Constructive Logic (15-317), Spring 2023
Recitation 6

March 14, 2023

1 Classical logic

Task 1. Find proof term M, such that

$$M : (A \supset B) \supset \neg B \supset \neg A$$

You are encouraged to find the proof term in two ways. You can either directly write out the proof term by understanding its computational meaning, or you can write out the derivation and give proof terms at each level.

Solution. We first provide the derivation. Annotated proof terms are omitted.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg B\ \textit{true}}\,kb \qquad \dfrac{\dfrac{}{A \supset B\ \textit{true}}\,f \qquad \dfrac{\dfrac{\dfrac{}{\neg\neg A\ \textit{true}}\,kka \qquad \dfrac{}{\neg A\ \textit{true}}\,ka}{A\ \textit{true}}\,\neg E}{A\ \textit{true}}\,CC^{ka}}{B\ \textit{true}}\,\supset E}{\neg A\ \textit{true}}\,\neg E}{\neg A\ \textit{true}}\,CC^{kka}}{\neg B \supset \neg A\ \textit{true}}\,\supset I^{kb}}{(A \supset B) \supset \neg B \supset \neg A\ \textit{true}}\,\supset I^{f}$$

Therefore, the proof term is

$$M = \lambda f.\lambda kb.\textbf{callcc}\ (kka.\textbf{throw}\ kb\ f(\textbf{callcc}\ (ka.\textbf{throw}\ kka\ ka)))$$

2 Predicate Calculus

Task 2. Prove the following judgment in Predict Calculus.

$$((\exists x.A(x)) \supset \bot) \supset \forall x.(A(x) \supset \bot)\ \textit{true}$$

Solution.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(\exists x.A(x)) \supset \bot\ \mathit{true}}\,x \qquad \dfrac{\dfrac{}{a\ \mathsf{elem}}\,a \quad \dfrac{}{A(a)\ \mathit{true}}\,y}{\exists x.A(x)\ \mathit{true}}\,\exists I}{\bot\ \mathit{true}}\,{\supset}E}{A(a) \supset \bot\ \mathit{true}}\,{\supset}I^y}{\forall x.(A(x) \supset \bot)\ \mathit{true}}\,\forall I^a}{((\exists x.A(x)) \supset \bot) \supset \forall x.(A(x) \supset \bot)\ \mathit{true}}\,{\supset}I^x$$

Task 3. Prove the following judgment in Heyting Arithmetic.

$$\forall x : \mathsf{nat}.\forall y : \mathsf{nat}.(x = y) \supset (y = x)\ \mathit{true}$$

Solution.

$$\dfrac{\dfrac{}{a : \mathsf{nat}}\,a \qquad \dfrac{\dfrac{\dfrac{}{b : \mathsf{nat}}\,b \quad \dfrac{\dfrac{}{0 = 0\ \mathit{true}}\,f}{(0 = 0) \supset (0 = 0)\ \mathit{true}}\,{\supset}I^f \quad \dfrac{\dfrac{\dfrac{}{0 = \mathbf{s}\ q\ \mathit{true}}\,g}{\mathbf{s}\ q = 0\ \mathit{true}}\,{=}E_{0s}}{(0 = \mathbf{s}\ q) \supset (\mathbf{s}\ q = 0)\ \mathit{true}}\,{\supset}I^g}{(0 = b) \supset (b = 0)\ \mathit{true}}\,\mathsf{nat}E^{q,v}}{\forall y : \mathsf{nat}.(0 = y) \supset (y = 0)\ \mathit{true}}\,\forall I^b \qquad \mathcal{D}}{\dfrac{\forall y : \mathsf{nat}.(a = y) \supset (y = a)\ \mathit{true}}{\forall x : \mathsf{nat}.\forall y : \mathsf{nat}.(x = y) \supset (y = x)\ \mathit{true}}\,\forall I^a}\,\mathsf{nat}E^{p,u}$$

where

$$\begin{array}{c}\mathcal{D} \\ \forall y : \mathsf{nat}.(\mathbf{s}\ p = y) \supset (y = \mathbf{s}\ p)\ \mathit{true}\end{array}$$

Now let's prove it.

$$\dfrac{\dfrac{}{c : \mathsf{nat}}\,c \quad \dfrac{\dfrac{\dfrac{}{\mathbf{s}\ p = 0\ \mathit{true}}\,h}{0 = \mathbf{s}\ p\ \mathit{true}}\,{=}E_{s0}}{(\mathbf{s}\ p = 0) \supset (0 = \mathbf{s}\ p)\ \mathit{true}}\,{\supset}I^h \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\forall y : \mathsf{nat}.(p = y) \supset (y = p)}\,u \quad \dfrac{}{w : \mathsf{nat}}\,w}{(p = w) \supset (w = p)\ \mathit{true}}\,\forall E \quad \dfrac{\dfrac{}{\mathbf{s}\ p = \mathbf{s}\ w\ \mathit{true}}\,j}{p = w\ \mathit{true}}\,{=}E_{ss}}{w = p\ \mathit{true}}\,{\supset}E}{\mathbf{s}\ w = \mathbf{s}\ p\ \mathit{true}}\,{=}I_{ss}}{(\mathbf{s}\ p = \mathbf{s}\ w) \supset (\mathbf{s}\ w = \mathbf{s}\ p)\ \mathit{true}}\,{\supset}I^j}{\dfrac{(\mathbf{s}\ p = c) \supset (c = \mathbf{s}\ p)\ \mathit{true}}{\forall y : \mathsf{nat}.(\mathbf{s}\ p = y) \supset (y = \mathbf{s}\ p)\ \mathit{true}}\,\forall I^c}\,\mathsf{nat}E^{r,w}$$