Readings:
K&F: 4.1, 4.2, 4.3, 4.4, 4.5

Undirected Graphical Models

Graphical Models – 10708

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Normalization for computing probabilities

- To compute actual probabilities, must compute normalization constant (also called partition function)

$$P(ABCD) = \frac{1}{Z} \phi_1(AB)\, \phi_2(BC)\, \phi_3(CD)\, \phi_4(DA)$$

$$Z = \sum_a \sum_b \sum_c \sum_d \phi_1(a,b)\, \phi_2(b,c)\, \phi_3(c,d)\, \phi_4(d,a)$$

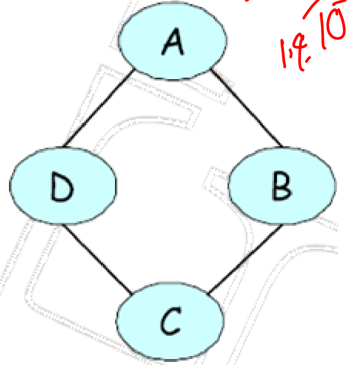
Potential

Assignment				Unnormalized	Normalized
a^0	b^0	c^0	d^0	300000	0.04
a^0	b^0	c^0	d^1	300000	0.04
a^0	b^0	c^1	d^0	300000	0.04
a^0	b^0	c^1	d^1	30	$4.1 \cdot 10^{-6}$
a^0	b^1	c^0	d^0	500	$6.9 \cdot 10^{-5}$
a^0	b^1	c^0	d^1	500	$6.9 \cdot 10^{-5}$
a^0	b^1	c^1	d^0	5000000 a lot	0.69
a^0	b^1	c^1	d^1	500	$6.9 \cdot 10^{-5}$
a^1	b^0	c^0	d^0	100	$1.4 \cdot 10^{-5}$
a^1	b^0	c^0	d^1	1000000	0.14
a^1	b^0	c^1	d^0	100	$1.4 \cdot 10^{-5}$
a^1	b^0	c^1	d^1	100	$1.4 \cdot 10^{-5}$
a^1	b^1	c^0	d^0	10	$1.4 \cdot 10^{-6}$
a^1	b^1	c^0	d^1	100000	0.014
a^1	b^1	c^1	d^0	100000	0.014
a^1	b^1	c^1	d^1	100000	0.014

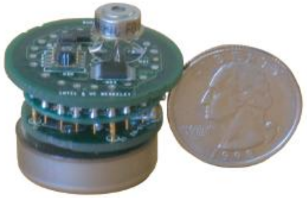
Sum = a lot = $10 / 1.4 \cdot 10^{-6}$

- Computing partition function is hard! ! Must sum over all possible assignments

Can use VE to compute Z if Markov Network has low tree width



Factorization in Markov networks

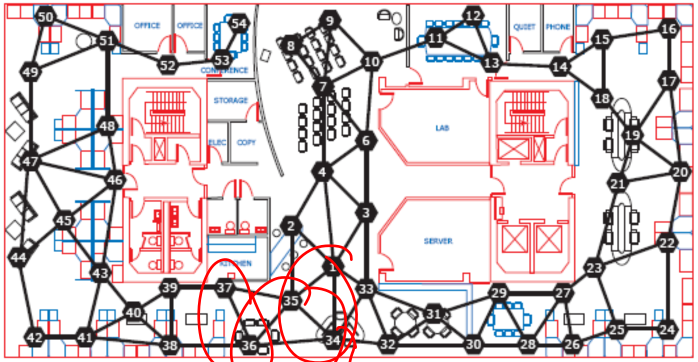


- Given an undirected graph *H* over variables **X**={X$_1$,...,X$_n$}

Di, Dj may overlap

- A distribution *P* **factorizes** over *H* if ~~∅~~ ∃
  - subsets of variables **D**$_1$⊆**X**,…, **D**$_m$⊆**X,** such that the **D**$_i$ ar *fully connected* in *H*
  - *non-negative potentials* (or factors) $\phi_1(\mathbf{D_1}),\ldots,\phi_m(\mathbf{D_m})$
    - also known as clique potentials
  - such that

