Constructive Logic (15-317), Spring 2023
Recitation 1
Sample Solution

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1 Dcheck

In this class, we will be using Dcheck to type out derivations. Therefore, we will spend part of this recitation familiarizing everyone with this system.

Here are some useful links related to Dcheck.

- The documentation for Dcheck is `http://www.cs.cmu.edu/~crary/dcheck/dcheck.pdf`
- An example file for Dcheck is `http://www.cs.cmu.edu/~crary/dcheck/example.deriv`

The main commands that you will need for Dcheck are below. All these commands are executed on the Andrew machine.

- Run the sanity checks. `~crary/bin/dsanity <filename>`

  This will check for basic syntactic errors. However, the correctness of your derivation will not be checked.

- Visualize your program. `~crary/bin/dvis <filename>`

  Running the visualizer will not run the sanity check, so it is recommended that you run the sanity check before the visualizer.

Dcheck syntax can be found in the documentation. In this recitation we will demonstrate how to convert natural deduction derivations into Dcheck.

Task 1. Write the derivation for the following judgment in Dcheck. You can find the corresponding derivations in the lecture notes[1].

$$((A \vee B) \supset C) \supset ((A \supset C) \wedge (B \supset C)) \textit{ true}$$

[1] `http://www.cs.cmu.edu/~fp/courses/15317-s23/lectures/02-natded.pdf`

2 Harmony

A connective is harmonious if its elimination rules are neither too strong nor too weak in relation to its introduction rules. The first property relates to local soundness, and the second property relates to local completeness.

Proving local soundness is usually done by local reduction. The main idea is to simplify a proof, or demonstrate a more direct proof.

Proving local completeness is usually done by local expansion. The main idea is to expand a proof into a proof which ends with an introduction rule.

Task 2. Consider the following new connective $\circ$.

$$\frac{A\ \mathit{true}}{A \circ B\ \mathit{true}}\circ I_1 \qquad \frac{B\ \mathit{true}}{A \circ B\ \mathit{true}}\circ I_2 \qquad \frac{A \circ B\ \mathit{true} \quad \begin{array}{c}\overline{A\ \mathit{true}}^{\,u}\\ \vdots \\ C\ \mathit{true}\end{array}}{C\ \mathit{true}}\circ E_1^u \qquad \frac{A \circ B\ \mathit{true} \quad \begin{array}{c}\overline{B\ \mathit{true}}^{\,v}\\ \vdots \\ C\ \mathit{true}\end{array}}{C\ \mathit{true}}\circ E_2^v$$

Answer the following questions.

a. If we want to show $\circ$ is locally sound, how many local reductions should we provide?

   Solution: 4. We need to make sure none of the local reductions allow you to gain more information, so we need to check each of the intro/elim pairs.

b. Is this connective locally sound? If so, provide the local reductions. If not, provide a derivation to show that this connective can prove judgments that are not provable, for instance $\perp true$.

   Solution: No, not sound. We could try doing all the reductions and see which ones fail (one such failed reduction is provided below), however, another way to approach this is to look at what the rules are doing and see if it makes logical sense. From the intros rules, we see that for $A \circ B$ to be true, we need $A\ true$ or $B\ true$. Now using this information and looking at the elim rules, we can come up with the following explanation for what the elim rules mean:

   For C to be true, we need A to be true, or B to be true, and for there to exist a proof of C from A OR for there to exist a proof C from B. (Since we can apply either elim rule).

   This is clearly not sound, since we can choose (for example) A to be true, B to be false, and then we can prove anything at all.

   Here is a proof of $\perp\ true$. The subscripts just point out which of these correlate with which variables, but have no logical meaning.

$$\frac{\dfrac{\overline{\top_A\ \mathit{true}}\ \top I}{\top_A \circ \perp_B\ \mathit{true}}\circ I_1 \qquad \dfrac{\overline{\perp_B\ \mathit{true}}^{\,u}}{\perp_C\ \mathit{true}}\perp E}{\perp_C\ \mathit{true}}\circ E_2^u$$

Here is one reduction that works and one that doesn't. The other two are symmetric.

$$
\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ A\ true\end{array}}{A \circ B\ true}\circ I_1 \qquad \begin{array}{c}\overline{A\ true}^{\,u}\\ \mathcal{E}\\ C\ true\end{array}}{C\ true}\circ E_1^u \Rightarrow_R \begin{array}{c}[\mathcal{D}/u]\mathcal{E}\\ C\ true\end{array} \quad \checkmark
$$

$$
\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ A\ true\end{array}}{A \circ B\ true}\circ I_1 \qquad \begin{array}{c}\overline{B\ true}^{\,u}\\ \mathcal{E}\\ C\ true\end{array}}{C\ true}\circ E_2^u \not\Rightarrow_R \quad \times
$$

c. If we want to show $\circ$ is locally complete, how many local expansions should we provide?

Solution: Just 1. We want to show that there exists such an expansion, so for showing completeness, you only need to provide 1 (even if more than one may exist).

d. Is this connective locally complete? If so, provide the local expansion. If not, briefly explain why local expansion fails.

Solution: Yes. Here is an expansion:

$$
\begin{array}{c}\mathcal{D}\\ A \circ B\ true\end{array} \Rightarrow_E \dfrac{\begin{array}{c}\mathcal{D}\\ A \circ B\ true\end{array} \qquad \dfrac{\overline{A\ true}^{\,u}}{A \circ B\ true}\circ I_1}{A \circ B\ true}\circ E_1^u
$$

This is similar to the disjunction case done in lecture, where, while it might look like this is an intros followed by an elim, this actually represents doing an elim to gain A, followed by intros.
Another valid expansion would be to use the the 2nd elim and 2nd intros rules.