# Resource Management for the Inverse Method in Linear Logic ⋆

Kaustuv Chaudhuri and Frank Pfenning

Carnegie Mellon University, Pittsburgh PA 15213, USA
{kaustuv,fp}@cs.cmu.edu

**Abstract.** One central aspect of proof search in linear logic is resource management. Strategies for efficient resource management have been developed for backward-directed calculi, such as top-down linear logic programming, tableaux calculi, and matrix methods. In this paper we consider resource management for forward-directed calculi, such as the inverse method, clausal resolution, and bottom-up linear logic programming. We focus on the inverse method for intuitionistic linear logic, and isolate the resource management problems. They turn out to come from exponentials, additive unit, and multiplicative unit, exhibiting some surprising differences from the backward-directed calculi. Our solution employs controlled contraction and weakening, and introduces affine hypotheses into sequents.

## 1 Introduction

In linear logic [5, 2], hypotheses are viewed as resources, with the number of occurrences playing a central role in the proof theory. This property allows natural encodings of theories that are difficult to express in the standard logic. Theorem proving in linear logic has an additional component of resource management which is not as critical for the standard logic. The kinds of these resource management problems are determined by the direction in which inference rules are read:

– For multi-premiss multiplicative rules, it is undetermined how to divide the linear context into the premisses when reading the rules backward ("multiplicative resource non-determinism").

$$\frac{\Delta_1 \Longrightarrow A \quad \Delta_2 \Longrightarrow B}{\Delta_1, \Delta_2 \Longrightarrow A \otimes B}$$

In the forward direction there is no non-determinism because the context $\Delta_1, \Delta_2$ is merely the result of adjoining the contexts $\Delta_1$ and $\Delta_2$.

⋆ This work has been partially supported by NSF Grant CCR-9988281 "*Logical and Meta-Logical Frameworks*". Draft submitted for publication on January 31, 2003

– For the additive units $\mathbf{0}$ and $\top$, the linear context is not structurally determined ("affine resource non-determinism").

$$\frac{}{\Delta \Longrightarrow \top} \qquad \frac{}{\Delta, \mathbf{0} \Longrightarrow C}$$

This causes non-determinism in both directions, but the nature is different in each case. In the forward direction, the contexts have to be invented; for a discussion of the issues in the backward direction, see [4].

– For the multiplicative unit $\mathbf{1}$ in the forward direction, or for resource dereliction in the backward direction, it is undetermined whether the introduced resource is necessary for the proof ("unknown-use resource non-determinism"). As a result, there is no control on the number of iterated applications of the rules.

$$\frac{\Delta \Longrightarrow C}{\Delta, \mathbf{1} \Longrightarrow C} \qquad \frac{!A, \Delta, A \Longrightarrow C}{!A, \Delta \Longrightarrow C}$$

The corresponding problems don't exist in the respective opposite directions: $\mathbf{1}$ can always be deleted safely from contexts when reasoning backward, and dereliction shrinks the context in the forward direction.

In the domain of top down linear logic programming – i.e., refining goals by applying inference rules in the *backward* direction until they are axiomatic, for example *Lolli* [7] or *Lygon* [14] – resource management approaches fall into two broad kinds. The first kind perform general search with constraint solving. For example, in [6], resource use is marked by boolean flags, and the use of an inference rule is guarded by constraints on these boolean flags. The second kind commit to a particular search strategy, which amount to particular solutions to the constraint satisfaction problems in resource management. For example, in [4] the judgements are refined with boolean "strictness" flags and a context of lax resources is added.

Neither of these approaches are directly applicable to forward reasoning because of the strikingly different nature of resource management. However, the essential idea of refining the sequents to expose the resource management problems is the gist of our approach. Our calculus is particularly suited for the inverse method [9, 13], which requires a forward sequent calculus with a strong subformula property, which our calculus enjoys. The subformula property is necessary to control iterations of rules like $\& L_1$ in the forward direction

$$\frac{\Gamma \,; \Delta, A \Longrightarrow C}{\Gamma \,; \Delta, A \,\&\, B \Longrightarrow C}\ \& L_1$$

The inverse method restricts rule application to only those instances where the principal formula is a subformula of the goal sequent, so the above rule would not be applied unless a negative instance of $A \,\&\, B$ is present in the goal. Resolution and the forward reasoning have been examined for classical linear logic in [10, 12], however without particular attention to affine or unknown-use resource non-determinism.

Our approach is sufficiently general and logically motivated that it is has wider application than just the inverse method. For example, the essential ideas are applicable to forward reasoning in intuitionistic *affine* logic in the style of [3], but with a closer attention to the resource management issues. Furthermore, our approach applies to classical inverse methods also, because classical linear logic is a simplification of intuitionistic linear logic.

## 2 Backward Sequent Calculus

We begin with a brief description of Gentzen-type backward sequent calculus for propositional intuitionistic linear logic without additive disjunction and quantifiers. See Sec.5 for a brief discussion of the issues involved.

Non-atomic propositions in this fragment are constructed using the following connectives: multiplicative and additive conjunctions together with their units, linear implication and the exponential modality:

| | | |
|---|---|---|
| Propositions | $A, B, \ldots ::= P$ | Atomic |
| | $\mid\ A \otimes B \mid \mathbf{1}$ | Mult. conj. and unit |
| | $\mid\ A \mathbin{\&} B \mid \top$ | Add. conj. and unit |
| | $\mid\ A \multimap B$ | Linear implication |
| | $\mid\ !A$ | Exponential modality |
| Contexts | $\Gamma, \Delta, \ldots ::= \cdot$ | Empty |
| | $\mid\ \Gamma, A$ | Adjoin |

In the sequent calculus we will treat unrestricted assumptions as a separate context of resources. Sequents will have two kinds of contexts: the *unrestricted context* and the *linear context*.

$$\underbrace{A_1, A_2, \ldots, A_j}_{\text{unrestricted}} ; \underbrace{B_1, B_2, \ldots, B_k}_{\text{linear}} \Longrightarrow C$$

We will use $\Gamma$ and $\Delta$ for the unrestricted and linear contexts, respectively. The interpretation is that the succedent $C$ is proved using the unrestricted resources arbitrarily often, and each linear resource exactly once.

The inference rules are given in Fig.1. Linear resources are used structurally in axiomatic initial sequents of the form $\Gamma ; A \Longrightarrow A$. A copy of any unrestricted resource can be transferred into the linear context using the dereliction rule, dl. For the multiplicative rules the unrestricted context is unchanged and the linear context is distributed among the premisses. For the additive rules, both contexts remain unchanged in the premisses. The $!L$ and $!R$ rules describe the exponential modality. Exchanging resources is valid within each kind of context but not between contexts.

**Theorem 1 (Structural properties).**

1. *(weakening) If $\Gamma ; \Delta \Longrightarrow C$ then $\Gamma, A ; \Delta \Longrightarrow C$.*

**Initial and Dereliction rules**

$$\frac{}{\Gamma \mathbin{;} A \Longrightarrow A}\ \text{init} \qquad \frac{\Gamma, A \mathbin{;} \Delta, A \Longrightarrow C}{\Gamma, A \mathbin{;} \Delta \Longrightarrow C}\ \text{dl}$$

**Multiplicative connectives**

$$\frac{\Gamma \mathbin{;} \Delta_1 \Longrightarrow A \quad \Gamma \mathbin{;} \Delta_2, B \Longrightarrow C}{\Gamma \mathbin{;} \Delta_1, \Delta_2, A \multimap B \Longrightarrow C}\ {\multimap}L \qquad \frac{\Gamma \mathbin{;} \Delta, A \Longrightarrow B}{\Gamma \mathbin{;} \Delta \Longrightarrow A \multimap B}\ {\multimap}R$$

$$\frac{\Gamma \mathbin{;} \Delta, A, B \Longrightarrow C}{\Gamma \mathbin{;} \Delta, A \otimes B \Longrightarrow C}\ {\otimes}L \qquad \frac{\Gamma \mathbin{;} \Delta_1 \Longrightarrow A \quad \Gamma \mathbin{;} \Delta_2 \Longrightarrow B}{\Gamma \mathbin{;} \Delta_1, \Delta_2 \Longrightarrow A \otimes B}\ {\otimes}R$$

$$\frac{\Gamma \mathbin{;} \Delta \Longrightarrow C}{\Gamma \mathbin{;} \Delta, \mathbf{1} \Longrightarrow C}\ \mathbf{1}L \qquad \frac{}{\Gamma \mathbin{;} \cdot \Longrightarrow \mathbf{1}}\ \mathbf{1}R$$

**Additive connectives**

$$\frac{\Gamma \mathbin{;} \Delta, A \Longrightarrow C}{\Gamma \mathbin{;} \Delta, A \mathbin{\&} B \Longrightarrow C}\ {\&}L_1 \qquad \frac{\Gamma \mathbin{;} \Delta, B \Longrightarrow C}{\Gamma \mathbin{;} \Delta, A \mathbin{\&} B \Longrightarrow C}\ {\&}L_2$$

$$\frac{\Gamma \mathbin{;} \Delta \Longrightarrow A \quad \Gamma \mathbin{;} \Delta \Longrightarrow B}{\Gamma \mathbin{;} \Delta \Longrightarrow A \mathbin{\&} B}\ {\&}R \qquad \frac{}{\Gamma \mathbin{;} \Delta \Longrightarrow \top}\ {\top}R$$

**Exponentials**

$$\frac{\Gamma, A \mathbin{;} \Delta \Longrightarrow C}{\Gamma \mathbin{;} \Delta, !A \Longrightarrow C}\ !L \qquad \frac{\Gamma \mathbin{;} \cdot \Longrightarrow A}{\Gamma \mathbin{;} \cdot \Longrightarrow !A}\ !R$$

**Fig. 1.** Inference rules for the backward calculus

2. *(contraction) If* $\Gamma, A, A \mathbin{;} \Delta \Longrightarrow C$ *then* $\Gamma, A \mathbin{;} \Delta \Longrightarrow C$.

