{array}{c}\mathcal{E}_1\\ \Delta_1 \vdash A_1\end{array} \quad \begin{array}{c}\mathcal{E}_2\\ \Delta_2, A_2 \vdash C\end{array}}{\Delta_1, \Delta_2, A_1 \multimap A_2 \vdash C} \multimap L}{\Delta', \Delta_1, \Delta_2 \vdash C} \mathsf{cut}_{A_1 \multimap A_2}$$

$$\longrightarrow_R \quad \dfrac{\dfrac{\begin{array}{c}\mathcal{E}_1\\ \Delta_1 \vdash A_1\end{array} \quad \begin{array}{c}\mathcal{D}'\\ \Delta', A_1 \vdash A_2\end{array}}{\Delta', \Delta_1 \vdash A_2} \mathsf{cut}_{A_1} \quad \begin{array}{c}\mathcal{D}_2\\ \Delta_2, A_2 \vdash C\end{array}}{\Delta', \Delta_1, \Delta_2 \vdash C} \mathsf{cut}_{A_2}$$

We don't write out the straightforward identity expansion.

Continuing our example, we can now express that \$10 will get you either of the two books from before.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\$10 \vdash \$10}\,\mathsf{id} \quad \dfrac{}{\mathsf{Al} \vdash \mathsf{Al}}\,\mathsf{id}}{\$10 \multimap \mathsf{Al}, \$10 \vdash \mathsf{Al}} \multimap L \quad \dfrac{\dfrac{}{\$10 \vdash \$10}\,\mathsf{id} \quad \dfrac{}{\mathsf{Bu} \vdash \mathsf{Bu}}\,\mathsf{id}}{\$10 \multimap \mathsf{Bu}, \$10 \vdash \mathsf{Bu}} \multimap L}{\$10 \multimap \mathsf{Al}, \$10 \multimap \mathsf{Bu}, \$10, \$10 \vdash \mathsf{Al} \otimes \mathsf{Bu}} \otimes R}{\$10 \multimap \mathsf{Al}, \$10 \multimap \mathsf{Bu}, \$10 \otimes \$10 \vdash \mathsf{Al} \otimes \mathsf{Bu}} \otimes L$$

There is something slightly odd here: it seems like there must be exactly one copy of the book, because the antecedent $\$10 \multimap \mathsf{Al}$ must be used exactly once. If there is a large supply of them, we need to use $!A$ from Section 4.

Disjunction ($A \oplus B$). The familiar conjunction from logic was split into two connectives in linear logic: the simultaneous and the alternative conjunction. Disjunction does not split the same way unless we introduce an explicit judgment for falsehood (which we will not pursue). The goal $A \oplus B$ can be achieved if we can achieve either A or B.

$$\frac{\Delta \vdash A}{\Delta \vdash A \oplus B} \oplus R_1 \qquad\qquad \frac{\Delta \vdash B}{\Delta \vdash A \oplus B} \oplus R_2$$

Disjunction is also called *internal choice* because the provider of the resource can decide whether to prove A or B.

Conversely, if we are given $A \oplus B$ as a resource, we do not know which of the two is true, so we have to account for both eventualities. Our proof splits into cases, and we have to show that we can achieve our goal in either case.

$$\frac{\Delta, A \vdash C \quad \Delta, B \vdash C}{\Delta, A \oplus B \vdash C}\ \oplus L$$

Again, it might appear as if linearity is violated due to the duplication of Δ and even C. However, only one of A or B will be true, so only one part of the plan represented by the two premises really applies, preserving linearity.

The cut reduction and identity expansion introduce no new ideas and therefore omitted here.

Falsehood (0). There is no way to prove falsehood 0, so there is no right rule for it. On the other hand, if we have 0 as an assumption we know we are really in an impossible state so we are permitted to succeed.

$$\text{(no } 0R \text{ rule)} \qquad\qquad \frac{}{\Delta, 0 \vdash C}\ 0L$$

We can also formally think of falsehood as a disjunction between zero alternatives and arrive at the same rule.

4 Validity and Reusable Resources

The system of linear logic so far is much less powerful than intuitionistic logic. However, we would like to see it as a *generalization* of intuitionistic logic, adding expressive power. Girard's key idea to achieve that was that if we can prove A without using any resources ($\cdot \vdash A$), then we can use A as many times as we want! Whenever we need an A, we just reproduce another copy since it does not require any resources to do so.

That by itself, though, it not quite sufficient. Those who did Miniproject 2 however will recognize this as the judgment of the validity of A. Rather than using the $\Gamma^\dagger$ notation from the miniproject, we use two different kind of antecedents: A *valid* (and therefore is a renewable resource) and A *res* (which must be used exactly once as before).

