Computational Foundations for ML

10-607

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Exercise

mini-sudoku

each square digit 1..4

every row
every column
every marked quadrant
→ contains 1..4

1			3
3			2
	3		1
	1		4

what goes here

Mini-sudoku as logic

$(\mathrm{val}(1,2) = 4) \to T/F$

$\forall r.\ \exists c.\ \mathrm{val}(r,c) = 1 \wedge 1 \le c \wedge c \le 4$

$= 2$

$= 3$

$= 4$

$\exists r.\ \exists c.\ p(r,c)$

$\exists r,c.\ p(r,c)$

$(\exists r)\ p(r,c)$ (alternate syntax)

↑ repeat

$\forall c.\ \exists r.\ (\sim)$

1			3
			2
	3		
	1		

repeat for other quadrants

$$\exists r.\ \exists c.\ (r=1 \lor r=2) \land (c=1 \lor c=2) \land \mathrm{val}(r,c)=1$$

$$\land\ \mathrm{val}(r,c)=2$$

$$\land\ \mathrm{val}(r,c)=3$$

$$\land\ \mathrm{val}(r,c)=4$$

$$\forall r.\ \forall c.\ (\mathrm{val}(r,c)=1) \lor (\mathrm{val}(r,c)=2) \lor (\ldots=3) \lor (\ldots 4)$$