$$P(X) = \frac{1}{Z}\prod_{i=1}^{m}\phi_i(D_i)$$



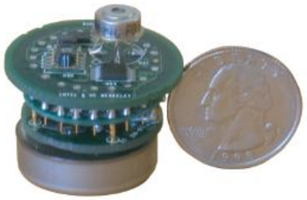
$D_1=\{1,34,35\}$

$D_2=\{36,37\}$

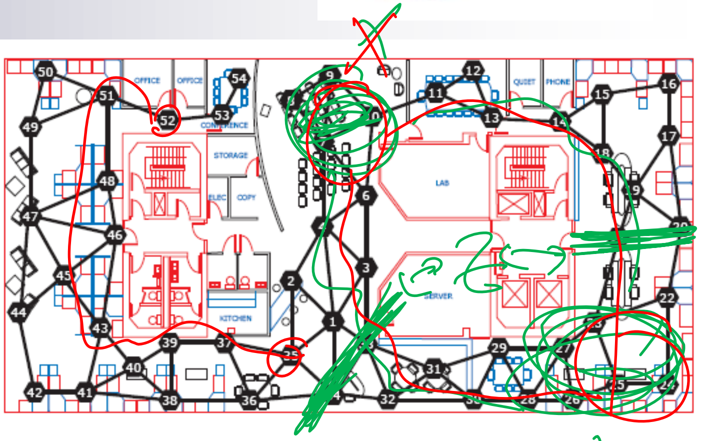
$D_3=\{34,35,36\}$

- Also called Markov random field *H,* or Gibbs distribution over *H*

Global Markov assumption in Markov networks



- A path $X_1 - \ldots - X_k$ is **active** when set of variables **Z** are observed if none of $X_i \in \{X_1,\ldots,X_k\}$ are observed (are part of **Z**)



- Variables **X** are **separated** from **Y** given **Z** in graph *H*, *sep*$_H$(**X**;**Y**|**Z**), if there is no active path between any $X \in$ **X** and any $Y \in$ **Y** given **Z**

$sep_H(X;Y|Z)$ Y

- The **global Markov assumption** for a Markov network *H* is

$sep_H(X;Y|Z) \Rightarrow X \perp Y | Z$

The BN Representation Theorem

$I(G) \subseteq I(P)$

If conditional independencies in BN are subset of conditional independencies in *P*

I-map

Obtain

Joint probability distribution:

$$P(X_1, \ldots, X_n) = \prod_{i=1}^{n} P\left(X_i \mid \mathbf{Pa}_{X_i}\right)$$

Important because:
Independencies are sufficient to obtain BN structure *G*

give you a BN

If joint probability distribution:

$$P(X_1, \ldots, X_n) = \prod_{i=1}^{n} P\left(X_i \mid \mathbf{Pa}_{X_i}\right)$$

Obtain

Then conditional independencies in BN are subset of conditional independencies in *P*

$I(G) \subseteq I(P)$

Important because:
Read independencies of *P* from BN structure *G*

Markov networks representation Theorem 1

If joint probability distribution *P*: & MN graph H → add edge between $X_u - X_v$, $\exists i$ if $X_u, X_v \in D_i$

$$P(X_1, \ldots, X_n) = \frac{1}{Z} \prod_{i=1}^{m} \phi_i(\mathbf{D}_i)$$

Then

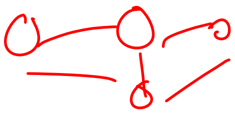
H is an I-map for *P*

$I(H) \subseteq I(P)$

- If you can write distribution as a normalized product of factors) Can read independencies from graph

What about the other direction for Markov networks ?

If H is an I-map for P



~~Then~~

joint probability distribution P:

$$P(X_1,\ldots,X_n) = \frac{1}{Z}\prod_{i=1}^{m}\phi_i(\mathbf{D}_i)$$

- Counter-example: $X_1,\ldots,X_4$ are binary, and only eight assignments have positive probability:

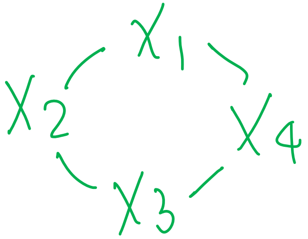
  1/8

  (0,0,0,0) (1,0,0,0) (1,1,0,0) (1,1,1,0)
  (0,0,0,1) (0,0,1,1) (0,1,1,1) (1,1,1,1)

  all other have prob. ∅

  need factors over all 4 vars

- For example, $X_1 \perp X_3 | X_2, X_4$: independencies
  - □ E.g., $P(X_1=0|X_2=0, X_4=0)$



I map for P

- But distribution doesn't factorize!!!

