Constructive Logic (15-317), Spring 2023
Recitation 4

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1 Cut Elimination

Theorem 1 (Cut Admissibility) *If* $\Gamma \Longrightarrow A$ *and* $\Gamma, A \Longrightarrow C$ *then* $\Gamma \Longrightarrow C$. *Alternatively, the rule*

$$\frac{\Gamma \Longrightarrow A \quad \Gamma, A \Longrightarrow C}{\Gamma \Longrightarrow C}\ \mathsf{cut}$$

is admissible.

The proof goes by nested inductions on the structure of A, the derivation $\mathcal{D}$ of $\Gamma \Longrightarrow A$ and $\mathcal{E}$ of $\Gamma, A \Longrightarrow C$. The proof is constructive: we show how to construct $\mathcal{F}$ of $\Gamma \Longrightarrow C$ from $\mathcal{D}$ and $\mathcal{E}$:

$$\frac{\overset{\mathcal{D}}{\Gamma \Longrightarrow A} \quad \overset{\mathcal{E}}{\Gamma, A \Longrightarrow C}}{\Gamma \Longrightarrow C}\ \mathsf{cut} \qquad \rightsquigarrow \qquad \overset{\mathcal{F}}{\Gamma \Longrightarrow C}$$

Show the cases of the proof of cut admissibility that involve $\vee$:

Task 1 *Principal case:* $\mathcal{D}$ *ends with an* $\vee R_1$ *(or* $\vee R_2$*) and* $\mathcal{E}$ *ends with an* $\vee L$*, where* $\vee L$ *operates on the* principal formula *(the one the cut introduces).*

Solution 1

$$\frac{\dfrac{\overset{\mathcal{D}_1}{\Gamma \Longrightarrow A_1}}{\Gamma \Longrightarrow A_1 \vee A_2}\ \vee R_1 \quad \dfrac{\overset{\mathcal{E}_1}{\Gamma, A_1, A_1 \vee A_2 \Longrightarrow C} \quad \overset{\mathcal{E}_2}{\Gamma, A_2, A_1 \vee A_2 \Longrightarrow C}}{\Gamma, A_1 \vee A_2 \Longrightarrow C}\ \vee L}{\Gamma \Longrightarrow C}\ \mathsf{cut}$$

$$\rightsquigarrow$$

$$\mathcal{F} = \dfrac{\overset{\mathcal{D}_1}{\Gamma \Longrightarrow A_1} \qquad \dfrac{\overset{\mathcal{D} \cup \{A_1\}}{\Gamma, A_1 \Longrightarrow A_1 \vee A_2} \quad \overset{\mathcal{E}_1}{\Gamma, A_1, A_1 \vee A_2 \Longrightarrow C}}{\Gamma, A_1 \Longrightarrow C}\ \textsf{i.h.}(\mathcal{D}, \mathcal{E}_1)}{\Gamma \Longrightarrow C}\ \textsf{i.h.}(A_1)$$

The other case is analogous.

Task 2 *$\mathcal{D}$ ends with an $\vee L$ and $\mathcal{E}$ is arbitrary.*

Solution 2

$$\dfrac{\dfrac{\overset{\mathcal{D}_1}{\Gamma, B_1, B_1 \vee B_2 \Longrightarrow A} \quad \overset{\mathcal{D}_2}{\Gamma, B_2, B_1 \vee B_2 \Longrightarrow A}}{\Gamma, B_1 \vee B_2 \Longrightarrow A}\ \vee L \qquad \overset{\mathcal{E}}{\Gamma, B_1 \vee B_2, A \Longrightarrow C}}{\Gamma, B_1 \vee B_2 \Longrightarrow C}\ \textsf{cut}$$

$$\rightsquigarrow$$

$$\mathcal{F} = \dfrac{\dfrac{\overset{\mathcal{D}_1}{\Gamma, B_1, B_1 \vee B_2 \Longrightarrow A} \quad \overset{\mathcal{E}}{\Gamma, B_1 \vee B_2, A \Longrightarrow C}}{\Gamma, B_1, B_1 \vee B_2 \Longrightarrow C}\ \textsf{i.h.}(A, \mathcal{D}_1, \mathcal{E}) \qquad \overset{\vdots}{\Gamma, B_2, B_1 \vee B_2 \Longrightarrow C}}{\Gamma, B_1 \vee B_2 \Longrightarrow C}\ \vee L$$

