Readings:
K&F: 3.3, 3.4

BN Semantics 3 –
Now it's personal!

Graphical Models – 10708
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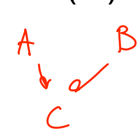
Independencies encoded in BN

- We said: All you need is the local Markov assumption
  - ($X_i \perp \text{NonDescendants}_{X_i} \mid \mathbf{Pa}_{X_i}$)
- But then we talked about other (in)dependencies
  - e.g., explaining away

$A \to B \to C \to D$
$A \perp D \mid B$



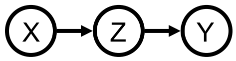
$A \perp B$
$\neg A \perp B \mid C$

- What are the independencies encoded by a BN?
  - Only assumption is local Markov
  - But many others can be derived using the algebra of conditional independencies!!!

Understanding independencies in BNs – BNs with 3 nodes

Local Markov Assumption:
A variable X is independent of its non-descendants given its parents and only its parents

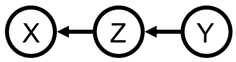
Indirect causal effect:



$Y \perp X \mid Z$

$\neg X \perp Y$

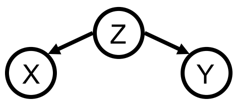
Indirect evidential effect:



$Y \perp X \mid Z$

$\neg X \perp Y$

Common cause:



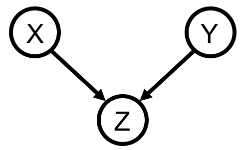
$Y \perp X \mid Z$

$\neg X \perp Y$

all represent same dist.

v-structures

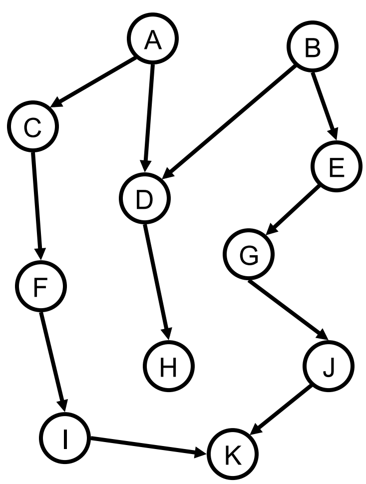
Common effect:



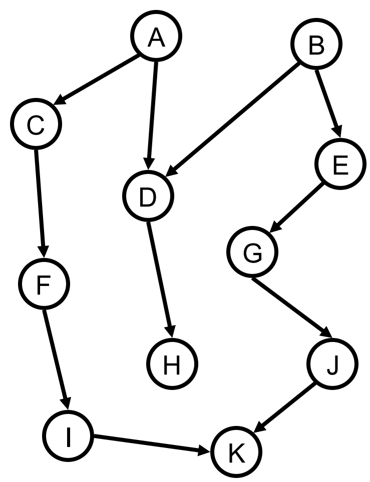
$X \perp Y$

$\neg X \perp Y \mid Z$

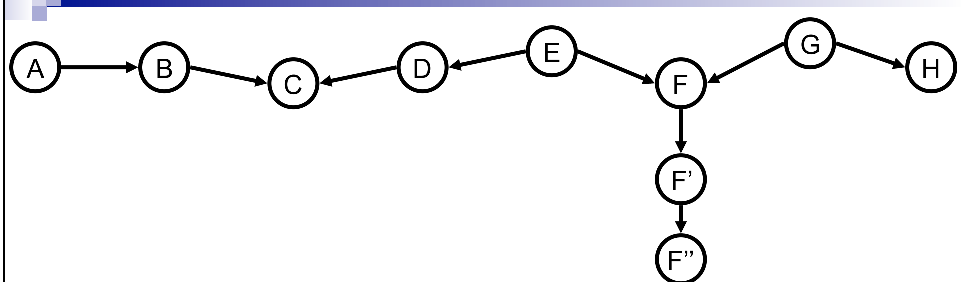
Understanding independencies in BNs – Some examples



Understanding independencies in BNs – Some more examples



An active trail – Example



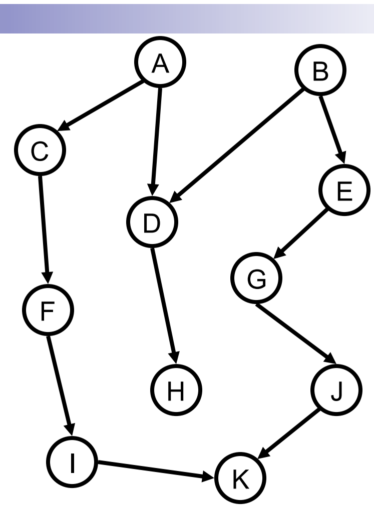
When are A and H independent?

