# Math Foundations for ML

## 10-606

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# Exercise: Gaussian elimination

- Suppose
  - $x + y + z = 3$
  - $2x + y = 5$
  - $-x + y - 2z = 4$
- What are x, y, z?

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ -1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}$$

$$x + y + z = 3$$

$$2x + y = 5$$

$$-x + y - 2z = 4$$

$$0 + 2y - z = 7$$

$$0 + -y - 2z = -1$$

$$0 + 0 - 5z = 5$$

$z = -1$

$-1 \to -3$ ↳ $y = 3$

$3 \to -1 \to 4 \to 1$ ↳ $x = 1$

$$R \overbrace{\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ -1 & 1 & -2 \end{pmatrix}}^{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = R \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}$$

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad RA = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 2 & -1 \end{pmatrix}$$

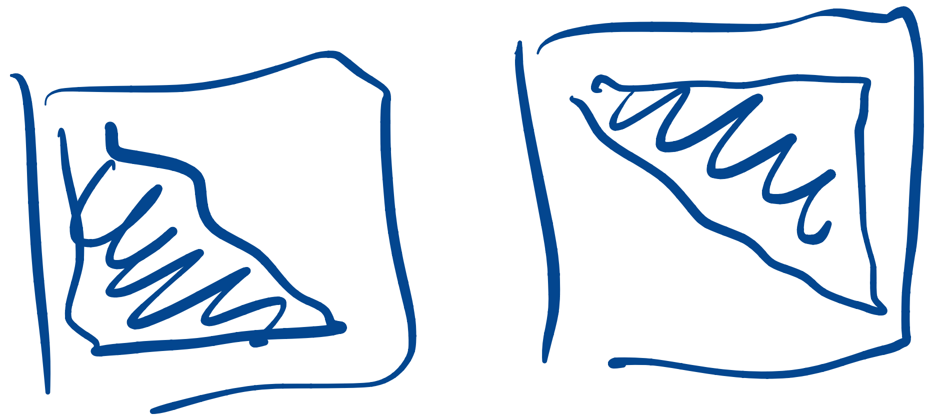
$$R^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$R_3 R_2 R_1 A = U$$

$$A = \underbrace{R_1^{-1} R_2^{-1} R_3^{-1}}_{L} U = LU$$

$m < n$

$\mathbb{R}^{m \times n}$



$A = A^T$ ?

$L = U^T$ sometimes

$$A = U \Sigma V^T$$

$U \in \mathbb{R}^{m \times m}$

$\Sigma$: diagonal, $\geq 0$

$V \in \mathbb{R}^{n \times n}$ orthonormal

# Exercise: LU decomposition

## submit on Canvas, or email me the answer if you're not on Canvas

- What is the LU decomposition of this matrix?

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \qquad RA = \begin{pmatrix} 1 & 3 \\ 0 & -6 \end{pmatrix} = U$$