Therefore sequents now have the form

$$B_1\ \mathit{valid}, \ldots, B_k\ \mathit{valid}\ ;\ A_1\ \mathit{res}, \ldots A_n\ \mathit{res} \vdash A\ \mathit{true}$$

We can *internalize* validity as the proposition $!A$ pronounced either "*of course* A" or "*bang* A". The reasoning from miniproject 2 yields the following three rules.

$$\frac{\Gamma\ ;\ \cdot \vdash A}{\Gamma\ ;\ \cdot \vdash\ !A}\ !R \qquad \frac{\Gamma, A\ ;\ \Delta \vdash C}{\Gamma\ ;\ \Delta, !A \vdash C}\ !L$$

$$\frac{\Gamma, A\ ;\ \Delta, A \vdash C}{\Gamma, A\ ;\ \Delta \vdash C}\ \mathsf{VR}$$

The succedent will always be of the form C *true*, and the VR rule transitions from A *valid* to A *res*, keeping an extra copy because valid resources are renewable.

We now need two forms of cut, namely:

$$\dfrac{\Gamma \,;\, \Delta_1 \vdash A \quad \Gamma \,;\, \Delta_2, A \vdash C}{\Gamma \,;\, \Delta_1, \Delta_2 \vdash C}\ \mathsf{cut} \qquad\qquad \dfrac{\Gamma \,;\, \cdot \vdash A \quad \Gamma, A \,;\, \Delta \vdash C}{\Gamma \,;\, \Delta \vdash C}\ \mathsf{cut!}$$

The principal case of $\mathsf{cut}_{!A}$ for $!A$ appeals to cut! on A. And cut! is propagated throughout the derivation and appeals to cut on A, possibly multiple times. That's because the only rule applicable to antecedents A *valid* is the VR rule. Overall, the proof of cut elimination still goes through if we order $\mathsf{cut}_A < \mathsf{cut!}_A$ and $\mathsf{cut!}_A < \mathsf{cut}_B$ if $A < B$.

Returning to our example, if we want to say that there is a (practically) unlimited supply of the books, we can precede the implications with a "!".

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\ldots\,;\,\$10 \vdash \$10}\ \mathsf{id} \quad \dfrac{}{\ldots\,;\,\mathsf{Al} \vdash \mathsf{Al}}\ \mathsf{id}}{\ldots\,;\,\$10 \multimap \mathsf{Al}, \$10 \vdash \mathsf{Al}}\ {\multimap}L}{\$10 \multimap \mathsf{Al}, \$10 \multimap \mathsf{Bu}\,;\,\$10, \$10 \vdash \mathsf{Al}}\ \mathsf{VR} \quad \dfrac{\dfrac{\dfrac{}{\ldots\,;\,\$10 \vdash \$10}\ \mathsf{id} \quad \dfrac{}{\ldots\,;\,\mathsf{Bu} \vdash \mathsf{Bu}}\ \mathsf{id}}{\ldots\,;\,\$10 \multimap \mathsf{Bu}, \$10 \vdash \mathsf{Bu}}\ {\multimap}L}{\$10 \multimap \mathsf{Al}, \$10 \multimap \mathsf{Bu}\,;\,\$10, \$10 \vdash \mathsf{Bu}}\ \mathsf{VR}}{\$10 \multimap \mathsf{Al}, \$10 \multimap \mathsf{Bu}\,;\,\$10, \$10 \vdash \mathsf{Al} \otimes \mathsf{Bu}}\ {\otimes}R}{\$10 \multimap \mathsf{Al}, \$10 \multimap \mathsf{Bu}\,;\,\$10 \otimes \$10 \vdash \mathsf{Al} \otimes \mathsf{Bu}}\ {\otimes}L}{\$10 \multimap \mathsf{Al}\,;\,!(\$10 \multimap \mathsf{Bu}), \$10 \otimes \$10 \vdash \mathsf{Al} \otimes \mathsf{Bu}}\ !L}{!(\$10 \multimap \mathsf{Al}), !(\$10 \multimap \mathsf{Bu}), \$10 \otimes \$10 \vdash \mathsf{Al} \otimes \mathsf{Bu}}\ !L$$

In the place where we wrote $\ldots$ we just repeat the valid propositions from the sequent below without change. These valid antecedents are no longer needed at that point.

5 Exercises

Exercise 1 *Prove that A res $\vdash A$ true for any proposition A.*

Exercise 2 *For each of the following purely linear entailments, give a proof that they hold or demonstrate that they do not hold because there is no deduction in our system. You do not need to prove formally that no deduction exists.*

i. $A \mathbin{\&} (B \oplus C) \vdash (A \mathbin{\&} B) \oplus (A \mathbin{\&} C)$

ii. $A \otimes (B \oplus C) \vdash (A \otimes B) \oplus (A \otimes C)$

iii. $A \oplus (B \mathbin{\&} C) \vdash (A \oplus B) \mathbin{\&} (A \oplus C)$

iv. $A \oplus (B \otimes C) \vdash (A \oplus B) \otimes (A \oplus C)$

Exercise 3 *Repeat Exercise 2 by checking the reverse linear entailments.*

Exercise 4 *For each of the following purely linear entailments, give a proof that they hold or demonstrate that they do not hold because there is no deduction in our system. You do not need to prove formally that no deduction exists.*