*Proof.* By structural induction on the derivations.

The backward sequent calculus enjoys the following substitution properties, often written as explicit cut rules.

**Theorem 2 (Cut).**

1. *If* $\Gamma \mathbin{;} \Delta_1 \Longrightarrow A$ *and* $\Gamma \mathbin{;} \Delta_2, A \Longrightarrow C$*, then* $\Gamma \mathbin{;} \Delta_1, \Delta_2 \Longrightarrow C$.
2. *If* $\Gamma \mathbin{;} \cdot \Longrightarrow A$ *and* $\Gamma, A \mathbin{;} \Delta \Longrightarrow C$*, then* $\Gamma \mathbin{;} \Delta \Longrightarrow C$.

*Proof.* See [11, 8].

## 3 Controlled Weakening

We will now turn to forward reasoning, i.e., with the aim of assembling the conclusion from the premisses. To summarize, the forward direction has the following resource management problems:

1. **Undetermined contexts**: the linear context in $\top R$, and the unrestricted contexts in init, $\mathbf{1}R$ and $\top R$ are not determined from the premisses.
2. **Uncontrolled application**: the $\mathbf{1}L$ rule may be iterated arbitrarily often. A defining property of linear logic is sensitivity to the number of occurrences of linear resources, so these uncontrolled applications can result in a potentially unbounded number of *structurally different* conclusions from a given set of premisses.

We will solve these problems with the rules in Fig.1 in stages. In the first stage, we will show how to eliminate undetermined contexts by making the introduction of such contexts implicit. Then, we will show how to control the application of $\mathbf{1}L$ by making all uses of that rule implicit in the backward direction. In the final stage we will turn the modified backward calculus into a forward *selection* calculus (Sec.4.3). The resulting calculus will be ready for use in the inverse method.

First we attack the problem of undetermined contexts. The resulting calculus will be significantly different from that of the backward direction, so we will use the $\longrightarrow$ sequent arrow to distinguish it.

For the context of unrestricted resources, we will change the interpretation from that of a weakenable set in the backward direction to that of a *strict* context with implicit contraction in the forward direction. It will contain only those resources that are actually required in a particular proof of the sequent. For example, the init rule does not use any unrestricted hypotheses, so we will remove the unrestricted context $\Gamma$ entirely from the conclusion to get:

$$\frac{}{\cdot \mathrel{;} A \longrightarrow A}\ \text{init}$$

As a result of strictness, the multi-premiss rules will have different unrestricted contexts in the premisses. In the conclusion, these contexts of unrestricted resources are unioned, and multiple occurrences of a resource are factored into a single occurrence.[1]

For the linear context, we will require a more careful enumeration of the ways in which resources are introduced in a proof.

1. "One use" introduction, as happens in the init rule. In this case, the linear resource is necessary for the proof.
2. "Zero use" introduction, as happens in the $\top R$ rule. In this case, the linear context is arbitrary. To illustrate, all of the following sequents have essentially the same proof: $\cdot \mathrel{;} \cdot \longrightarrow \top$ and $\cdot \mathrel{;} A \longrightarrow \top$ and $\cdot \mathrel{;} A, B \longrightarrow \top$.
3. "Undetermined use" introduction, as happens in the $\mathbf{1}L$ rule. During the introduction, it is unknown if the introduced $\mathbf{1}$ is required as a linear resource.

[1] In the presence of quantifiers we apply factoring by unifying unrestricted propositions as in the ordinary (non-linear) inverse method.

The final case is a symptom of a problem with uncontrolled application, whose solution we will defer until Sec.4.2. For the zero-use and one-use introductions, we will track the kind of introduction by means of a boolean flag on the sequent arrow. Our sequents will have the shape:

$$\Gamma \,;\, \Delta \longrightarrow^{w} C$$

where $w$ is a meta variable with value either 0 or 1. This is similar to the strictness annotation in [4], with the difference that the flag in our case denotes that the context $\Delta$ may be weakened. More precisely, the interpretation is:

| | | |
|---|---|---|
| $\Gamma; \Delta \longrightarrow^{0} C$ | corresponds to | $\Gamma'; \Delta \Longrightarrow C$ for any $\Gamma' \supseteq \Gamma$ |
| $\Gamma; \Delta \longrightarrow^{1} C$ | corresponds to | $\Gamma'; \Delta' \Longrightarrow C$ for any $\Delta' \supseteq \Delta$ and $\Gamma' \supseteq \Gamma$ |

We will call sequents with the annotation of 1 *weak sequents*. The linear context of weak sequents may be weakened when they are used as premisses of an inference rule. The complete set of rules are shown in Fig.2. Initial sequents are given the annotation 0 because the lone resource is the principal formula on both sides of the sequent arrow. Multiplicative and additive rules will disjoin and conjoin the flags, respectively. For the dereliction and exponential rules, the flag will remain untouched. The $\top R$ rule will now have no contexts at all, but the conclusion will be weak.

*A comment on $\&R$.* We define a conditional union of contexts $\Delta_1 \overset{w_1,w_2}{\cup} \Delta_2$:

$$\Delta_1 \overset{w_1,w_2}{\cup} \Delta_2 = \begin{cases} \Delta_1 & \text{with } \Delta_1 = \Delta_2 \text{ if } w_1 \vee w_2 = 0 \\ \Delta_1 & \text{with } \Delta_2 \subseteq \Delta_1 \text{ if } w_1 = 0 \text{ and } w_2 = 1 \\ \Delta_2 & \text{with } \Delta_1 \subseteq \Delta_2 \text{ if } w_1 = 1 \text{ and } w_2 = 0 \\ \Delta_1 \sqcup \Delta_2 & \text{if } w_1 \wedge w_2 = 1 \end{cases}$$

The $\sqcup$ operator in the final case is multiset join, i.e., the smallest multiset that contains both operands. These unions capture the essence of the weakening annotation by allowing the linear contexts of weak premisses to be smaller than other contexts. Weakening is, therefore, implicit in this rule.

*A comment on side conditions.* Some rules such as $\multimap R_1$ will carry an extra side condition to prevent unnecessary non-deterministic choices. For example, we never have to choose between the following applications:

$$\frac{\Gamma \,;\, A \longrightarrow^{1} B}{\Gamma \,;\, \cdot \longrightarrow^{1} A \multimap B} \multimap R \qquad \text{and} \qquad \frac{\Gamma \,;\, A \longrightarrow^{1} B}{\Gamma \,;\, A \longrightarrow^{1} A \multimap B} \multimap R'$$

It is complete to use the first of these rules only, because the conclusion is weak and the resource $A$ can be introduced later if necessary by implicit weakening. By similar reasoning, a second version of $\multimap L$ where $B$ is not required in the second premiss and $w_2 = 1$ is unnecessary, because the conclusion would be a weakened version of the second premiss.

**Initial and Dereliction rules**

$$\frac{}{\cdot \,;\, A \longrightarrow^0 A}\ \text{init} \qquad \frac{\Gamma \,;\, \Delta, A \longrightarrow^w C}{\Gamma, A \,;\, \Delta \longrightarrow^w C}\ \text{dl}$$

**Multiplicative connectives**

$$\frac{\Gamma_1 \,;\, \Delta_1 \longrightarrow^{w_1} A \quad \Gamma_2 \,;\, \Delta_2, B \longrightarrow^{w_2} C}{\Gamma_1 \cup \Gamma_2 \,;\, \Delta_1, \Delta_2, A \multimap B \longrightarrow^{w_1 \vee w_2} C}\ {\multimap}L$$

$$\frac{\Gamma \,;\, \Delta, A \longrightarrow^w C}{\Gamma \,;\, \Delta \longrightarrow^w A \multimap C}\ {\multimap}R \qquad \frac{\Gamma \,;\, \Delta \longrightarrow^1 C \quad A \notin \Delta}{\Gamma \,;\, \Delta \longrightarrow^1 A \multimap C}\ {\multimap}R_1$$

$$\frac{\Gamma \,;\, \Delta, A \longrightarrow^1 C \quad B \notin \Delta}{\Gamma \,;\, \Delta, A \otimes B \longrightarrow^1 C}\ \otimes L_1 \qquad \frac{\Gamma \,;\, \Delta, B \longrightarrow^1 C \quad A \notin \Delta}{\Gamma \,;\, \Delta, A \otimes B \longrightarrow^1 C}\ \otimes L_2$$

$$\frac{\Gamma \,;\, \Delta, A, B \longrightarrow^w C}{\Gamma \,;\, \Delta, A \otimes B \longrightarrow^w C}\ \otimes L \qquad \frac{\Gamma_1 \,;\, \Delta_1 \longrightarrow^{w_1} A \quad \Gamma_2 \,;\, \Delta_2 \longrightarrow^{w_2} B}{\Gamma_1 \cup \Gamma_2 \,;\, \Delta_1, \Delta_2 \longrightarrow^{w_1 \vee w_2} A \otimes B}\ \otimes R$$

$$\frac{\Gamma \,;\, \Delta \longrightarrow^0 C}{\Gamma \,;\, \Delta, \mathbf{1} \longrightarrow^0 C}\ \mathbf{1}L \qquad \frac{}{\cdot \,;\, \cdot \longrightarrow^0 \mathbf{1}}\ \mathbf{1}R$$

**Additive connectives**

$$\frac{\Gamma \,;\, \Delta, A \longrightarrow^w C}{\Gamma \,;\, \Delta, A \,\&\, B \longrightarrow^w C}\ \&L_1 \qquad \frac{\Gamma \,;\, \Delta, B \longrightarrow^w C}{\Gamma \,;\, \Delta, A \,\&\, B \longrightarrow^w C}\ \&L_2$$

$$\frac{\Gamma_1 \,;\, \Delta_1 \longrightarrow^{w_2} A \quad \Gamma_2 \,;\, \Delta_2 \longrightarrow^{w_2} B}{\Gamma_1 \cup \Gamma_2 \,;\, \Delta_1 \overset{w_1, w_2}{\cup} \Delta_2 \longrightarrow^{w_1 \wedge w_2} A \,\&\, B}\ \&R \qquad \frac{}{\cdot \,;\, \cdot \longrightarrow^1 \top}\ \top R$$

**Exponentials**

$$\frac{\Gamma, A \,;\, \Delta \longrightarrow^w C}{\Gamma \,;\, \Delta, !A \longrightarrow^w C}\ !L \qquad \frac{\Gamma \,;\, \Delta \longrightarrow^0 C \quad A \notin \Gamma}{\Gamma \,;\, \Delta, !A \longrightarrow^0 C}\ !L_1 \qquad \frac{\Gamma \,;\, \cdot \longrightarrow^0 A}{\Gamma \,;\, \cdot \longrightarrow^0 !A}\ !R$$

**Fig. 2.** Rules for the first forward calculus

*A comment on dereliction.* The dl rule should not be applied unless the transferred formula is actually used unrestrictedly, by either occuring in the unrestricted context in the goal, or being an operand of the ! modality. In the inverse method, labelling will flag such subformulas with a weight – *heavy* if it has an unrestricted use, and *light* otherwise. The dl rule will be used to transfer only heavy subformulas.