$$R^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = L$$

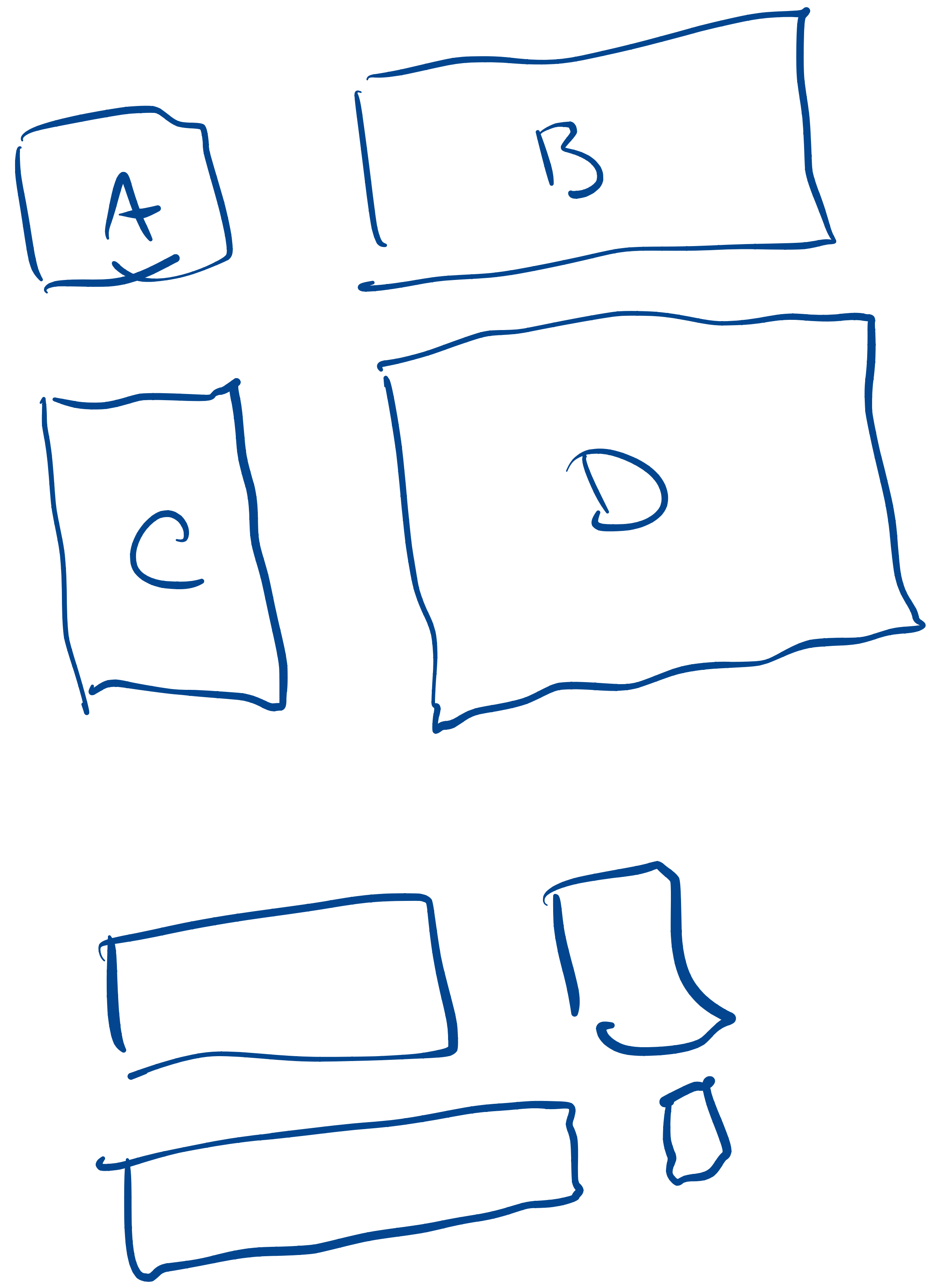
$$A = R^{-1} U \equiv LU$$

$$\begin{bmatrix} 1 & & 0 \\ I & 1 & \\ II & III & 1 \end{bmatrix}$$

order for 3x3

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$A_{(1,2),(1,3)} = \begin{pmatrix} 1 & 3 \\ 4 & 6 \end{pmatrix}$$