Markov networks representation Theorem 2 (Hammersley-Clifford Theorem)

If *H* is an I-map for *P*
and
P is a positive distribution

Then

joint probability
distribution *P*:

$$P(X_1,\ldots,X_n) = \frac{1}{Z}\prod_{i=1}^{m}\phi_i(\mathbf{D}_i)$$

- Positive distribution and independencies $\overset{\Rightarrow}{)}$ *P* factorizes over graph

$\forall x \quad P(x) > 0$

Representation Theorem for Markov Networks

If joint probability distribution *P*:

$$P(X_1,\ldots,X_n) = \frac{1}{Z}\prod_{i=1}^{m}\phi_i(\mathbf{D}_i)$$

Then

H is an I-map for *P*

If *H* is an I-map for *P*
and
P is a positive distribution

Then

joint probability distribution *P*:

$$P(X_1,\ldots,X_n) = \frac{1}{Z}\prod_{i=1}^{m}\phi_i(\mathbf{D}_i)$$

Completeness of separation in Markov networks

- **Theorem: Completeness of separation**
  - □ For “almost all” distributions that P factorize over Markov network H, we have that $\mathrm{I}(H) = \mathrm{I}(P)$
  - □ *“almost all” distributions*: except for a set of measure zero of parameterizations of the Potentials (assuming no finite set of parameterizations has positive measure)
- **Analogous to BNs**

What are the "local" independence assumptions for a Markov network?

- In a BN *G*:
  - □ local Markov assumption: variable independent of non-descendants given parents
  - □ d-separation defines global independence
  - □ Soundness: For all distributions:
- In a Markov net *H*:
  - □ **Separation** defines global independencies
  - □ What are the notions of local independencies?

Local independence assumptions for a Markov network

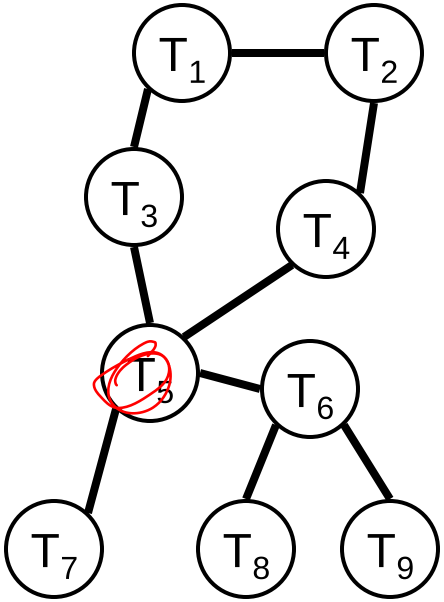
- **Separation** defines global independencies

- **Pairwise Markov Independence**:
  - □ Pairs of non-adjacent variables A,B are independent given all others

$A \perp B \mid \mathcal{X} - \{A, B\}$

- **Markov Blanket**: $MB(A) \equiv$ neighbors of A in H
  - □ Variable A independent of rest given its neighbors

$A \perp \mathcal{X} - MB(A) \mid MB(A)$



$T_5 \perp \{T_8, T_9, T_1, T_2\} \mid T_3, T_4, T_7, T_6$

Equivalence of independencies in Markov networks

- **Soundness Theorem**: For all positive distributions *P*, the following three statements are equivalent:
  - □ *P* entails the global Markov assumptions

    $\mathrm{Sep}_H(X, Y \mid Z) \Rightarrow X \perp Y \mid Z$

  - □ *P* entails the pairwise Markov assumptions

    $A \perp B \mid \mathcal{X} - \{A, B\}$

  - □ *P* entails the local Markov assumptions (Markov blanket)