*Soundness and completeness.* Because the linear context has vastly different behaviour depending on the weakening annotation of the sequent, the soundness theorem will have two different cases. In the case for weak sequents, soundness can be established for any weakening of the linear context.

**Theorem 3 (Soundness).**

1. *If $\Gamma \,;\, \Delta \longrightarrow^0 C$, then $\Gamma \,;\, \Delta \Longrightarrow C$.*
2. *If $\Gamma \,;\, \Delta \longrightarrow^1 C$, then $\Gamma \,;\, \Delta' \Longrightarrow C$ for any $\Delta' \supseteq \Delta$.*

*Proof.* By induction on the structure of the derivation of $\Gamma \mathbin{;} \Delta \longrightarrow^{w} C$.

For the completeness theorem, we allow for the possibility that the forward calculus will find proofs of sequents that do not have unnecessary resources.

**Theorem 4 (Completeness).** *If $\Gamma \mathbin{;} \Delta \Longrightarrow C$ then for some $\Gamma' \subseteq \Gamma$,*

1. *either $\Gamma' \mathbin{;} \Delta \longrightarrow^{0} C$;*
2. *or $\Gamma' \mathbin{;} \Delta' \longrightarrow^{1} C$ for some $\Delta' \subseteq \Delta$.*

*Proof.* By structural induction on the derivation of $\Gamma \mathbin{;} \Delta \Longrightarrow C$.

## 4 Controlling $\mathbf{1}L$

Uncontrolled application of $\mathbf{1}L$ presents our next obstacle; before proceeding further, a note on why this rule is interesting. In encoding of theories in linear logic, sometimes the exactness of the linear resource is too restrictive for the semantics, for example, if we require an at-most one use semantics. For these theories, it is a common idiom to wrap such resources in $\&\mathbf{1}$. Resources of this form have an *affine* interpretation; that is, the following choice of rules is applicable in order to introduce such a resource,

$$\frac{\Gamma \mathbin{;} \Delta, A \Longrightarrow C}{\Gamma \mathbin{;} \Delta, A \mathbin{\&} \mathbf{1} \Longrightarrow C} \qquad \frac{\Gamma \mathbin{;} \Delta, \mathbf{1} \Longrightarrow C}{\Gamma \mathbin{;} \Delta, A \mathbin{\&} \mathbf{1} \Longrightarrow C}$$

The first rule is the "one use" case, the second the "zero use" case, and together they encode the "at-most one use" semantics for $A$.

### 4.1 Removing $\mathbf{1}L$ from the Backward Calculus

First we will remove $\mathbf{1}L$ from the backward calculus of Sec.2. The $\mathbf{1}L$ rule may be viewed as a particular instance of an explicit weakening rule that introduces new $\mathbf{1}$ resources. We will first identify the scenarios where $\mathbf{1}$ is being used as the multiplicative unit in sequents. We say a formula is in $\mathbf{1}$-*normal form* (**1NF**) if it contains no occurrences of $\mathbf{1} \otimes A$ or $A \otimes \mathbf{1}$ or $\mathbf{1} \multimap A$ or $!\mathbf{1}$ as subformulas. A context of resources $\Gamma$ or $\Delta$ is in **1NF** if all its resources are in **1NF**, and it doesn't contain the resource $\mathbf{1}$. A sequent $\Gamma; \Delta \Longrightarrow C$ is in **1NF** if $\Gamma$, $\Delta$ and $C$ are in **1NF**.

Formulas are translated into **1NF** in the obvious way, by reducing unitary uses, $!\mathbf{1}$, etc. Fig.3 has a version of the calculus where the premisses and conclusions are assumed to be in **1NF**. All these are derivable rules of the original backward calculus (Sec.2), so soundness is immediate.

The cut and completeness theorems requires an update. The proof is immediate because we've removed only unitary uses of $\mathbf{1}$.

**Theorem 5 (Cut, Completeness).** *For $A \neq \mathbf{1}$,*

**Initial and Dereliction rules**

$$\frac{}{\Gamma \mathbin{;} A \Longrightarrow A}\ \text{init} \qquad \frac{\Gamma, A \mathbin{;} \Delta, A \Longrightarrow C}{\Gamma, A \mathbin{;} \Delta \Longrightarrow C}\ \text{dl}$$

**Multiplicative connectives**

$$\frac{\Gamma \mathbin{;} \Delta_1 \Longrightarrow A \quad \Gamma \mathbin{;} \Delta_2, B \Longrightarrow C \quad B \neq \mathbf{1}}{\Gamma \mathbin{;} \Delta_1, \Delta_2, A \multimap B \Longrightarrow C}\ {\multimap}L$$

$$\frac{\Gamma \mathbin{;} \Delta_1 \Longrightarrow A \quad \Gamma \mathbin{;} \Delta_2 \Longrightarrow C}{\Gamma \mathbin{;} \Delta_1, \Delta_2, A \multimap \mathbf{1} \Longrightarrow C}\ {\multimap}\mathbf{1}L \qquad \frac{\Gamma \mathbin{;} \Delta, A \Longrightarrow B}{\Gamma \mathbin{;} \Delta \Longrightarrow A \multimap B}\ {\multimap}R$$

$$\frac{\Gamma \mathbin{;} \Delta, A, B \Longrightarrow C}{\Gamma \mathbin{;} \Delta, A \otimes B \Longrightarrow C}\ {\otimes}L \qquad \frac{\Gamma \mathbin{;} \Delta_1 \Longrightarrow A \quad \Gamma \mathbin{;} \Delta_2 \Longrightarrow B}{\Gamma \mathbin{;} \Delta_1, \Delta_2 \Longrightarrow A \otimes B}\ {\otimes}R \qquad \frac{}{\Gamma \mathbin{;} \cdot \Longrightarrow \mathbf{1}}\ \mathbf{1}R$$

**Additive connectives**

$$\frac{\Gamma \mathbin{;} \Delta, A \Longrightarrow C \quad A \neq \mathbf{1}}{\Gamma \mathbin{;} \Delta, A \mathbin{\&} B \Longrightarrow C}\ \&L_1 \qquad \frac{\Gamma \mathbin{;} \Delta, B \Longrightarrow C \quad B \neq \mathbf{1}}{\Gamma \mathbin{;} \Delta, A \mathbin{\&} B \Longrightarrow C}\ \&L_2$$

$$\frac{\Gamma \mathbin{;} \Delta \Longrightarrow C}{\Gamma \mathbin{;} \Delta, A \mathbin{\&} \mathbf{1} \Longrightarrow C}\ \&\mathbf{1}L \qquad \frac{\Gamma \mathbin{;} \Delta \Longrightarrow C}{\Gamma \mathbin{;} \Delta, \mathbf{1} \mathbin{\&} A \Longrightarrow C}\ \mathbf{1} \mathbin{\&} L$$

$$\frac{\Gamma \mathbin{;} \Delta \Longrightarrow A \quad \Gamma \mathbin{;} \Delta \Longrightarrow B}{\Gamma \mathbin{;} \Delta \Longrightarrow A \mathbin{\&} B}\ \&R \qquad \frac{}{\Gamma \mathbin{;} \Delta \Longrightarrow \top}\ \top R$$

**Exponentials**

$$\frac{\Gamma, A \mathbin{;} \Delta \Longrightarrow C}{\Gamma \mathbin{;} \Delta, !A \Longrightarrow C}\ !L \qquad \frac{\Gamma \mathbin{;} \cdot \Longrightarrow A}{\Gamma \mathbin{;} \cdot \Longrightarrow !A}\ !R$$

**Fig. 3.** Inference rules for the backward calculus without $\mathbf{1}L$.

1. *If $\Gamma \mathbin{;} \Delta_1 \Longrightarrow A$ and $\Gamma \mathbin{;} \Delta_2, A \Longrightarrow C$, then $\Gamma \mathbin{;} \Delta_1, \Delta_2 \Longrightarrow C$.*
2. *If $\Gamma \mathbin{;} \Delta_1 \Longrightarrow \mathbf{1}$ and $\Gamma \mathbin{;} \Delta_2 \Longrightarrow C$, then $\Gamma \mathbin{;} \Delta_1, \Delta_2 \Longrightarrow C$.*
3. *If $\Gamma \mathbin{;} \cdot \Longrightarrow A$ and $\Gamma, A \mathbin{;} \Delta_2 \Longrightarrow C$, then $\Gamma \mathbin{;} \Delta_2 \Longrightarrow C$.*
4. *If $\Gamma \mathbin{;} \Delta \Longrightarrow C$ is in $\mathbf{1}$**NF** and it is provable in the original backward calculus, then it is provable in the modified calculus.*