$$W = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

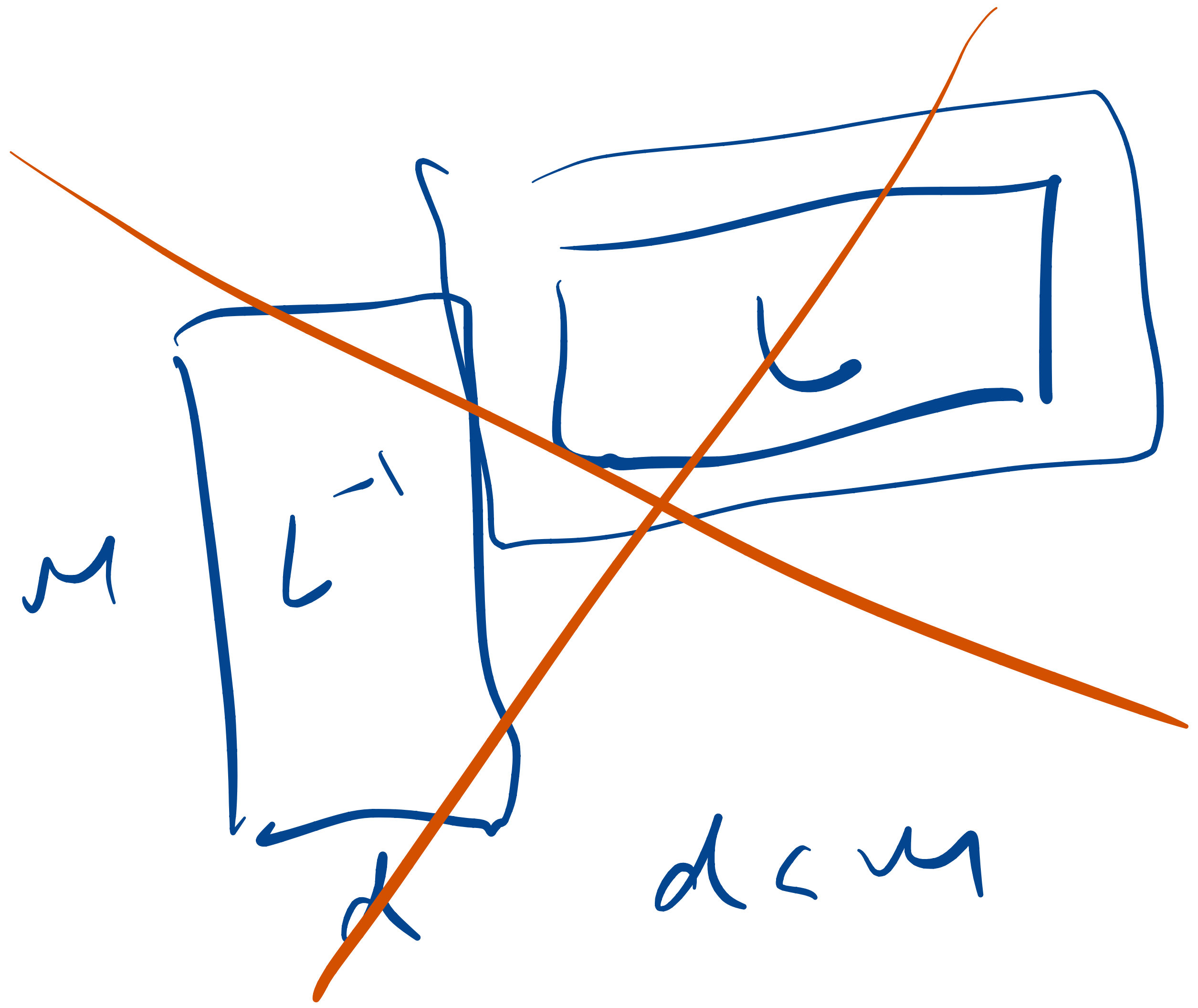
$L^*$ adjoint of $L$

$\longleftrightarrow$

$$\forall x, y \quad \langle x, Ly \rangle = \langle L^* x, y \rangle$$

$y = Lx \iff x = L^{-1}y \quad \forall\, x, y$

