

$$\frac{}{A \rightarrow A} \text{ id}$$

$$\frac{\Gamma_1 \rightarrow A_1 \quad \Gamma_2 \rightarrow A_2}{\Gamma_1, \Gamma_2 \rightarrow A_1 \wedge A_2} \wedge R$$

$$\frac{\Gamma, A \rightarrow C}{\Gamma, A \wedge B \rightarrow C}$$

already used

$$\frac{\Gamma, B \rightarrow C}{\Gamma, A \wedge B \rightarrow C}$$

$$\frac{}{\bullet \rightarrow T}$$

"modulo contraction"

$$\frac{\Gamma, A \rightarrow B}{\Gamma \rightarrow A \supset B} \supset R_1$$

$$\frac{\Gamma \rightarrow B}{\Gamma \rightarrow A \supset B} \supset R_1$$

$$\frac{\Gamma_1 \rightarrow A \quad \Gamma_2, B \rightarrow C}{\Gamma_1, \Gamma_2, A \supset B \rightarrow C}$$

$$\frac{\Gamma \rightarrow A}{\Gamma \rightarrow A \vee B}$$

$$\frac{\Gamma \rightarrow B}{\Gamma \rightarrow A \vee B}$$

$$\frac{}{\perp \rightarrow C}$$

Examples:

- $A \supset (A \supset B)$ doesn't hold
- $((P \supset Q) \supset R) \supset (P \vee R) \wedge (Q \supset R)$ also should not hold
- $((P \vee R) \wedge (Q \supset R)) \supset ((P \supset Q) \supset R)$ holds

$$\rightarrow A \vee (A \supset B)$$

$$(P \supset Q) \supset R \rightarrow (P \vee R) \wedge (Q \supset R)$$

$$(P \vee R) \wedge (Q \supset R) \rightarrow (P \supset Q) \supset R$$

$$\begin{array}{c} R \quad R \quad L \quad R \quad R \\ \rightarrow A \vee (A \supset B) \end{array}$$

$$\frac{}{A \rightarrow A} R_0 \quad \frac{\Gamma, A \rightarrow B}{\Gamma \rightarrow A \supset B} R_1 \quad \frac{\Gamma \rightarrow B}{\Gamma \rightarrow A \supset B} R_2$$

$$\frac{\Gamma \rightarrow A}{\Gamma \rightarrow A \vee (A \supset B)} R_3 \quad \frac{\Gamma \rightarrow A \supset B}{\Gamma \rightarrow A \vee (A \supset B)} R_4$$

$$(R_0) \quad A \rightarrow A \quad P_2$$

$$(R_3) \quad A \rightarrow \overbrace{A \vee (A \supset B)}$$

STUCK

$$\underbrace{\overbrace{LRR}^{LRR} \overbrace{LL}^{LL}}_{N_0} \supset R \longrightarrow \underbrace{\overbrace{RRR}^{RRR} R}_{N_1} \wedge \underbrace{\overbrace{LRR}^{LRR}}_{N_2}$$

$$\underbrace{M_0} \qquad \qquad \underbrace{M_1}$$

$$\frac{}{P \rightarrow P} R_0$$

$$\frac{}{Q \rightarrow Q} R_1$$

$$\frac{}{R \rightarrow R} R_2$$

$$\frac{\Gamma, Q \rightarrow R}{\Gamma \rightarrow \underbrace{Q \supset R}_{N_2}} R_3$$

$$\frac{\Gamma \rightarrow R}{\Gamma \rightarrow \underbrace{Q \supset R}_{N_2}} R_4$$

$$\frac{\Gamma \rightarrow P}{\Gamma \rightarrow \underbrace{P \vee R}_{N_1}} R_5$$

$$\frac{\Gamma \rightarrow R}{\Gamma \rightarrow \underbrace{P \vee R}_{N_1}} R_6$$

$$\frac{\Gamma, P \rightarrow Q}{\Gamma \rightarrow \underbrace{P \supset Q}_{N_0}} R_{7.1}$$

$$\frac{\Gamma_1 \rightarrow N_1 \quad \Gamma_2 \rightarrow N_2}{\Gamma_1 \Gamma_2 \rightarrow \underbrace{N_1 \wedge N_2}_{M_1}} R_8$$

$$\frac{\Gamma_1 \rightarrow N_0 \quad \Gamma_2, R \rightarrow \gamma}{\Gamma_1 \Gamma_2, \underbrace{N_0 \supset R}_{M_0} \rightarrow \gamma} R_9$$

$$\frac{\Gamma \rightarrow Q}{\Gamma \rightarrow \underbrace{P \supset Q}_{N_0}} R_{7.2}$$

