Constructive Logic (15-317), Spring 2023
Recitation 3

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1 Review: Verifications & Uses

Task 1. Prove the following: $(A \wedge B) \supset (B \wedge A) \uparrow$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \wedge B \downarrow}\,u}{B \downarrow}\,\wedge E_2}{B \uparrow}\,\uparrow\downarrow \qquad \dfrac{\dfrac{\dfrac{}{A \wedge B \downarrow}\,u}{A \downarrow}\,\wedge E_1}{A \uparrow}\,\uparrow\downarrow}{B \wedge A \uparrow}\,\wedge I}{(A \wedge B) \supset (B \wedge A) \uparrow}\,\supset I^u$$

2 Sequent Calculus

Task 2. Prove the following judgement: $\Longrightarrow (A \wedge B) \supset (B \wedge A)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \wedge B, A, B \Longrightarrow B}\,id(0) \qquad \dfrac{}{A \wedge B, A, B \Longrightarrow A}\,id(1)}{A \wedge B, A, B \Longrightarrow B \wedge A}\,\wedge R}{A \wedge B, A \Longrightarrow B \wedge A}\,\wedge L2(1)}{A \wedge B \Longrightarrow B \wedge A}\,\wedge L1(0)}{\Longrightarrow (A \wedge B) \supset (B \wedge A)}\,\supset R$$

Note that the order of applying $\wedge L1$ and $\wedge L2$ can be reversed in this example.

3 Presentations of Constructive Logic

Theorem 1 (Completeness of Sequent Calculus)**.** If D_1 true$, \ldots, D_n$ true $\vdash C$ true, then $D_1, ..., D_n \Longrightarrow C$.

The completeness of sequent calculus can be interpreted as asserting that anything we can show in ND can also be proven using SC.

The converse is also true:

Theorem 2. If$D_1, ..., D_n \Longrightarrow C$, then $D_1 \downarrow, \ldots, D_n \downarrow \ \vdash C \uparrow$.

The proof proceeds by induction on the derivation of $D_1, ..., D_n \Longrightarrow C$.

Task 3. Prove the $\wedge R$ case.

Recall the two rules of interest in this case:

$$\frac{\Delta \Longrightarrow A \quad \Delta \Longrightarrow B}{\Delta \Longrightarrow A \wedge B} \wedge R$$

$$\frac{\Gamma \vdash A \uparrow \quad \Gamma \vdash B \uparrow}{\Gamma \vdash A \wedge B \uparrow} \wedge I$$

Suppose $\Delta = D_1, ..., D_n$.

By the induction hypothesis, we know $D_1 \downarrow, \ldots, D_n \downarrow \ \vdash A \uparrow$.

Similarly, we know $D_1 \downarrow, \ldots, D_n \downarrow \ \vdash B \uparrow$.

With these two, we can show $D_1 \downarrow, \ldots, D_n \downarrow \ \vdash A \wedge B \uparrow$.

Task 4. Prove the $\vee L$ case.

$$\frac{\Delta, A \vee B, B \Longrightarrow C \quad \Delta, A \vee B, A \Longrightarrow C}{\Delta, A \vee B \Longrightarrow C} \vee L$$

Let $\Gamma = D_1 \downarrow, \ldots, D_n \downarrow, A \vee B \downarrow$

$$\frac{\Gamma \vdash A \vee B \downarrow \quad \Gamma, A \downarrow \ \vdash C \uparrow \quad \Gamma, B \downarrow \ \vdash C \uparrow}{\Gamma \vdash C \uparrow} \vee E$$

By the induction hypothesis, $D_1 \downarrow, \ldots, D_n \downarrow, A \vee B \downarrow, A \downarrow \vdash C \uparrow$

Symmetrically, $D_1 \downarrow, \ldots, D_n \downarrow, A \vee B \downarrow, B \downarrow \vdash C \uparrow$

The first judgement can be shown by identity.

So, $D_1 \downarrow, \ldots, D_n \downarrow, A \vee B \downarrow \vdash C \uparrow$