### 4.2 The Affine Context

Although we have removed the $\mathbf{1}L$ rule, we cannot directly adapt the calculus for the forward direction because the $\&\mathbf{1}L$ and $\mathbf{1}\&L$ (and no other) rules have "undetermined use" introductions of resources similar to $\mathbf{1}L$. We will add such resources into an affine context, $\Psi$, which will admit a structural weakening theorem (Thm.6). Our sequents now have the shape:

$$\Gamma \mathbin{;} \Psi \mathbin{;} \Delta \Longrightarrow C$$

**Initial, Dereliction and Crystallisation rules**

$$\frac{}{\Gamma \mathbin{;} \Psi \mathbin{;} A \Longrightarrow A}\ \text{init} \qquad \frac{\Gamma, A \mathbin{;} \Psi \mathbin{;} \Delta, A \Longrightarrow C}{\Gamma, A \mathbin{;} \Psi \mathbin{;} \Delta \Longrightarrow C}\ \text{dl} \qquad \frac{\Gamma \mathbin{;} \Psi \mathbin{;} \Delta, A \Longrightarrow C}{\Gamma \mathbin{;} \Psi, A \mathbin{;} \Delta \Longrightarrow C}\ \text{dl}_1$$

$$\frac{\Gamma \mathbin{;} \Psi, A \mathbin{;} \Delta \Longrightarrow C}{\Gamma \mathbin{;} \Psi \mathbin{;} \Delta, A \mathbin{\&} \mathbf{1} \Longrightarrow C}\ \text{cryst}_1 \qquad \frac{\Gamma \mathbin{;} \Psi, A \mathbin{;} \Delta \Longrightarrow C}{\Gamma \mathbin{;} \Psi \mathbin{;} \Delta, \mathbf{1} \mathbin{\&} A \Longrightarrow C}\ \text{cryst}_2$$

**Multiplicative connectives**

$$\frac{\Gamma \mathbin{;} \Psi_1 \mathbin{;} \Delta_1 \Longrightarrow A \quad \Gamma \mathbin{;} \Psi_2 \mathbin{;} \Delta_2, B \Longrightarrow C \quad B \neq \mathbf{1}}{\Gamma \mathbin{;} \Psi_1, \Psi_2 \mathbin{;} \Delta_1, \Delta_2, A \multimap B \Longrightarrow C}\ {\multimap}L$$

$$\frac{\Gamma \mathbin{;} \Psi_1 \mathbin{;} \Delta_1 \Longrightarrow A \quad \Gamma \mathbin{;} \Psi_2 \mathbin{;} \Delta_2 \Longrightarrow C}{\Gamma \mathbin{;} \Psi_1, \Psi_2 \mathbin{;} \Delta_1, \Delta_2, A \multimap \mathbf{1} \Longrightarrow C}\ {\multimap}\mathbf{1}L \qquad \frac{\Gamma \mathbin{;} \Psi \mathbin{;} \Delta, A \Longrightarrow B}{\Gamma \mathbin{;} \Psi \mathbin{;} \Delta \Longrightarrow A \multimap B}\ {\multimap}R$$

$$\frac{\Gamma \mathbin{;} \Psi \mathbin{;} \Delta, A, B \Longrightarrow C}{\Gamma \mathbin{;} \Psi \mathbin{;} \Delta, A \otimes B \Longrightarrow C}\ {\otimes}L \qquad \frac{\Gamma \mathbin{;} \Psi_1 \mathbin{;} \Delta_1 \Longrightarrow A \quad \Gamma \mathbin{;} \Psi_2 \mathbin{;} \Delta_2 \Longrightarrow B}{\Gamma \mathbin{;} \Psi_1, \Psi_2 \mathbin{;} \Delta_1, \Delta_2 \Longrightarrow A \otimes B}\ {\otimes}R$$

$$\frac{}{\Gamma \mathbin{;} \Psi \mathbin{;} \cdot \Longrightarrow \mathbf{1}}\ \mathbf{1}R$$

**Additive connectives**

$$\frac{\Gamma \mathbin{;} \Psi \mathbin{;} \Delta, A \Longrightarrow C \quad A \neq \mathbf{1}}{\Gamma \mathbin{;} \Psi \mathbin{;} \Delta, A \mathbin{\&} B \Longrightarrow C}\ {\&}L_1 \qquad \frac{\Gamma \mathbin{;} \Psi \mathbin{;} \Delta, B \Longrightarrow C \quad B \neq \mathbf{1}}{\Gamma \mathbin{;} \Psi \mathbin{;} \Delta, A \mathbin{\&} B \Longrightarrow C}\ {\&}L_2$$

$$\frac{\Gamma \mathbin{;} \Psi \mathbin{;} \Delta \Longrightarrow A \quad \Gamma \mathbin{;} \Psi \mathbin{;} \Delta \Longrightarrow B}{\Gamma \mathbin{;} \Psi \mathbin{;} \Delta \Longrightarrow A \mathbin{\&} B}\ {\&}R \qquad \frac{}{\Gamma \mathbin{;} \Psi \mathbin{;} \Delta \Longrightarrow \top}\ {\top}R$$

**Exponentials**

$$\frac{\Gamma, A \mathbin{;} \Psi \mathbin{;} \Delta \Longrightarrow C}{\Gamma \mathbin{;} \Psi \mathbin{;} \Delta, !A \Longrightarrow C}\ !L \qquad \frac{\Gamma \mathbin{;} \cdot \mathbin{;} \cdot \Longrightarrow A}{\Gamma \mathbin{;} \cdot \mathbin{;} \cdot \Longrightarrow !A}\ !R$$

**Fig. 4.** The backward sequent calculus with affine context

Resources in the affine context $\Psi$ may be transferred into the linear context by wrapping them in $\&\mathbf{1}$ or $\mathbf{1}\&$; these are written as a pair of *crystallisation* rules:

$$\frac{\Gamma \mathbin{;} \Psi, A \mathbin{;} \Delta \Longrightarrow C}{\Gamma \mathbin{;} \Psi \mathbin{;} \Delta, A \mathbin{\&} \mathbf{1} \Longrightarrow C}\ \text{cryst}_1 \qquad \frac{\Gamma \mathbin{;} \Psi, A \mathbin{;} \Delta \Longrightarrow C}{\Gamma \mathbin{;} \Psi \mathbin{;} \Delta, \mathbf{1} \mathbin{\&} A \Longrightarrow C}\ \text{cryst}_2$$

Furthermore, linear resources may be transferred into the affine context because any proof that uses a resource linearly also uses that resource affinely:

$$\frac{\Gamma \mathbin{;} \Psi \mathbin{;} \Delta, A \Longrightarrow C}{\Gamma \mathbin{;} \Psi, A \mathbin{;} \Delta \Longrightarrow C}\ \text{dl}_1$$

It is of course unsound to continue to carry the resource $A$ in the affine context in the premiss, unlike the dereliction rule for the unrestricted resources. If we need a new "zero use" copy of the resource, we make use of the admissible structural rule of weakening the affine context (Thm. 6 below). The rules for the backward calculus with the affine context are in Fig. 4.

Given an affine context $\Psi$, we will write $\Psi \mathbin{\&} \mathbf{1}$ for that context of resources formed by replacing each resource $A$ in $\Psi$ with $A \mathbin{\&} \mathbf{1}$. The characteristic structural property of the affine sequent is the admissibility of weakening.

**Theorem 6 (Weakening the affine context).** *If $\Gamma \mathbin{;} \Psi \mathbin{;} \Delta \Longrightarrow C$ then $\Gamma' \mathbin{;} \Psi' \mathbin{;} \Delta \Longrightarrow C$ for any $\Gamma' \supseteq \Gamma$ and $\Psi' \supseteq \Psi$.*

*Proof.* Structural Induction on the derivation of $\Gamma \mathbin{;} \Psi \mathbin{;} \Delta \Longrightarrow C$.

**Theorem 7 (Cut).** *For $A \neq \mathbf{1}$,*

1. *If $\Gamma \mathbin{;} \Psi_1 \mathbin{;} \Delta_1 \Longrightarrow A$ and $\Gamma \mathbin{;} \Psi_2 \mathbin{;} \Delta_2, A \Longrightarrow C$, then $\Gamma \mathbin{;} \Psi_1, \Psi_2 \mathbin{;} \Delta_1, \Delta_2 \Longrightarrow C$.*
2. *If $\Gamma \mathbin{;} \Psi_1 \mathbin{;} \Delta_1 \Longrightarrow \mathbf{1}$ and $\Gamma \mathbin{;} \Psi_2 \mathbin{;} \Delta_2 \Longrightarrow C$, then $\Gamma \mathbin{;} \Psi_1, \Psi_2 \mathbin{;} \Delta_1, \Delta_2 \Longrightarrow C$.*
3. *If $\Gamma \mathbin{;} \Psi_1 \mathbin{;} \cdot \Longrightarrow A$ and $\Gamma \mathbin{;} \Psi_2, A \mathbin{;} \Delta_2 \Longrightarrow C$, then $\Gamma \mathbin{;} \Psi_1, \Psi_2 \mathbin{;} \Delta_2 \Longrightarrow C$.*
4. *If $\Gamma \mathbin{;} \cdot \mathbin{;} \cdot \Longrightarrow A$ and $\Gamma, A \mathbin{;} \Psi_2 \mathbin{;} \Delta_2 \Longrightarrow C$, then $\Gamma \mathbin{;} \Psi_2 \mathbin{;} \Delta_2 \Longrightarrow C$.*

*Proof.* Similar to theorems 2 and 5.

Soundness and completeness with respect to the extended backward calculus of Sec.4.1 are fairly straightforward.

**Theorem 8 (Soundness and Completeness).**

1. *If $\Gamma \mathbin{;} \Psi \mathbin{;} \Delta \Longrightarrow C$, then $\Gamma \mathbin{;} \Psi \mathbin{\&} \mathbf{1}, \Delta \Longrightarrow C$.*
2. *If $\Gamma \mathbin{;} \Delta \Longrightarrow C$, then for any $\Psi$, $\Gamma \mathbin{;} \Psi \mathbin{;} \Delta \Longrightarrow C$.*

*Proof.* By structural induction on the respective derivations.