Task 3 *$\mathcal{D}$ is arbitrary and $\mathcal{E}$ ends with an $\vee R_1$ (or $\vee R_2$).*

Solution 3

$$\dfrac{\overset{\mathcal{D}}{\Gamma \Longrightarrow A} \qquad \dfrac{\overset{\mathcal{E}_1}{\Gamma, A \Longrightarrow C_1}}{\Gamma, A \Longrightarrow C_1 \vee C_2}\ \vee R_1}{\Gamma \Longrightarrow C_1 \vee C_2}\ \textsf{cut}$$

$$\rightsquigarrow$$

$$\mathcal{F} = \dfrac{\dfrac{\overset{\mathcal{D}}{\Gamma \Longrightarrow A} \quad \overset{\mathcal{E}_1}{\Gamma, A \Longrightarrow C_1}}{\Gamma \Longrightarrow C_1}\ \textsf{i.h.}(A, \mathcal{D}, \mathcal{E}_1)}{\Gamma \Longrightarrow C_1 \vee C_2}\ \vee R_1$$

The other case is analogous.

Task 4 *$\mathcal{D}$ is arbitrary and $\mathcal{E}$ ends with an $\vee L$, where $\vee L$ operates on a* side formula *(a formula that is not the principal one).*

Solution 4

$$\dfrac{\dfrac{\mathcal{D}}{\Gamma, B_1 \vee B_2 \Longrightarrow A} \quad \dfrac{\dfrac{\mathcal{E}_1}{\Gamma, B_1, B_1 \vee B_2, A \Longrightarrow C} \quad \dfrac{\mathcal{E}_2}{\Gamma, B_2, B_1 \vee B_2, A \Longrightarrow C}}{\Gamma, B_1 \vee B_2, A \Longrightarrow C} \vee L}{\Gamma, B_1 \vee B_2 \Longrightarrow C} \mathsf{cut}$$

$$\rightsquigarrow$$

$$\mathcal{F} = \dfrac{\dfrac{\dfrac{\mathcal{D} \cup \{B_1\}}{\Gamma, B_1, B_1 \vee B_2 \Longrightarrow A} \quad \dfrac{\mathcal{E}_1}{\Gamma, B_1, B_1 \vee B_2, A \Longrightarrow C}}{\Gamma, B_1, B_1 \vee B_2 \Longrightarrow C} \mathsf{i.h.}(A, \mathcal{D}, \mathcal{E}_1) \quad \begin{array}{c}\vdots \\ \Gamma, B_2, B_1 \vee B_2 \Longrightarrow C\end{array}}{\Gamma, B_1 \vee B_2 \Longrightarrow C} \vee L$$

2 Rule Induction

Rule induction is when we use induction on the structure of the derivation. With it, we can apply the induction hypothesis if the derivation is smaller. Prove the following theorem **using rule induction** (do **not** use admissibility principles previously proved).

Task 5 *Prove that if $\Gamma \Longrightarrow A_1 \supset A_2$ then $\Gamma A_1 \Longrightarrow A_2$ by rule induction. Alternatively, prove the admissibility of*

$$\dfrac{\dfrac{\mathcal{D}}{\Gamma \Longrightarrow A_1 \supset A_2}}{\Gamma, A_1 \Longrightarrow A_2} \mathsf{inv}(\supset R)$$

by induction on the structure of $\mathcal{D}$.

Solution 5 *Cases: left rule operating on a formula inside Γ, or $\supset R$ on the conclusion. I show some of the cases:*

Case: last rule on $\mathcal{D}$ is $\wedge L$.