Goal: $M_0 \longrightarrow M_1$

1. $P \rightarrow P$ (R_1)
2. $Q \rightarrow Q$ (R_2)
3. $R \rightarrow R$ (R_3)
4. $R \rightarrow N_2$ (R_4) 3
5. $P \rightarrow N_1$ (R_5) \perp
6. $R \rightarrow N_1$ (R_6) 3
7. $Q \rightarrow N_0$ ($R_{7.2}$) 2
8. $P, R \rightarrow M_1$ (R_8) 5 4
9. $R, R \rightarrow M_1$ (R_8) 6 4
10. $Q, M_0 \rightarrow R$ (R_9) 7 3
11. $Q, M_0 \rightarrow N_2$ (R_9) 7 4
12. $Q, M_0 \rightarrow N_1$ (R_9) 7 6
13. $M_0 \rightarrow N_2$ (R_3) 10
14. $M_0, Q \rightarrow N_2^x$ (R_4) 10
15. $Q, M_0 \rightarrow N_1^x$ (R_6) 10
16. $Q, M_0 \rightarrow M_1$ (R_9) 7 9
17. $Q, M_0, P \rightarrow M_1$ (R_9) 7 8
18. $Q, M_0 \rightarrow M_1$ (R_8) 12 11
19. $Q, M_0, R \rightarrow M_1$ (R_8) 12 4 • R_9
20. $Q, M_0, P \rightarrow M_1$ (R_8) 11 5
21. $Q, M_0, R \rightarrow M_1$ (R_8) 11 6 • R_9

} R_8

$$22. \quad \mathcal{Q}, M_0 \rightarrow M, \times$$

$$(R_8) \text{ is } 14$$

$$23. \quad \mathcal{Q}, M_0 \rightarrow M, \times$$

$$(R_9) \text{ } \neq (19|21)$$

$$24. \quad \mathcal{Q}, M_0 \rightarrow M, \times$$

STUCK .

$$(P \vee R) \wedge (Q \supset R) \supset (P \supset Q) \supset R$$

Optimization: inversions.

↙ I decided to stop here, but
VL is invertible.

$$(P \vee R), (Q \supset R), (P \supset Q) \rightarrow R$$

$$(P \vee R) \wedge (Q \supset R), (P \supset Q) \rightarrow R$$

$$(P \vee R) \wedge (Q \supset R) \rightarrow (P \supset Q) \supset R$$

$$\rightarrow (P \vee R) \wedge (Q \supset R) \supset (P \supset Q) \supset R$$

cont.

$$\underbrace{LLL}_{N_0} (P \vee R), \underbrace{RLL}_{N_1} (Q \supset R), \underbrace{RLL}_{N_2} (P \supset Q) \longrightarrow R$$

$$\frac{}{P \longrightarrow P} R_0 \quad \frac{}{Q \longrightarrow Q} R_1 \quad \frac{}{R \longrightarrow R} R_2$$

$$\frac{\Gamma_1, P \longrightarrow \gamma \quad \Gamma_2, R \longrightarrow \gamma}{\Gamma_1, \Gamma_2, \underbrace{P \vee R}_{N_0} \longrightarrow \gamma} R_3$$

$$\frac{\Gamma_1 \longrightarrow Q \quad \Gamma_2, R \longrightarrow \gamma}{\Gamma_1, \Gamma_2, \underbrace{Q \supset R}_{N_1} \longrightarrow \gamma} R_4$$

$$\frac{\Gamma_1 \longrightarrow P \quad \Gamma_2, Q \longrightarrow \gamma}{\Gamma_1, \Gamma_2, \underbrace{P \supset Q}_{N_2} \longrightarrow \gamma} R_5$$

1. $P \longrightarrow P$	4. $(R_4) \ 2 \ 3 \quad Q, N_1 \longrightarrow R$
2. $Q \longrightarrow Q$	5. $(R_5) \ 1 \ 2 \quad P, N_2 \longrightarrow Q$
3. $R \longrightarrow R$	

$$7. (R_4) \ 5 \ 3 \quad P, N_2, N_1 \longrightarrow R$$

$$8. (R_5) \ 1 \ 4 \quad P, N_1, N_2 \longrightarrow R^*$$

$$9. (R_3) \ 7 \ 3 \quad N_1, N_2, N_0 \longrightarrow R^*$$