Active trails formalized

- A trail $X_1 - X_2 - \cdots - X_k$ is an **active trail** when variables $\boldsymbol{O} \subseteq \{X_1,\ldots,X_n\}$ are observed if for each consecutive triplet in the trail:
  - $X_{i-1} \rightarrow X_i \rightarrow X_{i+1}$, and X_i is **not observed** ($X_i \notin \boldsymbol{O}$)
  - $X_{i-1} \leftarrow X_i \leftarrow X_{i+1}$, and X_i is **not observed** ($X_i \notin \boldsymbol{O}$)
  - $X_{i-1} \leftarrow X_i \rightarrow X_{i+1}$, and X_i is **not observed** ($X_i \notin \boldsymbol{O}$)
  - $X_{i-1} \rightarrow X_i \leftarrow X_{i+1}$, and X_i **is observed** ($X_i \in \boldsymbol{O}$), or **one of its descendents**

Active trails and independence?

- **Theorem**: Variables $\mathbf{X_i}$ **and** $\mathbf{X_j}$ **are independent given** $\boldsymbol{Z} \subseteq \{X_1,\ldots,X_n\}$ if the is **no active trail** between X_i and X_j when variables $\boldsymbol{Z} \subseteq \{X_1,\ldots,X_n\}$ are observed

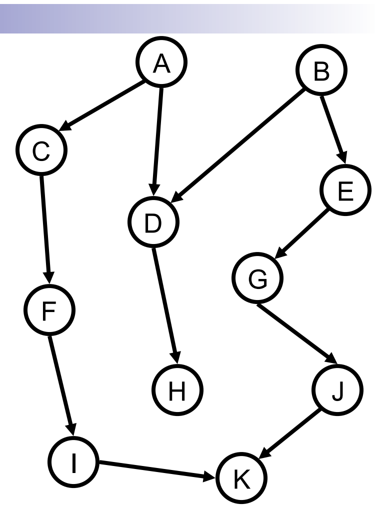


More generally: Soundness of d-separation

- Given BN structure G
- Set of independence assertions obtained by d-separation:
  - $I(G) = \{(\mathbf{X} \perp \mathbf{Y} | \mathbf{Z}) : \text{d-sep}_G(\mathbf{X};\mathbf{Y}|\mathbf{Z})\}$
- **Theorem: Soundness of d-separation**
  - If P factorizes over G then $I(G) \subseteq I(P)$
- **Interpretation:** d-separation only captures true independencies
- Proof discussed when we talk about undirected models

Existence of dependency when not d-separated

- **Theorem:** If X and Y are not d-separated given **Z**, then X and Y are dependent given **Z** under some P that factorizes over G
- **Proof sketch**:
  - Choose an active trail between X and Y given **Z**
  - Make this trail dependent
  - Make all else uniform (independent) to avoid "canceling" out influence



More generally: Completeness of d-separation

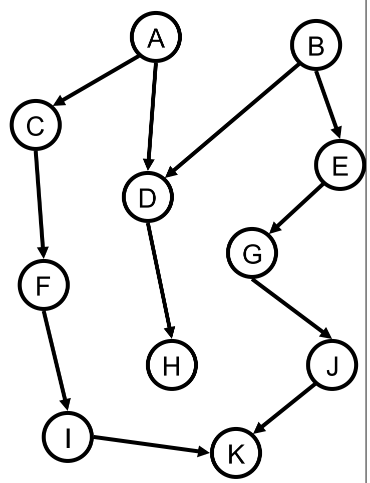
- **Theorem: Completeness of d-separation**
  - For "almost all" distributions where *P* factorizes over to *G*, we have that I(*G*) = I(*P*)
    - *"almost all" distributions*: except for a set of measure zero of parameterizations of the CPTs (assuming no finite set of parameterizations has positive measure)
    - Means that if all sets **X** & **Y** that are not d-separated given **Z**, then ¬(**X**⊥**Y**|**Z**)
- **Proof sketch for very simple case:**