### 4.3 The Forward Selection Calculus

We will now extend the first forward sequent calculus (Sec.3) to remove the $\mathbf{1}L$ rule, using a similar approach as the previous section. We extend the forward sequent with an affine context:

$$\Gamma \mathbin{;} \Psi \mathbin{;} \Delta \longrightarrow^{w} C$$

Particular care is needed for the crystallisation rules. For illustration, this selection of rules would contain unacceptable non-determinism:

$$\frac{\Gamma \mathbin{;} \Psi, A \mathbin{;} \Delta \longrightarrow^{w} C}{\Gamma \mathbin{;} \Psi \mathbin{;} \Delta, A \mathbin{\&} \mathbf{1} \longrightarrow^{w} C}\ \mathrm{cryst}_1 \qquad \frac{\Gamma \mathbin{;} \Psi \mathbin{;} \Delta \longrightarrow^{w} C \quad A \notin \Psi}{\Gamma \mathbin{;} \Psi \mathbin{;} \Delta, A \mathbin{\&} \mathbf{1} \longrightarrow^{w} C}\ \mathrm{cryst}'_1$$

$$\frac{\Gamma \mathbin{;} \Psi, A \mathbin{;} \Delta \longrightarrow^{w} C}{\Gamma \mathbin{;} \Psi \mathbin{;} \Delta, \mathbf{1} \mathbin{\&} A \longrightarrow^{w} C}\ \mathrm{cryst}_2 \qquad \frac{\Gamma \mathbin{;} \Psi \mathbin{;} \Delta \longrightarrow^{w} C \quad A \notin \Psi}{\Gamma \mathbin{;} \Psi \mathbin{;} \Delta, \mathbf{1} \mathbin{\&} A \longrightarrow^{w} C}\ \mathrm{cryst}'_2$$

The reason is that there is no control on the application of $\mathrm{cryst}'_1$ and $\mathrm{cryst}'_2$. In fact, a similar problem existed in already with regard to the $!L_1$ rule, which is uncontrolled in this form:

$$\frac{\Gamma \mathbin{;} \Psi \mathbin{;} \Delta \longrightarrow^{0} C \quad A \notin \Gamma}{\Gamma \mathbin{;} \Psi \mathbin{;} \Delta, !A \longrightarrow^{0} C}\ !L_1$$

We will solve these problems by uniting all rules that require membership of resources in contexts using a *selection* judgement, which will perform crystallisation and $\mathbf{1}L$ implicitly. We will write this selection judgement as:

$$(\Gamma \mathbin{;} \Psi \mathbin{;} \Delta) \rhd (\Gamma' \mathbin{;} \Psi' \mathbin{;} \Delta') \mid \Upsilon$$

read "the resources $\Upsilon$ are selected from the contexts $\Gamma ; \Psi ; \Delta$, leaving $\Gamma' ; \Psi' ; \Delta'$". $\Upsilon$ is a multiset of resources, just like $\Delta$. We will use the meta variable $\Omega$ to abbreviate $\Gamma ; \Psi ; \Delta$. The rules for this selection judgement are as follows:

$$\frac{}{\Omega \rhd \Omega \mid \cdot} \rhd \text{triv} \qquad \frac{\Gamma ; \Psi ; \Delta \rhd \Omega \mid \Upsilon}{(\Gamma ; \Psi ; \Delta, A) \rhd \Omega \mid \Upsilon, A} \rhd \text{linear}$$

$$\frac{(\Gamma ; \Psi ; \Delta) \rhd \Omega \mid \Upsilon}{(\Gamma ; \Psi, A ; \Delta) \rhd \Omega \mid \Upsilon, A \mathbin{\&} \mathbf{1}} \rhd \text{aff1} \qquad \frac{(\Gamma ; \Psi ; \Delta) \rhd \Omega \mid \Upsilon}{(\Gamma ; \Psi, A ; \Delta) \rhd \Omega \mid \Upsilon, \mathbf{1} \mathbin{\&} A} \rhd \text{aff2}$$

$$\frac{(\Gamma ; \Psi ; \Delta) \rhd \Omega \mid \Upsilon \quad A \notin \Psi}{(\Gamma ; \Psi ; \Delta) \rhd \Omega \mid \Upsilon, A \mathbin{\&} \mathbf{1}} \rhd \text{aff3} \qquad \frac{(\Gamma ; \Psi ; \Delta) \rhd \Omega \mid \Upsilon \quad A \notin \Psi}{(\Gamma ; \Psi ; \Delta) \rhd \Omega \mid \Upsilon, \mathbf{1} \mathbin{\&} A} \rhd \text{aff4}$$

$$\frac{(\Gamma ; \Psi ; \Delta) \rhd \Omega \mid \Upsilon}{(\Gamma, A ; \Psi ; \Delta) \rhd \Omega \mid \Upsilon, !A} \rhd !1 \qquad \frac{(\Gamma ; \Psi ; \Delta) \rhd \Omega \mid \Upsilon \quad A \notin \Gamma}{(\Gamma ; \Psi ; \Delta) \rhd \Omega \mid \Upsilon, !A} \rhd !2$$

These rules are *not* intended to be read in the forward direction. Instead, they define a bottom-up match for queries of the context $\Upsilon$. If for $\Omega$ and $\Upsilon$, we cannot establish $\Omega \rhd \Omega' \mid \Upsilon$ for any $\Omega'$, then we write $\Omega \not\rhd \Upsilon$.

**Theorem 9 (Selection).** *If $\Omega \Longrightarrow C$ and $\Omega \rhd (\Gamma ; \Psi ; \Delta) \mid \Upsilon$, then $\Gamma ; \Psi ; \Delta, \Upsilon \Longrightarrow C$.*

*Proof.* Structural induction of the selection $\Omega \rhd (\Gamma ; \Psi ; \Delta) \mid \Upsilon$, using theorems 6 and 1 as necessary.

Like with $\Psi \mathbin{\&} \mathbf{1}$, we write $!\Gamma$ for the context formed by replacing every resource $A$ in $\Gamma$ with $!A$. The following are some simple and useful properties of selection

**Lemma 1 (Characterising Selection).** *If $(\Gamma ; \Psi ; \Delta) \rhd (\Gamma' ; \Psi' ; \Delta') \mid \Upsilon$ then*

1. *$\Gamma' \subseteq \Gamma$, $\Psi' \subseteq \Psi$ and $\Delta' \subseteq \Delta$, and*
2. *$!(\Gamma \backslash \Gamma'), (\Psi \backslash \Psi') \mathbin{\&} \mathbf{1}, (\Delta \backslash \Delta') \subseteq \Upsilon$.*

*Proof.* By straightforward structural induction on the derivation of $(\Gamma ; \Psi ; \Delta) \rhd (\Gamma' ; \Psi' ; \Delta') \mid \Upsilon$.

**Lemma 2 (Weakening a Selection).** *If $(\Gamma_1 ; \Psi_1 ; \Delta_1) \rhd (\Gamma_2 ; \Psi_2 ; \Delta_2) \mid \Upsilon$, then for any $\Delta$, we have $(\Gamma_1 ; \Psi_1 ; \Delta_1, \Delta) \rhd (\Gamma_2 ; \Psi_2 ; \Delta_2, \Delta) \mid \Upsilon$*

*Proof.* By straightforward induction and lemma 1.

Using selection, we can now successfully remove the $\mathbf{1}L$ rule from the forward calculus. Our goal is the following correspondence:

$$\begin{array}{lll} \Gamma \,;\, \Psi \,;\, \Delta \longrightarrow^0 C & \text{corresponds to} & \Gamma' \,;\, \Psi', \Delta' \Longrightarrow C \\ & \multicolumn{2}{l}{\text{for } \Delta' \supseteq \Delta \text{ where elements of } \Delta' \backslash \Delta} \\ & \multicolumn{2}{l}{\text{are of the form } A \,\&\, \mathbf{1} \text{ or } \mathbf{1} \,\&\, A \text{ or } !A} \\ \Gamma \,;\, \Psi \,;\, \Delta \longrightarrow^1 C & \text{corresponds to} & \Gamma' \,;\, \Psi', \Delta' \Longrightarrow C \text{ for any } \Delta' \supseteq \Delta \end{array}$$

for any $\Psi' \supseteq \Psi$ and $\Gamma' \supseteq \Gamma$. The full set of rules is shown in Fig. 5.

### 4.4 Soundness and Completeness

Soundness for the forward selection calculus is not much harder than that for the forward calculus of Sec.3.

**Theorem 10 (Soundness).**

1. *If $\Gamma \,;\, \Psi \,;\, \Delta \longrightarrow^0 C$, then $\Gamma \,;\, \Psi \,;\, \Delta \Longrightarrow C$*
2. *If $\Gamma \,;\, \Psi \,;\, \Delta \longrightarrow^1 C$, then $\Gamma \,;\, \Psi \,;\, \Delta' \Longrightarrow C$ for any $\Delta' \supseteq \Delta$.*

*Proof.* Structural induction on the derivation of $\Gamma \,;\, \Psi \,;\, \Delta \longrightarrow^w C$, using Lem. 2 and Thm. 1 as necessary.

Completeness cannot be proven directly like for Thm. 4. Instead, the induction hypothesis must be strengthened.