    $A \perp \mathcal{X} - MB(A) \mid MB(A)$

$A - B$ may be dependent given $\mathcal{X} - \{A, B\}$

for almost all distributions $\neg A \perp B \mid \mathcal{X} - \{A, B\}$

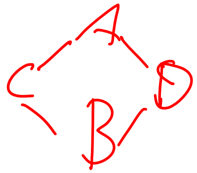
Minimal I-maps and Markov Networks

- A fully connected graph is an I-map
- Remember minimal I-maps?
  - □ A “simplest” I-map ! ⇒ Deleting an edge makes it no longer an I-map
- In a BN, there is no unique minimal I-map
- Theorem: For positive distributions & **Markov network, minimal I-map is unique!!**
- Many ways to find minimal I-map, e.g.,
  - □ Take pairwise Markov assumption: A not connected to B ⇒ $A \perp B \mid \mathcal{X} - \{A,B\}$
  - □ If *P* doesn’t entail it, add edge:

$$P \not\models A \perp B \mid \mathcal{X} - \{A,B\} \text{, add edge } A - B$$

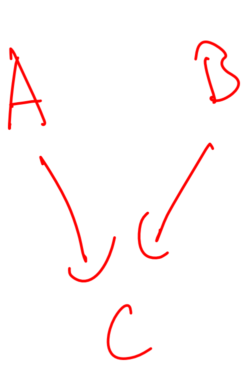
How about a perfect map?

- Remember perfect maps?
  - □ independencies in the graph are exactly the same as those in *P*
- For BNs, doesn't always exist
  - □ counter example: Swinging Couples



- How about for Markov networks?

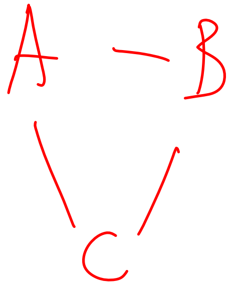
No!!



$A \perp B$

$\neg A \perp B \mid C$

minimal I-map MN



not a p-map

Unifying properties of BNs and MNs

- BNs:
  - give you: V-structures, CPTs are conditional probabilities, can directly compute probability of full instantiation
  - but: require acyclicity, and thus no perfect map for swinging couples
- MNs:
  - give you: cycles, and perfect maps for swinging couples
  - but: don't have V-structures, cannot interpret potentials as probabilities, requires partition function
- Remember PDAGS???
  - skeleton + immoralities
  - provides a (somewhat) unified representation
  - see book for details

What you need to know so far about Markov networks

- Markov network representation:
  - undirected graph
  - potentials over cliques (or sub-cliques)
  - normalize to obtain probabilities
  - need partition function
- Representation Theorem for Markov networks
  - if P factorizes, then it's an I-map
  - if P is an I-map, only factorizes for positive distributions
- Independence in Markov nets:
  - active paths and separation
  - pairwise Markov and Markov blanket assumptions
  - equivalence for positive distributions
- Minimal I-maps in MNs are unique
- Perfect maps don't always exist

Some common Markov networks and generalizations

- Pairwise Markov networks
- A very simple application in computer vision
- Logarithmic representation
- Log-linear models
- Factor graphs

Pairwise Markov Networks

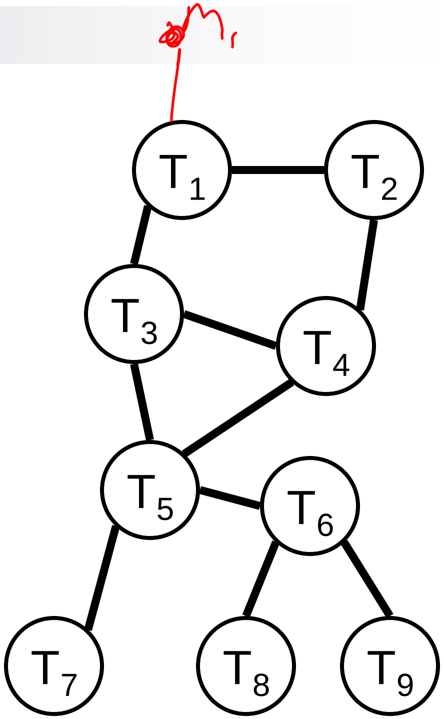
- All factors are over single variables or pairs of variables:
  - Node potentials $\phi_i(x_i)$
  - Edge potentials $\phi_{ij}(x_i,x_j)$ if i,j connected in graph
- Factorization:

$$P(x) = \frac{1}{Z} \prod_i \phi_i(x_i) \prod_{(i,j)\in H} \phi_{ij}(x_i,x_j)$$

often $\phi_i(x_i, m_i)$, a little less often $\phi_{ij}(x_i,x_j,m_i,m_j)$

more generally $\phi_{ij}(x_i,x_j,m_{1:n})$

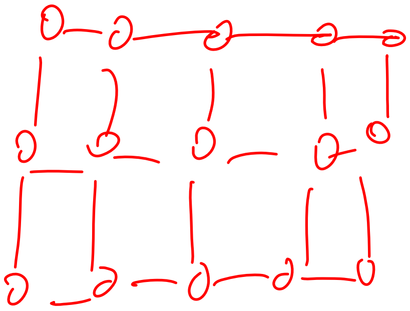
- Note that there may be bigger cliques in the graph, but only consider pairwise potentials



