Constructive Logic (15-317), Spring 2023
Recitation 7

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1 Review: Invertibility

Recall from lecture that invertibility of a rule means that not only does the top imply the bottom as is true for all inference rules, but also that the bottom implies the top as well.

Task 1. Show that $\top R$ and $\bot L$ are invertible.

The rules are:

$$\frac{}{\Delta \rightarrow \top}\ \top R$$

$$\frac{}{\Delta, \bot \rightarrow C}\ \bot L$$

They are vacuously invertible; all premises are derivable.

Task 2. (Review) As demonstrated in lecture, show that $\wedge R$ is invertible.

$$\frac{\Delta \rightarrow A \quad \Delta \rightarrow B}{\Delta \rightarrow A \wedge B}\ \wedge R$$

Suppose $\Delta \rightarrow A \wedge B$.

Observe that

$$\frac{\overline{\Delta, A, B \rightarrow A}\ \text{id}}{\Delta, A \wedge B \rightarrow A}\ \wedge L$$

By the admissibility of cut, we can obtain $\Delta \rightarrow A$ from $\Delta \rightarrow A \wedge B$ and $\Delta, A \wedge B \rightarrow A$.

We can apply the same process to also obtain $\Delta \rightarrow B$.

Task 3. Show that $\top \supset L$ is invertible.

$$\frac{\Delta, A \rightarrow D}{\Delta, \top \supset A \rightarrow D} \wedge R$$

In this proof tree, we will use weakening in addition to cut elimination.

$$\cfrac{\cfrac{\cfrac{}{\Delta, A, \top \rightarrow A}\ \text{id}}{\Delta, A \rightarrow \top \supset A}\ \supset\! R \qquad \cfrac{\cfrac{}{\Delta, \top \supset A \rightarrow D}\ \text{assumption}}{\Delta, A, \top \supset A \rightarrow D}\ \text{weakening}}{\Delta, A \rightarrow D}\ \text{cut}$$

Task 4. Show that $\supset\! L$ is not invertible.

$$\frac{\Delta, A \supset B \rightarrow A \quad \Delta, B \rightarrow C}{\Delta, A \supset B \rightarrow C} \supset\! L$$

Note that by using $\supset\! L$, we are essentially committing to having $A\ true$.

So, consider the case when we let $A = \perp$, and all other metavariables are set to $\top$.

In this case, we would be trying to show $\top, \perp \supset \top \rightarrow \perp$ (the premise on the left). This is obviously unprovable.

2 Examining Logic Programming

Recall `datatype nat = z | s of nat.`

How do we know whether a natural number is even, or equivalently that $even(z)$ is true?

Let us consider the two following rules for even-hood, where $n : nat$:

$$\frac{}{even(z)}\ evz$$

$$\frac{even(n)}{even(s(s(n)))}\ evs$$

Task 5. Show that $even(4)$ is provable.

First, we note that 4 is really $s(s(s(s(z))))$.

Now, we consider the rules in order.

Do $even(z)$ and $even(s(s(s(s(z)))))$ unify? Obviously not, so consider the next possible rule:

Do $even(s(s(n)))$ and $even(s(s(s(s(z)))))$ unify? Yes, they do, and we try to work with evs.

$$\frac{even(s(s(n)))}{even(s(s(s(s(z)))))}\ evs$$

We now change our focus to proving $even(s(s(n)))$.

Similarly, we note that this obviously does not unify with $even(z)$.

Thankfully, thought, this does unify with $even(s(s(n)))$. Let's try evs again, and shift to proving $even(z)$:

$$\cfrac{\cfrac{even(z)}{even(s(s(z)))}\ evs}{even(s(s(s(s(z)))))}\ evs$$

$even(z)$ and $even(z)$ do unify, and now there is nothing left to prove.

$$\cfrac{\cfrac{\cfrac{}{even(z)}\ evz}{even(s(s(z)))}\ evs}{even(s(s(s(s(z)))))}\ evs$$

We have succeeded in showing that $even(4)$ is provable.

Task 5. Show that $even(5)$ is not provable.

We proceed similarly, and eventually get to

$$\cfrac{\cfrac{\overline{even(s(z))}}{even(s(s(s(z))))}\ evs}{even(s(s(s(s(s(z))))))}\ evs$$

At this point, however, this does not unify with either $even(z)$ or $even(s(s(n)))$.

Thus, no rules apply.

We have failed, which means that $even(s(s(s(s(s(z))))))$, or $even(5)$, is not provable.