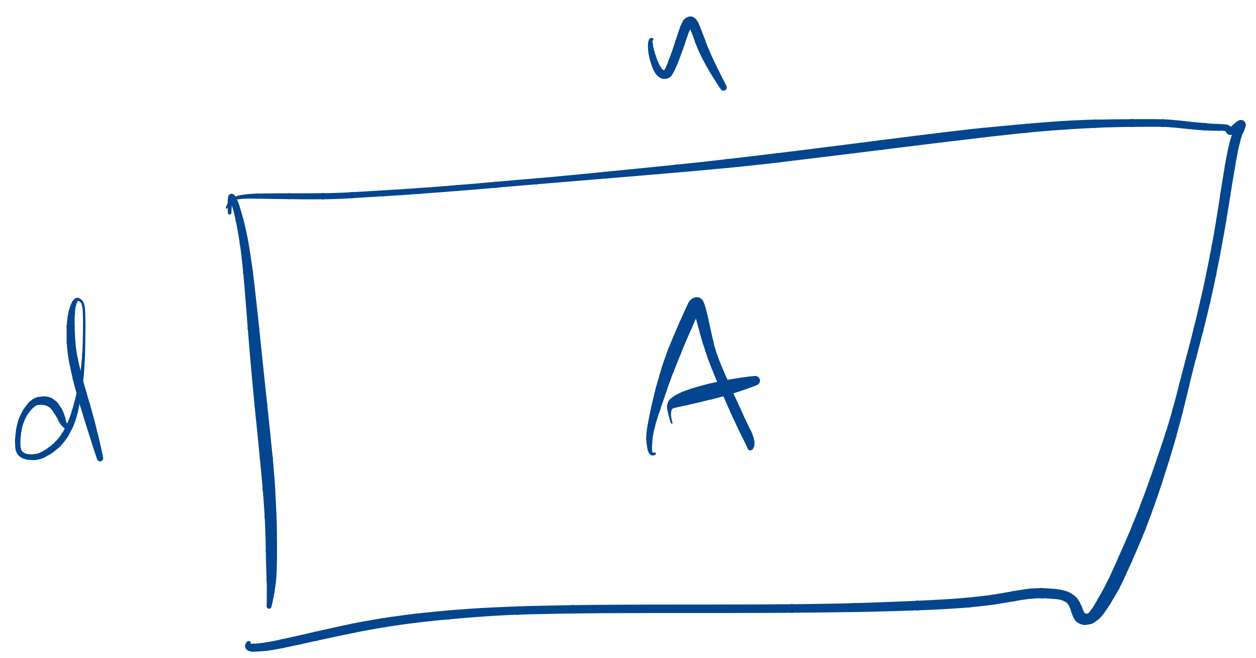
$L^{-1}$ may not exist




$LL^{-1} = I = L^{-1}L$

$I$ rank $m$

$L\underset{\sim}{x} = \underset{\sim}{e_j} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{pmatrix} \leftarrow j^{th}$ pos