Interpretation of completeness

- **Theorem: Completeness of d-separation**
  - For "almost all" distributions that *P* factorize over to *G*, we have that I(*G*) = I(*P*)
- BN graph is usually sufficient to capture all independence properties of the distribution!!!!
- But only for complete independence:
  - *P* ➔(**X**=**x**⊥**Y**=**y** | **Z**=**z**), ∀ **x**∈Val(**X**), **y**∈Val(**Y**), **z**∈Val(**Z**)
- Often we have context-specific independence (CSI)
  - ∃ **x**∈Val(**X**), **y**∈Val(**Y**), **z**∈Val(**Z**): *P* ➔(**X**=**x**⊥**Y**=**y** | **Z**=**z**)
  - Many factors may affect your grade
  - But if you are a frequentist, all other factors are irrelevant ☺

Algorithm for d-separation

- How do I check if X and Y are d-separated given **Z**
  - There can be exponentially-many trails between X and Y
- Two-pass linear time algorithm finds all d-separations for X
- 1. Upward pass
  - Mark descendants of **Z**
- 2. Breadth-first traversal from X
  - Stop traversal at a node if trail is "blocked"
  - (Some tricky details apply – see reading)



What you need to know

- d-separation and independence
  - sound procedure for finding independencies
  - existence of distributions with these independencies
  - (almost) all independencies can be read directly from graph without looking at CPTs

Announcements

- Homework 1:
  - Due next Wednesday – **beginning of class!**
  - It's hard – start early, ask questions
- Audit policy
  - No sitting in, official auditors only, see course website

Building BNs from independence properties

- From d-separation we learned:
  - Start from local Markov assumptions, obtain all independence assumptions encoded by graph
  - For most *P*'s that factorize over *G*, I(*G*) = I(*P*)
  - All of this discussion was for a given *G* that is an I-map for *P*
- Now, give me a *P*, how can I get a *G*?
  - i.e., give me the independence assumptions entailed by *P*
  - Many *G* are "equivalent", how do I represent this?
  - Most of this discussion is not about practical algorithms, but useful concepts that will be used by practical algorithms
    - Practical algs next time

Minimal I-maps

- One option:
  - *G* is an I-map for *P*
  - *G* is as simple as possible

- *G* is a **minimal I-map** for *P* if deleting any edges from *G* makes it no longer an I-map

Obtaining a minimal I-map

- Given a set of variables and conditional independence assumptions
- Choose an ordering on variables, e.g., $X_1, \ldots, X_n$
- For i = 1 to n
  - Add X_i to the network
  - Define parents of X_i, $\mathbf{Pa}_{X_i}$, in graph as the minimal subset of $\{X_1,\ldots,X_{i-1}\}$ such that local Markov assumption holds – X_i independent of rest of $\{X_1,\ldots,X_{i-1}\}$, given parents $\mathbf{Pa}_{X_i}$
  - Define/learn CPT – $P(X_i | \mathbf{Pa}_{X_i})$

Flu, Allergy, SinusInfection, Headache

Minimal I-map not unique (or minimum)

- Given a set of variables and conditional independence assumptions
- Choose an ordering on variables, e.g., $X_1, \ldots, X_n$
- For i = 1 to n
  - Add X_i to the network
  - Define parents of X_i, $\mathbf{Pa}_{X_i}$, in graph as the minimal subset of $\{X_1,\ldots,X_{i-1}\}$ such that local Markov assumption holds – X_i independent of rest of $\{X_1,\ldots,X_{i-1}\}$, given parents $\mathbf{Pa}_{Xi}$
  - Define/learn CPT – $P(X_i| \mathbf{Pa}_{Xi})$

Flu, Allergy, SinusInfection, Headache

Perfect maps (P-maps)

- I-maps are not unique and often not simple enough
- Define "simplest" *G* that is I-map for *P*
  - A BN structure *G* is a **perfect map** for a distribution *P* if I(*P*) = I(*G*)
- Our goal:
  - Find a perfect map!
  - Must address equivalent BNs

Inexistence of P-maps 1

- XOR (this is a hint for the homework)