**Theorem 11 (Completeness).** *If $\Gamma \,;\, \Psi \,;\, \Delta \Longrightarrow C$, then one of the following is true:*

1. *either $\Gamma_1 \,;\, \Psi_1 \,;\, \Delta, !\Gamma_2, \Psi_2 \,\&\, \mathbf{1} \longrightarrow^0 C$,*
2. *or $\Gamma_1 \,;\, \Psi_1 \,;\, \Delta', !\Gamma_2, \Psi_2 \,\&\, \mathbf{1} \longrightarrow^1 C$ for some $\Delta' \subseteq \Delta$,*

*for some $\Gamma_1 \cup \Gamma_2 \subseteq \Gamma$ and $\Psi_1, \Psi_2 \subseteq \Psi$.*

*Proof.* Structural induction on the derivation of $\Gamma \,;\, \Psi \,;\, \Delta \Longrightarrow C$.

## 5 Conclusion

We have presented a forward sequent calculus for the multiplicative-additive-exponential fragment of linear logic (Sec.4.3). Our calculus has the following properties from the perspective of resource management:

- No "zero use" resources. We identify sequents whose linear context is subject to weakening, and introduce such resources implicitly.
- Controlled "undetermined use" resources. All resource introductions are structural, and controlled by tight selection criteria.

**Initial, Dereliction**

$$\dfrac{}{\bullet \fatsemi \bullet \fatsemi A \longrightarrow^{0} A}\;\text{init}$$

$$\dfrac{\Omega \,\triangleright\, (\Gamma \fatsemi \Psi \fatsemi \Delta) \mid A \quad \Omega \longrightarrow^{w} C}{\Gamma, A \fatsemi \Psi \fatsemi \Delta \longrightarrow^{w} C}\;\text{dl} \qquad \dfrac{\Omega \,\triangleright\, (\Gamma \fatsemi \Psi \fatsemi \Delta) \mid A \quad \Omega \longrightarrow^{w} C}{\Gamma \fatsemi \Psi, A \fatsemi \Delta \longrightarrow^{w} C}\;\text{dl}_1$$

**Multiplicative connectives**

$$\dfrac{\Gamma_1 \fatsemi \Psi_1 \fatsemi \Delta_1 \longrightarrow^{w_1} A \quad \Omega \,\triangleright\, (\Gamma_2 \fatsemi \Psi_2 \fatsemi \Delta_2) \mid B \quad \Omega \longrightarrow^{w_2} C}{\Gamma_1 \cup \Gamma_2 \fatsemi \Psi_1, \Psi_2 \fatsemi \Delta_1, \Delta_2, A \multimap B \longrightarrow^{w_1 \vee w_2} C}\;{\multimap}L$$

$$\dfrac{\Gamma_1 \fatsemi \Psi_1 \fatsemi \Delta_1 \longrightarrow^{w_1} A \quad \Gamma_2 \fatsemi \Psi_2 \fatsemi \Delta_2 \longrightarrow^{w_2} C}{\Gamma_1 \cup \Gamma_2 \fatsemi \Psi_1, \Psi_2 \fatsemi \Delta_1, \Delta_2, A \multimap \mathbf{1} \longrightarrow^{w_1 \vee w_2} C}\;{\multimap}\mathbf{1}L$$

$$\dfrac{\Omega \,\triangleright\, (\Gamma \fatsemi \Psi \fatsemi \Delta) \mid A \quad \Omega \longrightarrow^{w} C}{\Gamma \fatsemi \Psi \fatsemi \Delta \longrightarrow^{w} A \multimap C}\;{\multimap}R \qquad \dfrac{\Omega \longrightarrow^{1} C \quad \Omega \ntriangleright A}{\Omega \longrightarrow^{1} A \multimap C}\;{\multimap}R_1$$

$$\dfrac{\Omega \,\triangleright\, (\Gamma \fatsemi \Psi \fatsemi \Delta) \mid A, B \quad \Omega \longrightarrow^{w} C}{\Gamma \fatsemi \Psi \fatsemi \Delta, A \otimes B \longrightarrow^{w} C}\;{\otimes}L$$

$$\dfrac{\Omega \,\triangleright\, (\Gamma \fatsemi \Psi \fatsemi \Delta) \mid B \quad \Omega \ntriangleright A \quad \Omega \longrightarrow^{1} C}{\Gamma \fatsemi \Psi \fatsemi \Delta, A \otimes B \longrightarrow^{1} C}\;{\otimes}L_1$$

$$\dfrac{\Omega \,\triangleright\, (\Gamma \fatsemi \Psi \fatsemi \Delta) \mid A \quad \Omega \ntriangleright B \quad \Omega \longrightarrow^{1} C}{\Gamma \fatsemi \Psi \fatsemi \Delta, A \otimes B \longrightarrow^{1} C}\;{\otimes}L_2$$

$$\dfrac{\Gamma_1 \fatsemi \Psi_1 \fatsemi \Delta_1 \longrightarrow^{w_1} A \quad \Gamma_2 \fatsemi \Psi_2 \fatsemi \Delta_2 \longrightarrow^{w_2} B}{\Gamma_1 \cup \Gamma_2 \fatsemi \Psi_1, \Psi_2 \fatsemi \Delta_1, \Delta_2 \longrightarrow^{w_1 \vee w_2} A \otimes B}\;{\otimes}R \qquad \dfrac{}{\bullet \fatsemi \bullet \fatsemi \bullet \longrightarrow^{0} \mathbf{1}}\;\mathbf{1}R$$

**Additive connectives**

$$\dfrac{\Omega \,\triangleright\, (\Gamma \fatsemi \Psi \fatsemi \Delta) \mid A \quad \Omega \longrightarrow^{w} C \quad B \neq \mathbf{1}}{\Gamma \fatsemi \Psi \fatsemi \Delta, A \,\&\, B \longrightarrow^{w} C}\;\&L_1$$

$$\dfrac{\Omega \,\triangleright\, (\Gamma \fatsemi \Psi \fatsemi \Delta) \mid B \quad \Omega \longrightarrow^{w} C \quad A \neq \mathbf{1}}{\Gamma \fatsemi \Psi \fatsemi \Delta, A \,\&\, B \longrightarrow^{w} C}\;\&L_2$$

$$\dfrac{\Gamma_1 \fatsemi \Psi_2 \fatsemi \Delta_1 \longrightarrow^{w_2} A \quad \Gamma_2 \fatsemi \Psi_2 \fatsemi \Delta_2 \longrightarrow^{w_2} B}{\Gamma_1 \cup \Gamma_2 \fatsemi \Psi_1 \cup \Psi_2 \fatsemi \Delta_1 \overset{w_1, w_2}{\cup} \Delta_2 \longrightarrow^{w_1 \wedge w_2} A \,\&\, B}\;\&R \qquad \dfrac{}{\bullet \fatsemi \bullet \fatsemi \bullet \longrightarrow^{1} \top}\;\top R$$

**Exponentials**

$$\dfrac{\Gamma \fatsemi \bullet \fatsemi \bullet \longrightarrow^{0} A}{\Gamma \fatsemi \bullet \fatsemi \bullet \longrightarrow^{0} {!A}}\;{!R}$$

**Fig. 5.** The forward selection calculus

We believe our framework to be sufficiently general that some extensions can be made readily. As remarked earlier, external choice and its unit $\mathbf{0}$ do not present any significant challenges. $\mathbf{0}$ on the left behaves like $\top$ on the right, so sequents which have $\mathbf{0}$ as a resource will be weak. This will require the right hand side of sequents to be possibly empty, so sequents will have the shape: $\Gamma \,;\, \Psi \,;\, \Delta \longrightarrow^{w} \gamma$, where $\gamma$ is either $\cdot$ or a formula $C$. The $\mathbf{0}L$ rule then becomes simply:

$$\frac{}{\cdot \,;\, \cdot \,;\, \mathbf{0} \longrightarrow^{1} \cdot}\ \mathbf{0}L$$

Extending the calculus to first order connectives requires relaxing equalities to unification. It is relatively straightforward to extend the rules to the first order case, but the negative existence conditions in the selection rules $\triangleright$ aff3, $\triangleright$ aff4 and $\triangleright$ !2 require special consideration.

In order to complete a practical implemention of an inverse method theorem prover that uses this forward calculus, the two most important requirements are (1) a version of the sequent calculus that incorporates focused derivations in the sense of Andreoli [1, 8], and (2) an efficient indexing mechanism. Focused derivations impose further controls on rule application, allowing the creation of big-step derived inference rules, and thereby cuts down on the number of new sequents. We are presently examining the interactions between inversion, focusing, and selection.

## References


1. Jean Marc Andreoli. Logic programming with focussing proofs in linear logic. *Journal of Logic and Computation*, 2, 1992.
2. Andrew Barber and Gordon Plotkin. Dual intuitionistic linear logic. Technical Report ECS-LFCS-96-347, University of Edinburgh, 1996.
3. Marco Bozzano. *A Logic-Based Approach to Model Checking of Parameterized and Infinite-State Systems*. PhD thesis, DISI, Università di Genova, 2002.
4. Iliano Cervesato, Joshua S. Hodas, and Frank Pfenning. Efficient resource management for linear logic proof search. In R. Dyckhoff, H. Herre, and P. Schroeder-Heister, editors, *Proceedings of the 5th International Workshop on Extensions of Logic Programming*, pages 67–81, Leipzig, Germany, March 1996. Springer-Verlag LNAI 1050.
5. J-Y. Girard. Linear logic. *Theoretical Computer Science*, 50:1–102, 1987.
6. James Harland and David J. Pym. Resource-distribution via boolean constraints. In W. McCune, editor, *Proceedings of CADE-14*, pages 222–236, Townsville, Australia, July 1997. Springer-Verlag LNAI 1249.
7. Joshua S. Hodas. Lolli: an extension of $\lambda$Prolog with linear logic context management. In Dale Miller, editor, *Proceedings of the 1992 workshop on the $\lambda$Prolog programming language*, Philadelphia, 1992.
8. Jakob M. Howe. *Proof search issues in some non-classical logics*. PhD thesis, University of St. Andrews, September 1999.
9. S. Maslov. The inverse method of establishing deducibility in the classical predicate calculus. *Soviet Mathematical Doklady*, 5:1420–1424, 1964.