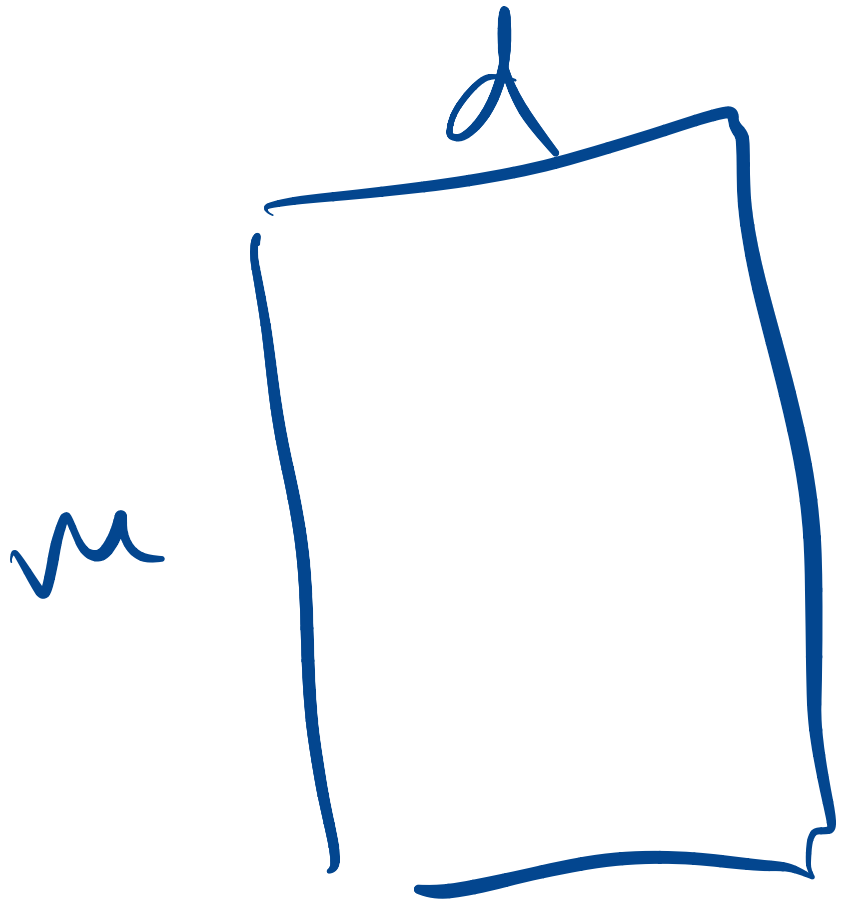

$d < n$

$L^{left} L = I$

$\text{rank}(A) \le d$

$L L^{Right} = I$

if $\dim(\text{range}(A)) < d \Rightarrow A$ singular




$d < m$

$\text{rank}(A) \le d$

if $\text{rank}(A) < d$

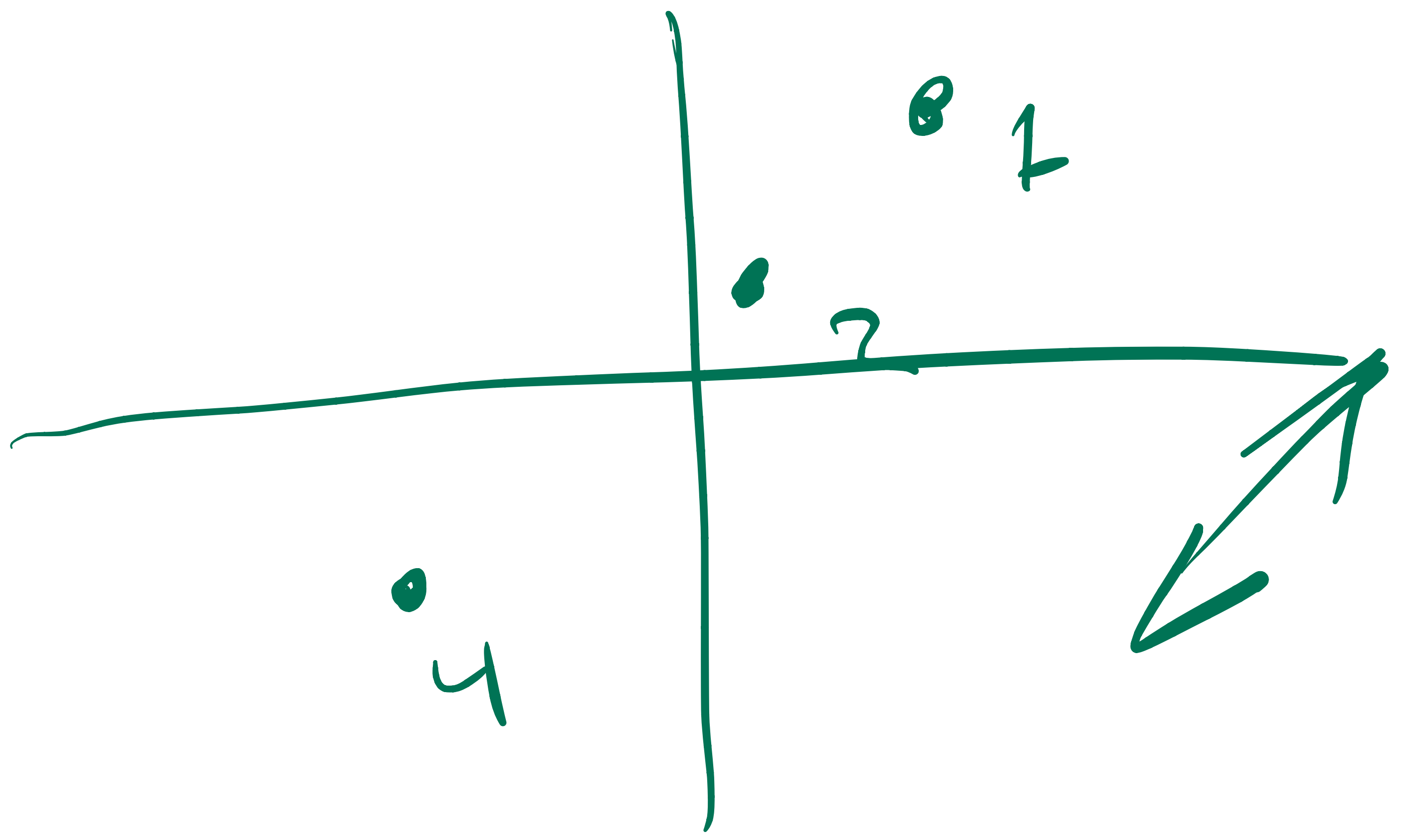
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ x & x & 1 & 0 \\ x & x & x & 1 \end{pmatrix}$$