Inexistence of P-maps 2

- (Slightly un-PC) swinging couples example

Obtaining a P-map

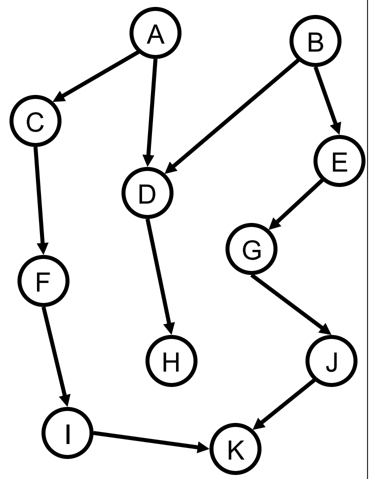
- Given the independence assertions that are true for *P*
- Assume that there exists a perfect map G^*
  - Want to find G^*
- Many structures may encode same independencies as G^*, when are we done?
  - Find all equivalent structures simultaneously!

I-Equivalence

- Two graphs G_1 and G_2 are **I-equivalent** if $I(G_1) = I(G_2)$
- **Equivalence class** of BN structures
  - Mutually-exclusive and exhaustive partition of graphs
- How do we characterize these equivalence classes?

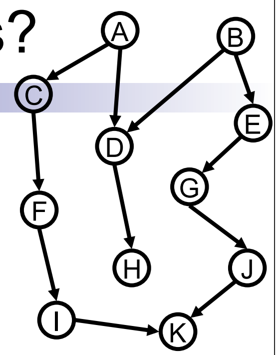
Skeleton of a BN

- **Skeleton** of a BN structure *G* is an **undirected graph** over the same variables that has an edge X–Y for every X→Y or Y→X in *G*



- (Little) **Lemma:** Two I -equivalent BN structures must have the same skeleton

What about V-structures?



- **V-structures are key property of BN structure**
- **Theorem:** If G_1 and G_2 have the same skeleton and V-structures, then G_1 and G_2 are I-equivalent

Same V-structures not necessary

- **Theorem:** If G_1 and G_2 have the same skeleton and V-structures, then G_1 and G_2 are I-equivalent
- Though sufficient, same V-structures not necessary

Immoralities & I-Equivalence

- Key concept not V-structures, but "immoralities" (unmarried parents ☺)
  - $X \rightarrow Z \leftarrow Y$, with no arrow between X and Y
  - Important pattern: X and Y independent given their parents, but not given Z
  - (If edge exists between X and Y, we have *covered* the V-structure)
- **Theorem:** G_1 and G_2 have the same skeleton and immoralities if and only if G_1 and G_2 are I-equivalent

Obtaining a P-map

- Given the independence assertions that are true for *P*
  - Obtain skeleton
  - Obtain immoralities

- From skeleton and immoralities, obtain every (and any) BN structure from the equivalence class

Identifying the skeleton 1

- When is there an edge between X and Y?

- When is there no edge between X and Y?

Identifying the skeleton 2

- Assume d is max number of parents (d could be n)

- For each X_i and X_j
  - $E_{ij} \leftarrow$ true
  - For each $\mathbf{U} \subseteq \mathbf{X} - \{X_i, X_j\}$, $|\mathbf{U}| \leq d$
    - Is $(X_i \perp X_j \mid \mathbf{U})$?
      - $E_{ij} \leftarrow$ false
  - If E_{ij} is true
    - Add edge X – Y to skeleton

Identifying immoralities

- Consider X – Z – Y in skeleton, when should it be an immorality?

- Must be $X \rightarrow Z \leftarrow Y$ (immorality):
  - When X and Y are **never independent** given **U**, if $Z \in \mathbf{U}$

- Must **not** be $X \rightarrow Z \leftarrow Y$ (not immorality):
  - When there exists **U** with $Z \in \mathbf{U}$, such that X and Y are **independent** given **U**

From immoralities and skeleton to BN structures

- Representing BN equivalence class as a **partially-directed acyclic graph** (PDAG)
- Immoralities force direction on some other BN edges
- Full (polynomial-time) procedure described in reading

What you need to know

- Minimal I-map
  - every *P* has one, but usually many
- Perfect map
  - better choice for BN structure
  - not every *P* has one
  - can find one (if it exists) by considering I-equivalence
  - Two structures are I-equivalent if they have same skeleton and immoralities