A very simple vision application

- Image segmentation: separate foreground from background
- Graph structure:
  - □ pairwise Markov net
  - □ grid with one node per pixel

If I use only node potentials: "salt & pepper noise"



grid MN

one var per pixel

$X_i \in \{fg, bg\}$

- Node potential:
  - □ "background color" v. "foreground color"

color of pixel i

$\mu_{fg} \equiv$ avg fg color

$\mu_{bg} \equiv$ avg bg color

$$\phi_i(x_i=fg) = e^{-\frac{\|m_i-\mu_{fg}\|^2}{\sigma^2}}$$

$$\phi_i(x_i=bg) = e^{-\frac{\|m_i-\mu_{bg}\|^2}{\sigma^2}}$$

- Edge potential:
  - □ neighbors like to be of the same class

"attractive potential" $\Rightarrow \phi_i(x_i, x_j)$

$x_i \backslash x_j$	fg	bg
fg	10	1
bg	1	10

Logarithmic representation

- Standard model: $$P(X_1,\ldots,X_n) = \frac{1}{Z}\prod_{i=1}^{m}\phi_i(\mathbf{D}_i)$$

- Log representation of potential (assuming positive potential):
  - also called the energy function

$$P(X) = \frac{1}{Z}\prod_i \phi_i(D_i) = \frac{1}{Z}\prod_i e^{\log \phi_i(D_i)} = \frac{1}{Z} e^{\sum_i \log \phi_i(D_i)}$$

" comes from physics"

$$\text{Energy}(X) \equiv \sum_i \psi_i(D_i)$$

- Log representation of Markov net:

$$\psi_i(D_i) = -\log \phi_i(D_i)$$

$$\boxed{P(X) = \frac{1}{Z} e^{-\sum_i \psi_i(D_i)}}$$

states with high energy have low probability

Log-linear Markov network (most common representation)

- **Feature** is some function f [**D**] for some subset of variables **D**
  - e.g., indicator function

$f(D) \equiv \mathbb{I}(D=d)$

- **Log-linear model** over a Markov network *H*:
  - a set of features $f_1[\mathbf{D}_1],\ldots, f_k[\mathbf{D}_k]$
    - each $\mathbf{D}_i$ is a subset of a clique in *H*
    - two f's can be over the same variables
  - a set of weights $w_1,\ldots,w_k$
    - usually learned from data
  - $$P(X_1,\ldots,X_n) = \frac{1}{Z}\exp\left[\sum_{i=1}^{k} w_i f_i(\mathbf{D}_i)\right]$$

it's ok for $D_i = D_j$

e.g., pairwise log-linear model $D_i = \{X_u, X_v\}$

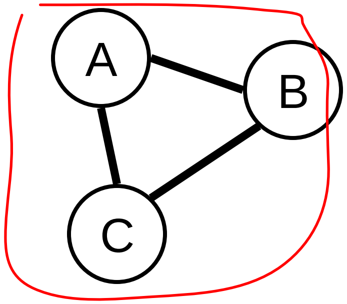
log linear, because log P is linear in w (risky business wrt log $\frac{1}{Z}$)

exactly equivalent to MN with $p(x) > 0 \ \forall x$

if $P(x) = 0$ for some x, then risky business!! ;)

Structure in cliques

- Possible potentials for this graph:

