Constructive Logic (15-317), Spring 2023
Recitation 12

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1 Linear logic

Task 1. Prove that the following judgement is not derivable in linear sequent calculus.

$$\cdot \longrightarrow A \multimap B \multimap A$$

Solution. This is the only possible derivation.

$$\dfrac{\dfrac{A, B \longrightarrow A}{A \longrightarrow B \multimap A} \multimap R}{\cdot \longrightarrow A \multimap B \multimap A} \multimap R$$

However, we get stuck here, because no further rules can be applied. Therefore, the original judgement is not provable.

Task 2. Recall the linear sequent cut.

Theorem 1 (Linear sequent cut)**.** If $\Gamma_1 \longrightarrow A$ and $\Gamma_2, A \longrightarrow C$, then $\Gamma_1, \Gamma_2 \longrightarrow C$.

Provide a counter example to show why the following cut theorem is incorrect in linear sequent calculus.

If $\Gamma \longrightarrow A$ and $\Gamma, A \longrightarrow C$, then $\Gamma \longrightarrow C$.

Solution. Assume A is some atom, and $\Gamma = A$, $C = A \otimes A$. Clearly both $A \longrightarrow A$ and $A, A \longrightarrow A \otimes A$ are derivable. However, $A \longrightarrow A \otimes A$ is not.

Task 3. Prove the left commuting case $(\otimes L/*)$.

Solution. Assume the following derivations.

$$\dfrac{\begin{matrix}\mathcal{D}_1 \\ \Gamma_1, A_1, A_2 \longrightarrow A\end{matrix}}{\Gamma_1, A_1 \otimes A_2 \longrightarrow A} \otimes L \qquad \begin{matrix}\mathcal{E} \\ \Gamma_2, A \longrightarrow C\end{matrix}$$

Apply I.H. on $\mathcal{D}_1$ and $\mathcal{E}$ with principal formula A, we have $\Gamma_1, \Gamma_2, A_1, A_2 \longrightarrow C$. Use $\otimes L$, to derive $\Gamma_1, \Gamma_2, A_1 \otimes A_2 \longrightarrow C$.

2 Linear application

Blocks World is a class of scenarios in which there is a table, some number of blocks which can be stacked on top of each other, and a robotic arm which can pick up and move blocks. The following atomic predicates are used:

a. $ontable(x)$ means that the block x is sitting directly on the table.

b. $on(x, y)$ means that the block x is directly on top of the block y.

c. $clear(x)$ means that the block x does not have anything on top of it.

d. $empty$ means that the robotic arm's hand is empty.

e. $holds(x)$ means that the robotic arm's hand is holding x.

Task 4. Write out the following axioms in linear logic.

a. The robot can place a block x on the table directly.

b. The robot can place a block x directly on block y.

c. The robot can pick a block x from the table directly.

d. The robot can pick a block x from the top of block y.

Solution.

a. $\forall x.holds(x) \multimap ontable(x) \otimes clear(x) \otimes empty$

b. $\forall x.\forall y.clear(y) \otimes holds(x) \multimap on(x, y) \otimes clear(x) \otimes empty$

c. $\forall x.ontable(x) \otimes clear(x) \otimes empty \multimap holds(x)$

d. $\forall x.\forall y.on(x, y) \otimes clear(x) \otimes empty \multimap clear(y) \otimes holds(x)$