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\Gamma, B_1 \wedge B_2, B_1 \Longrightarrow A_1 \supset A_2}}{\Gamma, B_1 \wedge B_2 \Longrightarrow A_1 \supset A_2} \wedge L_1}{\Gamma, B_1 \wedge B_2, A_1 \Longrightarrow A_2} \mathsf{inv}(\supset R) \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\Gamma, B_1 \wedge B_2, B_1 \Longrightarrow A_1 \supset A_2}}{\Gamma, B_1 \wedge B_2, B_1, A_1 \Longrightarrow A_2} \mathsf{i.h.}(\mathcal{D}_1)}{\Gamma, B_1 \wedge B_2, A_1 \Longrightarrow A_2} \wedge L_1$$

Case: last rule on $\mathcal{D}$ is $\supset L$.

$$
\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\Gamma, B_1 \supset B_2 \Longrightarrow B_1} \quad \dfrac{\mathcal{D}_2}{\Gamma, B_1 \supset B_2, B_2 \Longrightarrow A_1 \supset A_2}}{\Gamma, B_1 \supset B_2 \Longrightarrow A_1 \supset A_2}\ \supset L_1}{\Gamma, B_1 \supset B_2, A_1 \Longrightarrow A_2}\ \mathsf{inv}(\supset R)
$$

$$\rightsquigarrow$$

$$
\dfrac{\dfrac{\mathcal{D}_1 \cup \{A_1\}}{\Gamma, B_1 \supset B_2, A_1 \Longrightarrow B_1} \quad \dfrac{\dfrac{\mathcal{D}_2}{\Gamma, B_1 \supset B_2, B_2 \Longrightarrow A_1 \supset A_2}}{\Gamma, B_1 \supset B_2, B_2, A_1 \Longrightarrow A_2}\ \mathsf{i.h.}(\mathcal{D}_2)}{\Gamma, B_1 \supset B_2, A_1 \Longrightarrow A_2}\ \supset L_1
$$

Case: last rule on $\mathcal{D}$ is $\supset R$.

$$
\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\Gamma, A_1 \Longrightarrow A_2}}{\Gamma \Longrightarrow A_1 \supset A_2}\ \supset R}{\Gamma, A_1 \Longrightarrow A_2}\ \mathsf{inv}(\supset R) \quad \rightsquigarrow \quad \dfrac{\mathcal{D}_1}{\Gamma, A_1 \Longrightarrow A_2}
$$

3 Rules

$$\frac{\Gamma \Longrightarrow A \quad \Gamma, A \Longrightarrow C}{\Gamma \Longrightarrow C}\ \mathsf{cut} \qquad \frac{}{\Gamma, A \Longrightarrow A}\ \mathsf{id}$$

$$\frac{\Gamma \Longrightarrow C}{\Gamma, A \Longrightarrow C}\ \mathsf{weaken} \qquad \frac{\Gamma, A, A \Longrightarrow C}{\Gamma, A \Longrightarrow C}\ \mathsf{contract}$$

$$\frac{}{\Gamma, P \Longrightarrow P}\ \mathsf{id}^*$$

$$\frac{\Gamma \Longrightarrow A \quad \Gamma \Longrightarrow B}{\Gamma \Longrightarrow A \wedge B}\ {\wedge}R \qquad \frac{\Gamma, A \wedge B, A \Longrightarrow C}{\Gamma, A \wedge B \Longrightarrow C}\ {\wedge}L_1 \quad \frac{\Gamma, A \wedge B, B \Longrightarrow C}{\Gamma, A \wedge B \Longrightarrow C}\ {\wedge}L_2$$

$$\frac{\Gamma, A \Longrightarrow B}{\Gamma \Longrightarrow A \supset B}\ {\supset}R \qquad \frac{\Gamma, A \supset B \Longrightarrow A \quad \Gamma, [A \supset B], B \Longrightarrow C}{\Gamma, A \supset B \Longrightarrow C}\ {\supset}L$$

$$\frac{\Gamma \Longrightarrow A}{\Gamma \Longrightarrow A \vee B}\ {\vee}R_1 \quad \frac{\Gamma \Longrightarrow B}{\Gamma \Longrightarrow A \vee B}\ {\vee}R_2 \qquad \frac{\Gamma, [A \vee B], A \Longrightarrow C \quad \Gamma, [A \vee B], B \Longrightarrow C}{\Gamma, A \vee B \Longrightarrow C}\ {\vee}L$$

$$\frac{}{\Gamma \Longrightarrow \top}\ {\top}R \qquad \text{no rule } {\top}L$$

$$\text{no rule } {\bot}R \qquad \frac{}{\Gamma, \bot \Longrightarrow C}\ {\bot}L$$