[Main Idea:] apply all invertible rules first, then apply the non-invertible ones.

- invertible rules → can apply eagerly without even needing to backtrack.
- noninvertible rules → correspond to a choice. If it fails, you need to backtrack!

first, apply right invertible rules

$$\Gamma ; \Omega \overset{R}{\Longrightarrow} A$$

until you reach a noninvertible succedent

$$\frac{\Gamma ; \Omega \overset{L}{\Longrightarrow} C^+}{\Gamma ; \Omega \overset{R}{\Longrightarrow} C^+}$$ ← not right invertible

Then apply all left invertible rules in order (Ω is ordered, Γ is not). If a formula is not left invertible just transport it to Γ!

$$\frac{\Gamma, A^- ; \Omega \overset{L}{\Longrightarrow} C^+}{\Gamma ; A^- ; \Omega \overset{L}{\Longrightarrow} C^+}$$

Do that until $\Omega = \varepsilon$ (is empty). Then start making choices

$$\frac{\Gamma \overset{C}{\Longrightarrow} C^+}{\Gamma ; \varepsilon \overset{L}{\Longrightarrow} C^+}$$

{backtracking needed}

After making a choice we will want to apply inversion rules again, so choices usually continue as $\overset{L}{\Longrightarrow}$ or $\overset{R}{\Longrightarrow}$. We keep doing that until all branches fail or we reach an axiom like:

$$\frac{}{\Gamma ; \Omega, \bot \overset{L}{\Rightarrow} C^+} \quad \text{OR} \quad \frac{}{\Gamma ; \Omega \overset{R}{\Rightarrow} \top}$$

OR

$$\frac{P \in \Gamma}{\Gamma \overset{C}{\Longrightarrow} P}$$

Right Invertible $(\wedge, \top, \supset)$

$$\frac{\Gamma ; \Omega \overset{R}{\Longrightarrow} A \quad \Gamma ; \Omega \overset{R}{\Longrightarrow} B}{\Gamma ; \Omega \overset{R}{\Longrightarrow} A \wedge B}$$

$$\frac{}{\Gamma ; \Omega \overset{R}{\Longrightarrow} \top}$$

$$\frac{\Gamma ; \Omega, A \overset{R}{\Longrightarrow} B}{\Gamma ; \Omega \overset{R}{\Longrightarrow} A \supset B}$$

Not Right Inv $(P, \vee, \bot)$

$$\frac{\Gamma ; \Omega \overset{L}{\Longrightarrow} P}{\Gamma ; \Omega \overset{R}{\Longrightarrow} P}$$

$$\frac{\Gamma ; \Omega \overset{L}{\Longrightarrow} A \vee B}{\Gamma ; \Omega \overset{R}{\Longrightarrow} A \vee B}$$

$$\frac{\Gamma ; \Omega \overset{L}{\Longrightarrow} \bot}{\Gamma ; \Omega \overset{R}{\Longrightarrow} \bot}$$

Left Invertible ($\wedge$, $\top$, $\vee$, $\bot$, $\top\supset$, $\bot\supset$, $\wedge\supset$, $\vee\supset$)

$$\frac{\Gamma ; \Omega, B, A \xRightarrow{L} C^+}{\Gamma ; \Omega, A \wedge B \xRightarrow{L} C^+}$$

$$\frac{\Gamma ; \Omega \xRightarrow{L} C^+}{\Gamma ; \Omega, \top \xRightarrow{L} C^+}$$

$$\frac{\Gamma ; \Omega, A \xRightarrow{L} \quad \Gamma ; \Omega, B \Longrightarrow C^+}{\Gamma ; \Omega, A \vee B \xRightarrow{L} C^+}$$

$$\frac{}{\Gamma ; \Omega, \bot \xRightarrow{L} C^+}$$

$$\frac{\Gamma ; \Omega \xRightarrow{L} C}{\Gamma ; \Omega, \bot \supset A \xRightarrow{L} C^+}$$

$$\frac{\Gamma ; \Omega, A \xRightarrow{L} D^+}{\Gamma ; \Omega, \top \supset A \xRightarrow{L} D^+}$$

$$\frac{\Gamma ; \Omega, A \supset (B \supset C) \xRightarrow{L} D^+}{\Gamma ; \Omega, (A \wedge B) \supset C \xRightarrow{L} D^+}$$

$$\frac{\Gamma ; \Omega, A \supset C, B \supset C \xRightarrow{L} D^+}{\Gamma ; \Omega, (A \vee B) \supset C \xRightarrow{L} D^+}$$

Not Left Invertible $(P,\ P\supset,\ \supset\supset)$

$$\frac{\Gamma, P\ ;\ \Omega \overset{L}{\Longrightarrow} C^+}{\Gamma\ ;\ \Omega, P \overset{L}{\Longrightarrow} C^+}$$

$$\frac{\Gamma, P\supset A\ ;\ \Omega \overset{L}{\Longrightarrow} C^+}{\Gamma\ ;\ \Omega, P\supset A \overset{L}{\Longrightarrow} C^+}$$

$$\frac{\Gamma, (A\supset B)\supset C\ ;\ \Omega \overset{L}{\Longrightarrow} D^+}{\Gamma\ ;\ \Omega, (A\supset B)\supset C \overset{L}{\Longrightarrow} D^+}$$

Nothing in Ω:

$$\frac{\Gamma \overset{C}{\Longrightarrow} C^+}{\Gamma\ ;\ \varepsilon \overset{L}{\Longrightarrow} C^+}$$

Choice – right noninvertible ($\vee$, $\mathbf{0}$)

$$\frac{\Gamma ; \varepsilon \xRightarrow{R} A}{\Gamma \xRightarrow{c} A \vee B} \qquad \frac{\Gamma ; \varepsilon \xRightarrow{R} B}{\Gamma \xRightarrow{c} A \vee B}$$

$$\frac{P \in \Gamma}{\Gamma \xRightarrow{c} P}$$

Choice – left noninvertible

$$\frac{P \in \Gamma \qquad \Gamma ; A \xRightarrow{L} C^+}{\Gamma, P \supset A \xRightarrow{c} C^+}$$

$$\frac{\Gamma ; C \xRightarrow{L} D^+ \qquad \Gamma ; B \supset C, A \xRightarrow{R} B}{\Gamma, (A \supset B) \supset C \xRightarrow{c} D^+}$$

(the end.)