not necessary

Perceptrons

$x_t, y_t \rightarrow \{-1, 1\}$

$\hookrightarrow \mathbb{R}^d$

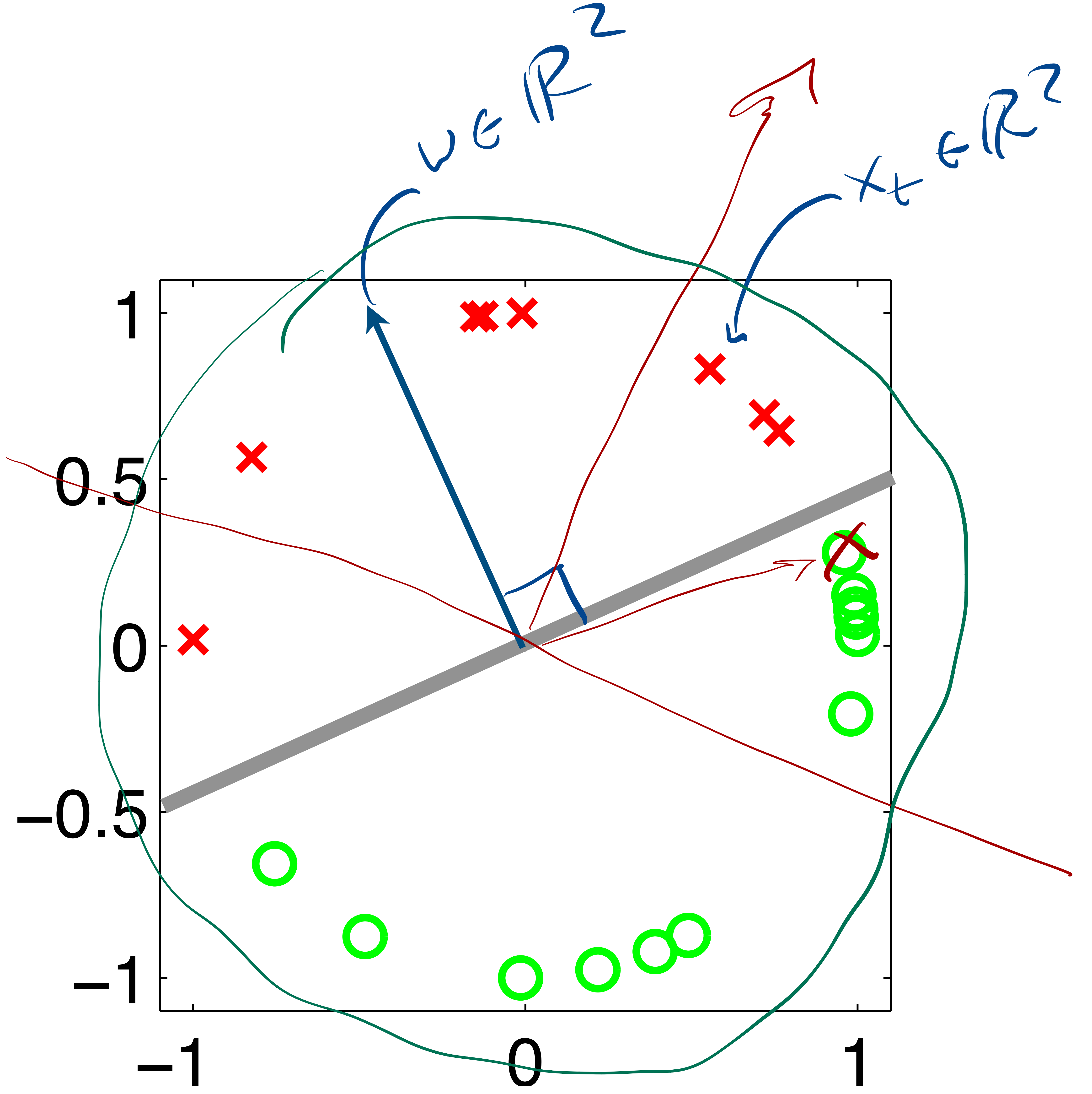
parameter $w \in \mathbb{R}^d$

predict $y_t = \mathrm{sgn}(x_t \cdot w)$

if mistake

$y_t = 1: \quad w_{t+1} \leftarrow w_t + x_t$

$y_t = -1: \quad w_{t+1} \leftarrow w_t - x_t$



$y_t = 1$: $x_t \cdot w_{t+1} = x_t \cdot w_t + x_t \cdot x_t > x_t \cdot w_t$

Assume $\exists w^* . \forall t . (y_t = 1) \to (w^* \cdot x_t \geq \epsilon) \wedge$
$(y_t = -1) \to (w^* \cdot x_t \leq -\epsilon)$

$\forall t . \|x_t\| \leq U$

Prove : fewer than $\dfrac{U^2 \|w^*\|^2}{\epsilon^2}$ mistakes

Hölder's inequality : $a \cdot b \leq \|a\| \, \|b\|$

I/ lower bound $w_t \cdot w^*$

no mistake: $w_{t+1} \cdot w^* = w_t \cdot w^*$ $\qquad M_t = M_{t+1}$

mistake $y_t = 1$: $w_{t+1} = w_t + x_t$ $\qquad M_{t+1} = 1 + M_t$

$$w_{t+1} \cdot w^* = w_t \cdot w^* + \overbrace{x_t \cdot w^*}^{\geq \epsilon}$$

$$w_{t+1} \cdot w^* \geq w_t + w^* + \epsilon$$

mistake $y_t = -1$: $w_{t+1} = w_t - x_t$

$$w_{t+1} \cdot w^* = w_t \cdot w^* \overbrace{- \overbrace{x_t \cdot w^*}^{\leq -\epsilon}}^{\geq \epsilon}$$

$$w_{t+1} \cdot w^* \geq w_t \cdot w^* + \epsilon \qquad M_{t+1} = 1 + M_t$$

$$\boxed{\forall t.\ w_t \cdot w^* \geq \epsilon M_t} \leftarrow \text{by induction}$$

base case $t=1$ $\quad w_1 \cdot w^* = 0 \cdot w^* = 0$
$M_1 = 0$ $\quad 0 \geq 0$

inductive: 3 cases above

II/ show $\boxed{\forall t: \|w_t\|^2 \leq M_t U^2}$

no mistake: $w_{t+1} = w_t \qquad M_{t+1} = M_t$

mistake $y_t = +1$:

$$w_{t+1} = w_t + x_t$$

$$\|w_{t+1}\|^2 = w_{t+1} \cdot w_{t+1} = w_t \cdot w_t + \overbrace{2 w_t \cdot x_t}^{\leq 0} + \overbrace{x_t \cdot x_t}^{\leq U^2}$$

$$\le w_t \cdot w_t + u^2 \qquad M_{t+1} = 1 + M_t$$

mistake $y_t = -1$:

$$w_{t+1} = w_t - x_t$$

$$\|w_{t+1}\|^2 = w_t \cdot w_t \underbrace{- 2x_t \cdot w_t}_{\le 0} + \overbrace{x_t \cdot x_t}^{\le u^2}$$

$$\le w_t \cdot w_t + u^2 \qquad M_{t+1} = 1 + M_t$$

induction!

III

$$w_t \cdot w^{*2} \geq \epsilon^2 M_t^2$$

$$M_t^2 \leq \left(\frac{w_t \cdot w^*}{\epsilon}\right)^2 \leq \left(\frac{\|w_t\| \, \|w^*\|}{\epsilon}\right)^2$$

$$\leq \frac{\|w^*\|^2 M_t U^2}{\epsilon^2}$$

$$M_t \leq \frac{\|w^*\|^2 U^2}{\epsilon^2}$$