i. $A \multimap (B \multimap C) \vdash (A \otimes B) \multimap C$

ii. $(A \otimes B) \multimap C \vdash A \multimap (B \multimap C)$

iii. $A \multimap (B \,\&\, C) \vdash (A \multimap B) \,\&\, (A \multimap C)$

iv. $(A \multimap B) \,\&\, (A \multimap C) \vdash A \multimap (B \,\&\, C)$

v. $(A \oplus B) \multimap C \vdash (A \multimap C) \,\&\, (A \multimap C)$

vi. $(A \multimap C) \,\&\, (A \multimap C) \vdash (A \oplus B) \multimap C$

Exercise 5 *For each of the following purely linear entailments, give a proof that they hold or demonstrate that they do not hold because there is no deduction in our system. You do not need to prove formally that no deduction exists.*

i. $C \vdash 1 \multimap C$

ii. $1 \multimap C \vdash C$

iii. $A \multimap \top \vdash \top$

iv. $\top \vdash A \multimap \top$

v. $0 \multimap C \vdash \top$

vi. $\top \vdash 0 \multimap C$

Exercise 6 *For each of the following purely linear entailments, give a proof that they hold or demonstrate that they do not hold because there is no deduction in our system. You do not need to prove formally that no deduction exists.*

i. $!(A \otimes B) \vdash !A \otimes !B$

ii. $!A \otimes !B \vdash !(A \otimes B)$

iii. $!(A \,\&\, B) \vdash !A \otimes !B$

iv. $!A \otimes !B \vdash !(A \,\&\, B)$

v. $!\top \vdash 1$

vi. $1 \vdash !\top$

vii. $!1 \vdash \top$

viii. $\top \vdash !1$

ix. $!!A \vdash !A$

x. $!A \vdash !!A$

6 Appendix: Summary of Intuitionistic Linear Logic

In the rules below, we show the unrestricted assumptions Γ only where affected by the rule. In all other rules it is propagated unchanged from the conclusion to all the premisses. Also recall that the order of hypotheses is irrelevant, and Δ_1, Δ_2 stands for the multiset union of two collections of linear assumptions, which are shown here in the simpler notation.

References

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Judgmental Rules

$$\frac{}{P \vdash P}\ \mathsf{id} \qquad \frac{\Gamma, A \,;\, \Delta, A \vdash C}{\Gamma, A \,;\, \Delta \vdash C}\ \mathsf{VR}$$

Multiplicative Connectives

$$\frac{\Delta_1 \vdash A \quad \Delta_2 \vdash B}{\Delta_1, \Delta_2 \vdash A \otimes B}\ \otimes R \qquad \frac{\Delta, A, B \vdash C}{\Delta, A \otimes B \vdash C}\ \otimes L$$

$$\frac{}{\cdot \vdash 1}\ 1R \qquad \frac{\Delta \vdash C}{\Delta, 1 \vdash C}\ 1L$$

$$\frac{\Delta, A \vdash B}{\Delta \vdash A \multimap B}\ \multimap R \qquad \frac{\Delta_1 \vdash A \quad \Delta_2, B \vdash C}{\Delta_1, \Delta_2, A \multimap B \vdash C}\ \multimap L$$

Additive Connectives

$$\frac{\Delta \vdash A \quad \Delta \vdash B}{\Delta \vdash A \mathbin{\&} B}\ \& R \qquad \frac{\Delta, A \vdash C}{\Delta, A \mathbin{\&} B \vdash C}\ \& L_1 \qquad \frac{\Delta, B \vdash C}{\Delta, A \mathbin{\&} B \vdash C}\ \& L_2$$

$$\frac{}{\Delta \vdash \top}\ \top R \qquad \text{no } \top L \text{ rule}$$

$$\frac{\Delta \vdash A}{\Delta \vdash A \oplus B}\ \oplus R_1 \qquad \frac{\Delta \vdash B}{\Delta \vdash A \oplus B}\ \oplus R_2 \qquad \frac{\Delta, A \vdash C \quad \Delta, B \vdash C}{\Delta, A \oplus B \vdash C}\ \oplus L$$

$$\text{no } 0R \text{ rule} \qquad \frac{}{\Delta, 0 \vdash C}\ 0L$$

Exponential Connective

$$\frac{\Gamma \,;\, \cdot \vdash A}{\Gamma \,;\, \cdot \vdash {!A}}\ !R \qquad \frac{\Gamma, A \,;\, \Delta \vdash C}{\Gamma \,;\, \Delta, {!A} \vdash C}\ !L$$

Figure 1: Intuitionistic Linear Logic