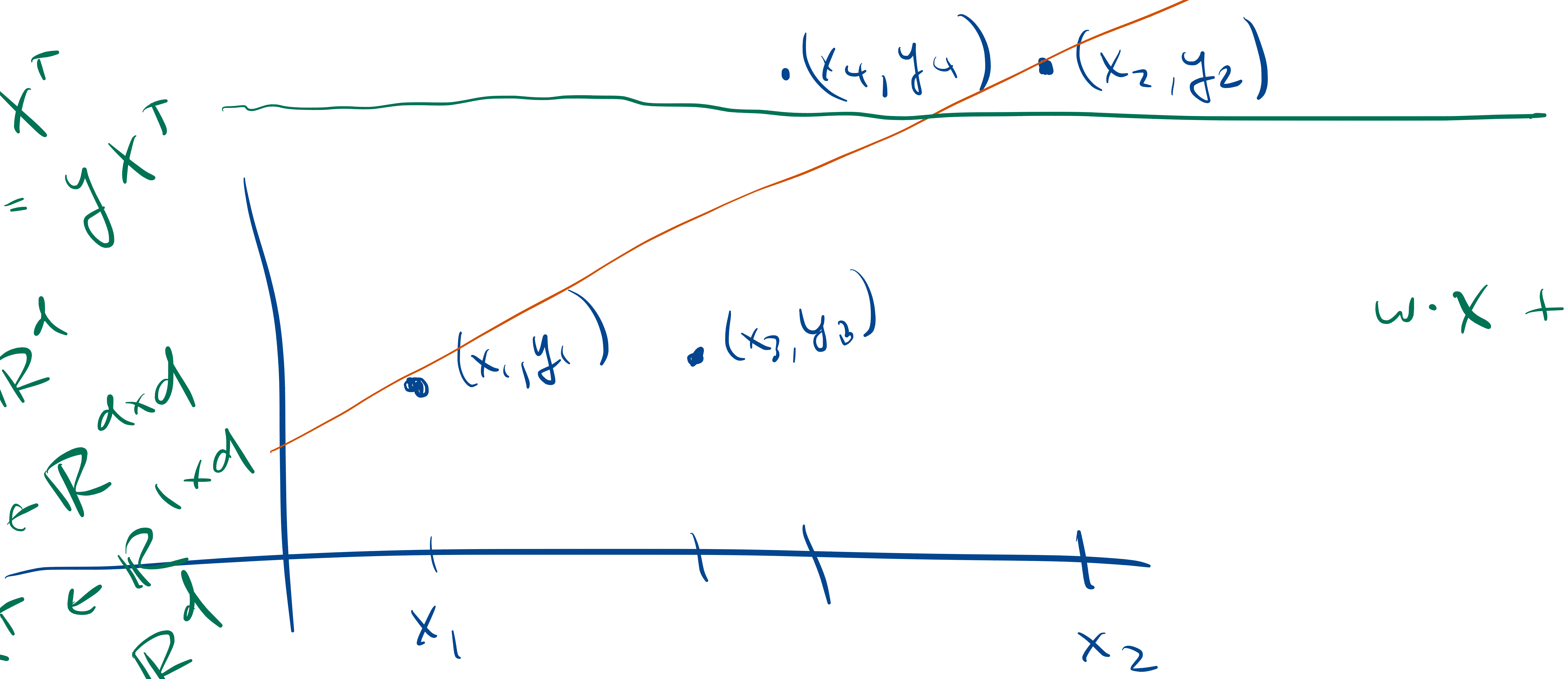
$U$ $V$ are orthonormal

$$U^T U = I$$

$$V V^T = I$$

$w \cdot x + b$






$w^T X X^T = y X^T$

$x \in \mathbb{R}^d$

$X X^T \in \mathbb{R}^{d\times d}$

$y X^T \in \mathbb{R}^{1\times d}$

$w \in \mathbb{R}^d$

$w \cdot x + b$

$\lambda \geq 0$

$$\min_{w,b} \left( \sum_{i=1}^{4} \left(y_i - (w \cdot x_i + b)\right)^2 + \lambda\, w \cdot w \right)$$

$(w \cdot x_i + b) \to \hat{y}_i$

$w \cdot w = \|w\|^2$

$\hat{y}_i = w \cdot x_i + b$

$$\begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_T \\ | & | & & | \end{bmatrix} = X$$

$y = (y_1 \;\; y_2 \;\; \cdots \;\; y_T)$