10. Grigori Mints. Resolution calculus for the first order linear logic. *Journal of Logic, Language and Information*, 2(1):59–83, 1993.
11. Frank Pfenning. Structural cut elimination in linear logic. Technical Report CMU-CS-94-222, Carnegie Mellon University, December 1994.
12. Tannel Tammet. Proof strategies in linear logic. *Journal of Automated Reasoning*, 12(3):273–304, 1994.
13. Andrei Voronkov. Theorem proving in non-standard logics based on the inverse method. In D. Kapur, editor, *Proceedings of the CADE-11*, pages 648–662, Saratoga Springs, New York, 1992. Springer-Verlag LNCS 607.
14. H. Winikoff and James Harland. Implementing the linear logic programming language Lygon. In *Proceedings of the International Logic Programming Symposium (ILPS)*, pages 66–80, December 1995.

## A. Proofs

We will now present the proofs of the theorems 3, 4, 5, 10, and 11.

*Proof (Thm. 3).* By induction on the structure of the derivation $\mathcal{D}$ of $\Gamma \fatsemi \Delta \longrightarrow^{w} C$. We have the following characteristic cases:

<u>*Case*</u>. dl, i.e.,

$$\mathcal{D} \quad = \quad \dfrac{\overset{\mathcal{D}_1}{\Gamma \fatsemi \Delta, A \longrightarrow^{w} C}}{\Gamma, A \fatsemi \Delta \longrightarrow^{w} C}\ \text{dl}$$

$$\begin{array}{rlll}
w = 0: & (1) & \Gamma \fatsemi \Delta, A \Longrightarrow C & \text{IH: } \mathcal{D}_1 \\
 & (2) & \Gamma, A \fatsemi \Delta, A \Longrightarrow C & \text{weak: 1} \\
 & (3) & \Gamma, A \fatsemi \Delta \Longrightarrow C & \text{dl: 2} \\
w = 1: & \multicolumn{3}{l}{\text{for any } \Delta', A \supseteq \Delta, A,} \\
 & (1) & \Gamma \fatsemi \Delta', A \Longrightarrow C & \text{IH: } \mathcal{D}_1, \\
 & (2) & \Gamma, A \fatsemi \Delta', A \Longrightarrow C & \text{weak: 1} \\
 & (3) & \Gamma, A \fatsemi \Delta' \Longrightarrow C & \text{dl: 2}
\end{array}$$

<u>*Case*</u>. rules that require the weakened form of case (2) for the induction to hold, for example $\multimap R_1$:

$$\mathcal{D} \quad = \quad \dfrac{\overset{\mathcal{D}_1}{\Gamma \fatsemi \Delta \longrightarrow^{1} C} \quad A \notin \Delta}{\Gamma \fatsemi \Delta \longrightarrow^{1} A \multimap C}\ \multimap R_1$$

Given $\Delta' \supseteq \Delta$, we have:

$$\begin{array}{lll}
(1) & \Gamma \fatsemi \Delta', A \Longrightarrow C & \text{IH: } \mathcal{D}_1 \\
(2) & \Gamma \fatsemi \Delta' \Longrightarrow A \multimap C & \multimap R\text{: 1}
\end{array}$$

<u>*Case*</u>. $\multimap L$, i.e.,

$$\mathcal{D} \quad = \quad \dfrac{\overset{\mathcal{D}_1}{\Gamma_1 \fatsemi \Delta_1 \longrightarrow^{w_1} A} \quad \overset{\mathcal{D}_2}{\Gamma_2 \fatsemi \Delta_2, B \longrightarrow^{w_2} C}}{\Gamma_1 \cup \Gamma_2 \fatsemi \Delta_1, \Delta_2, A \multimap B \longrightarrow^{w_1 \vee w_2} C}\ \multimap L$$

$$\begin{array}{rlll}
w_1 \vee w_2 = 0: & (1) & \Gamma_1 \fatsemi \Delta_1 \Longrightarrow A & \text{IH: } \mathcal{D}_1 \\
 & (2) & \Gamma_2 \fatsemi \Delta_2, B \Longrightarrow C & \text{IH: } \mathcal{D}_2 \\
 & (3) & \Gamma_1 \cup \Gamma_2 \fatsemi \Delta_1 \Longrightarrow A & \text{weak: 1} \\
 & (4) & \Gamma_1 \cup \Gamma_2 \fatsemi \Delta_2, B \Longrightarrow C & \text{weak 2} \\
 & (5) & \Gamma_1 \cup \Gamma_2 \fatsemi \Delta_1, \Delta_2, A \multimap B \Longrightarrow C & \multimap L\text{: 3, 4} \\
w_1 = 1: & \multicolumn{3}{l}{\text{in which case for any } \Delta_1', \Delta_2, A \multimap B \supseteq \Delta_1, \Delta_2, A \multimap B,} \\
 & (1) & \Gamma_1 \fatsemi \Delta_1' \Longrightarrow A & \text{IH: } \mathcal{D}_1 \\
 & (2) & \Gamma_1 \cup \Gamma_2 \fatsemi \Delta_1' \Longrightarrow A & \text{weak: 1} \\
 & (3) & \Gamma_2 \fatsemi \Delta_2, B \Longrightarrow C & \text{IH: } \mathcal{D}_2 \\
 & (4) & \Gamma_1 \cup \Gamma_2 \fatsemi \Delta_2, B \Longrightarrow C & \text{weak: 3} \\
 & (5) & \Gamma_1 \cup \Gamma_2 \fatsemi \Delta_1', \Delta_2, A \multimap B \Longrightarrow C & \multimap L\text{: 1, 3}
\end{array}$$

Other cases are similar. $\square$

*Proof (Thm. 4).* by induction on the structure of the derivation $\mathcal{D}$ of $\Gamma \mathbin{;} \Delta \Longrightarrow C$. We have the following cases for the right rule of $\mathcal{D}$:

<u>*Case*</u>. init, i.e.,

$$\mathcal{D} \quad = \quad \frac{}{\Gamma \mathbin{;} A \Longrightarrow A}\ \text{init}$$

In this case, $\bullet \mathbin{;} A \longrightarrow^0 A$, so part (1) holds.

<u>*Case*</u>. dl, i.e.,

$$\mathcal{D} \quad = \quad \frac{\begin{array}{c}\mathcal{D}_1\\ \Gamma, A \mathbin{;} \Delta, A \Longrightarrow C\end{array}}{\Gamma, A \mathbin{;} \Delta \Longrightarrow C}\ \text{dl}$$

Applying the induction hypothesis for $\mathcal{D}_1$, we have the following possibilities:

*a.* $\Gamma' \mathbin{;} \Delta, A \longrightarrow^0 C$ for some $\Gamma' \subseteq \Gamma, A$. Then we apply dl, and satisfy part (1) of the theorem.

*b.* $\Gamma' \mathbin{;} \Delta' \longrightarrow^1 C$ for some $\Gamma' \subseteq \Gamma, A$ and $\Delta' \subseteq \Delta, A$. If $A \notin \Delta'$ we satisfy part (2), so assume that $\Delta' = \Delta'', A$. Now, we apply dl to get $\Gamma', A \mathbin{;} \Delta'' \longrightarrow^1 C$, which satisfies part (2).

<u>*Case*</u>. $\multimap L$, i.e.,

$$\mathcal{D} \quad = \quad \frac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \mathbin{;} \Delta_1 \Longrightarrow A\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \Gamma \mathbin{;} \Delta_2, B \Longrightarrow C\end{array}}{\Gamma \mathbin{;} \Delta_1, \Delta_2, A \multimap B \Longrightarrow C}\ \multimap L$$

Here we have:

(1) $\Gamma_1 \mathbin{;} \Delta_1 \longrightarrow^0 A$ and $\Gamma_2 \mathbin{;} \Delta_2, B \longrightarrow^0 C$
assume IH(1): $\mathcal{D}_1$ and $\mathcal{D}_2$
$\Gamma_1$ and $\Gamma_2 \subseteq \Gamma$

(2) $1 \vdash \Gamma_1 \cup \Gamma_2 \mathbin{;} \Delta_1, \Delta_2, A \multimap B \longrightarrow^0 C$
$\multimap L$: 1, satisfies (1)

(3) $\Gamma_1 \mathbin{;} \Delta_1' \longrightarrow^1 A$ and $\Gamma_2 \mathbin{;} \Delta_2, B \longrightarrow^0 C$
assume IH(2): $\mathcal{D}_1$ and IH(1): $\mathcal{D}_2$
for some $\Gamma_1, \Gamma_2 \subseteq \Gamma$ and $\Delta_1' \subseteq \Delta_1$

(4) $3 \vdash \Gamma_1 \cup \Gamma_2 \mathbin{;} \Delta_1', \Delta_2, A \multimap B \longrightarrow^1 C$
$\multimap L$: 3, satisfies (2)

(5) $\Gamma_1 \mathbin{;} \Delta_1 \longrightarrow^0 A$ and $\Gamma_2 \mathbin{;} \Delta_2', B \longrightarrow^1 C$
assume IH(1): $\mathcal{D}_1$ and IH(2): $\mathcal{D}_2$
for some $\Gamma_1, \Gamma_2 \subseteq \Gamma$ and $\Delta_2' \subseteq \Delta_2$

(6) $5 \vdash \Gamma_1 \cup \Gamma_2 \mathbin{;} \Delta_1, \Delta_2', A \multimap B \longrightarrow^1 C$
$\multimap L$: 5, satisfies (2)

(7) $\Gamma_1 \mathbin{;} \Delta_1' \longrightarrow^0 A$ and $\Gamma_2 \mathbin{;} \Delta_2', B \longrightarrow^1 C$
assume IH(1): $\mathcal{D}_1$ and IH(2): $\mathcal{D}_2$

(8) $7 \vdash \Gamma_1 \cup \Gamma_2 \,\text{;}\, \Delta_1', \Delta_2', A \multimap B \longrightarrow^1 C$ — for some $\Delta_1' \subseteq \Delta_1$ and $\Delta_2' \subseteq \Delta_2$ and $\Gamma_1 \cup \Gamma_2 \subseteq \Gamma$; $\multimap L$: 7, satisfies (2)

If none of the assumptions in 1, 3, 5 or 7 hold, then the conclusion in $\multimap L$ would be a weakening of one of the premisses, so part (2) is immediately true of that premiss.

Most of the rules will require a similar enumeration of possibilities.

<u>*Case*</u>. $\multimap R$, i.e.,

$$\mathcal{D} \quad = \quad \frac{\Gamma \,\text{;}\, \Delta, A \Longrightarrow B}{\Gamma \,\text{;}\, \Delta \Longrightarrow A \multimap B} \multimap R$$

If the annotation is not 0 when applying the IH to the premiss, then we use $\multimap R$ or $\multimap R_1$ depending on whether $A$ is in the linear context or not; otherwise we apply $\multimap R$.

<u>*Case*</u>. $\otimes L$, i.e.,

$$\mathcal{D} \quad = \quad \frac{\Gamma \,\text{;}\, \Delta, A, B \Longrightarrow C}{\Gamma \,\text{;}\, \Delta, A \otimes B \Longrightarrow C} \otimes L$$

If the annotation is 0 when applying the IH to the premiss, then we use $\otimes L$. Otherwise we use $\otimes L$, $\otimes L_B$ or $\otimes L_A$ depending on whether both $A$ and $B$, only $A$, or only $B$ are in the linear context, respectively; if neither are in the context, then the premiss itself satisfies part (2).

<u>*Case*</u>. $\otimes R$, i.e.,

$$\mathcal{D} \quad = \quad \frac{\Gamma \,\text{;}\, \Delta_1 \Longrightarrow A \quad \Gamma \,\text{;}\, \Delta_2 \Longrightarrow B}{\Gamma \,\text{;}\, \Delta_1, \Delta_2 \Longrightarrow A \otimes B} \otimes R$$

The conclusion of this rule is structurally weaker than that of the corresponding rule in $\otimes R$, so the result is straightforward.

<u>*Case*</u>. $\mathbf{1}L$ or $\mathbf{1}R$, both of which are trivial.

<u>*Case*</u>. $\& L_1$: this rule is structurally identical to $\& L_1$, so the result holds trivially; similarly with $\& L_2$.

<u>*Case*</u>. $\& R$, i.e.,

$$\mathcal{D} \quad = \quad \frac{\Gamma \,\text{;}\, \Delta \Longrightarrow A \quad \Gamma \,\text{;}\, \Delta \Longrightarrow B}{\Gamma \,\text{;}\, \Delta \Longrightarrow A \,\&\, B} \& R$$

Once again, the conclusion of this rule is weaker than that of the corresponding sequent in $\& R$, so the result holds.

The rules for the exponentials are structurally identical to their forward analogues because the forward rules ignore the weakening annotation. Therefore these cases are trivial. □

*Proof (Thm. 5).* The cut cases (1-3) are similar to Thm.2. For case 4, we proceed by induction on the derivation $\mathcal{D}$ of $\Gamma ; \Delta \Longrightarrow C$ in the original calculus (of Sec.2). We have the following cases:

<u>*Case*</u>. init, dl, etc., which are identical in the two calculi. For example, for dl:

$$\mathcal{D} \quad = \quad \dfrac{\overset{\mathcal{D}'}{\Gamma, A ; \Delta, A \Longrightarrow C}}{\Gamma, A ; \Delta \Longrightarrow C}\ \mathrm{dl}$$

We use the IH on $\mathcal{D}'$ to get $\Gamma, A ; \Delta, A \Longrightarrow C$ in the modified calculus, then use dl in the modified calculus.

<u>*Case*</u>. the final rule in $\mathcal{D}$ is $\multimap L$, i.e.

$$\mathcal{D} \quad = \quad \dfrac{\overset{\mathcal{D}_1}{\Gamma ; \Delta_1 \Longrightarrow A} \quad \overset{\mathcal{D}_2}{\Gamma ; \Delta_2, B \Longrightarrow C}}{\Gamma ; \Delta_1, \Delta_2, A \multimap B \Longrightarrow C}\ \multimap R$$

In this case $A \neq \mathbf{1}$ because otherwise $A \multimap B$ is not in **1NF**. We use the IH for $\mathcal{D}_1$ and $\mathcal{D}_2$, and based on whether $B = \mathbf{1}$ or not, either $\multimap L$ or $\multimap \mathbf{1} L$. Most of the rules require an analysis of this flavour. The remaining interesting cases are for the $\& L$ rules.

<u>*Case*</u>. the final rule in $\mathcal{D}$ is $\& L_1$, i.e.,

$$\mathcal{D} \quad = \quad \dfrac{\overset{\mathcal{D}'}{\Gamma ; \Delta, A \Longrightarrow C}}{\Gamma ; \Delta, A \mathbin{\&} B \Longrightarrow C}\ \& L_1$$

If $A = \mathbf{1}$, then we know from the IH for $D'$ that $\Gamma ; \Delta \Longrightarrow C$, so we use $\mathbf{1} \mathbin{\&} L$. Otherwise, we have $\Gamma ; \Delta, A \Longrightarrow C$, in which case we use $\& L_1$. The case for $\& L_2$ is similar. □

*Proof (Thm. 10).* By induction on the structure of the derivation $\mathcal{D}$ of $\Gamma ; \Psi ; \Delta \longrightarrow^{w} C$, using the selection Thm. 9 as required. All the rules that use selection on premisses require analyses of the same flavour. For illustration, here is the case for the dereliction rules:

$$\mathcal{D} \quad = \quad \dfrac{\Omega \rhd \Gamma ; \Psi ; \Delta \mid A \quad \overset{\mathcal{D}'}{\Omega \longrightarrow^{w} C}}{\Gamma, A ; \Psi ; \Delta \longrightarrow^{w} C}\ \mathrm{dl}$$

If $w = 0$, then by the induction hypothesis for $\mathcal{D}'$ and the selection theorem (9), $\Gamma ; \Psi ; \Delta, A \Longrightarrow C$, so by dl, $\Gamma, A ; \Psi ; \Delta \Longrightarrow C$. If $w = 1$, then let $\Delta' = \Delta, \Delta_1$ be given. If $\Omega = \Gamma_\Omega ; \Psi_\Omega ; \Delta_\Omega$, then by the selection weakening lemma (2), $\Gamma_\Omega ; \Psi_\Omega ; \Delta_\Omega, \Delta_1 \ \rhd\ \Gamma ; \Psi ; \Delta, \Delta_1 \mid A$. By the induction hypothesis for $\mathcal{D}'$, $\Gamma_\Omega ; \Psi_\Omega ; \Delta_\Omega, \Delta_1 \Longrightarrow C$, and so by the selection theorem, $\Gamma ; \Psi ; \Delta, \Delta_1, A \Longrightarrow C$, so $\Gamma, A ; \Psi ; \Delta, \Delta_1 \Longrightarrow C$. □

*Proof (Thm. 11).* By induction on the structure of the derivation $\mathcal{D}$ of $\Gamma ; \Psi ; \Delta \Longrightarrow C$. We have the following characteristic cases:

*Case*. the last rule in $\mathcal{D}$ is init, dl, $\text{dl}_1$, $\mathbf{1}R$, $\top R$ or $!R$: these cases follow immediately from the corresponding forward rule, which is structurally identical.

*Case*. the last rule in $\mathcal{D}$ is one of the crystallisation rules, say $\text{cryst}_1$:

$$\mathcal{D} \qquad = \qquad \cfrac{\begin{array}{c}\mathcal{D}' \\ \Gamma \,;\, (\Psi, A) \,;\, \Delta \Longrightarrow C\end{array}}{\Gamma \,;\, \Psi \,;\, \Delta, A \,\&\, \mathbf{1} \Longrightarrow C}\ \text{cryst}_1$$

Given $\Gamma_1 \cup \Gamma_2 \subseteq \Gamma$ and $\Psi_1, \Psi_2 \subseteq \Psi, A$:

(a) if $\Gamma_1 \,;\, \Psi_1 \,;\, \Delta, !\Gamma_1, \Psi_1 \,\&\, \mathbf{1} \longrightarrow^0 C$, then if $A \,\&\, \mathbf{1} \in \Psi_1 \,\&\, \mathbf{1}$ then we are done. Otherwise, $\Psi_1, \Psi_2 \subseteq \Psi$.

(b) otherwise $\Gamma_1 \,;\, \Psi_2 \,;\, \Delta', !\Gamma_1, \Psi_2 \,\&\, \mathbf{1} \longrightarrow^1 C$ for some $\Delta' \subseteq \Delta$. Regardless of the constitution of $\Delta'$, the argument of the previous case still applies.

*Case*. the last rule in $\mathcal{D}$ is $\multimap R$:

$$\mathcal{D} \qquad = \qquad \cfrac{\begin{array}{c}\mathcal{D}' \\ \Gamma \,;\, \Psi \,;\, \Delta, A \Longrightarrow B\end{array}}{\Gamma \,;\, \Psi \,;\, \Delta \Longrightarrow A \multimap B}\ \multimap R$$

Given $\Gamma_1 \cup \Gamma_2 \subseteq \Gamma$ and $\Psi_1, \Psi_2 \subseteq \Psi$:

(a) if $\Gamma_1 \,;\, \Psi_1 \,;\, \Delta, A, !\Gamma_2, \Psi_2 \,\&\, \mathbf{1} \longrightarrow^0 C$, then we use $\multimap R_1$.

(b) otherwise, $\Gamma_1 \,;\, \Psi_2 \,;\, \Delta', !\Gamma_1, \Psi_2 \,\&\, \mathbf{1} \longrightarrow^1 C$ for some $\Delta' \subseteq \Delta$.

All rules require a similar analysis in order to extend the corresponding case from the proof of Thm. 4. $\square$