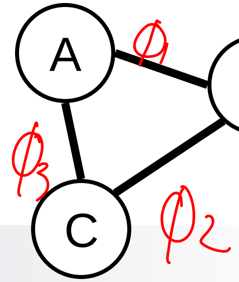


Can't look at graph & tell the differnce

full MN → $\phi(ABC)$

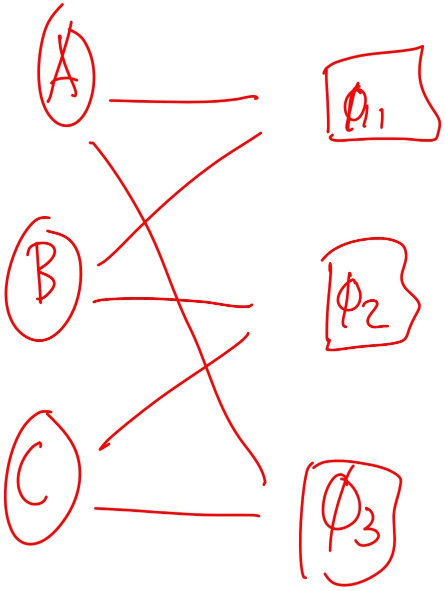
pairwise → $\phi(A,B), \phi(B,C)$
$\phi(A,C)$

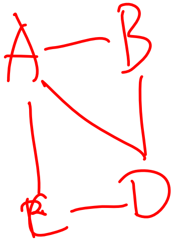
Factor graphs



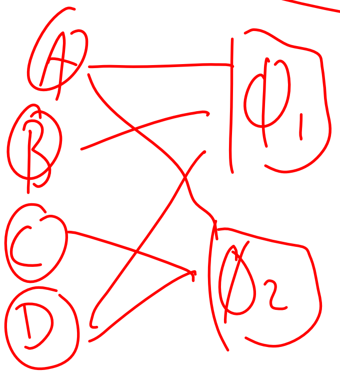
e.g., pairwise MN

- Very useful for approximate inference
  - □ Make factor dependency explicit
- Bipartite graph:
  - □ variable nodes (ovals) for $X_1,\ldots,X_n$
  - □ factor nodes (squares) for $\phi_1,\ldots,\phi_m$
  - □ edge $X_i – \phi_j$ if $X_i \in \text{Scope}[\phi_j]$





$$P(ABCD) = \frac{1}{Z}\,\phi_1(ABD)\,\phi_2(ACD)$$



Exact inference in MNs and Factor Graphs

- Variable elimination algorithm presented in terms of factors ! exactly the same VE algorithm can be applied to MNs & Factor Graphs
- Junction tree algorithms also applied directly here:
  - □ triangulate MN graph as we did with moralized graph
  - □ each factor belongs to a clique
  - □ same message passing algorithms

Summary of types of Markov nets

- Pairwise Markov networks
  - very common
  - potentials over nodes and edges
- Log-linear models
  - log representation of potentials
  - linear coefficients learned from data
  - most common for learning MNs
- Factor graphs
  - explicit representation of factors
    - you know exactly what factors you have
  - very useful for approximate inference

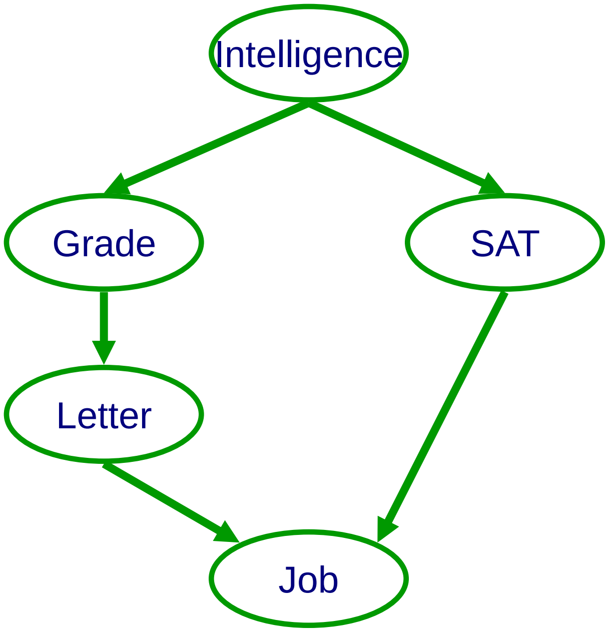
What you learned about so far

- ✓ Bayes nets
- ✓ Junction trees
- ✓ (General) Markov networks
- ✓ Pairwise Markov networks
- ✓ Factor graphs

- How do we transform between them?
- More formally:
  - □ I give you an graph in one representation, find an **I-map** in the other

From Bayes nets to Markov nets

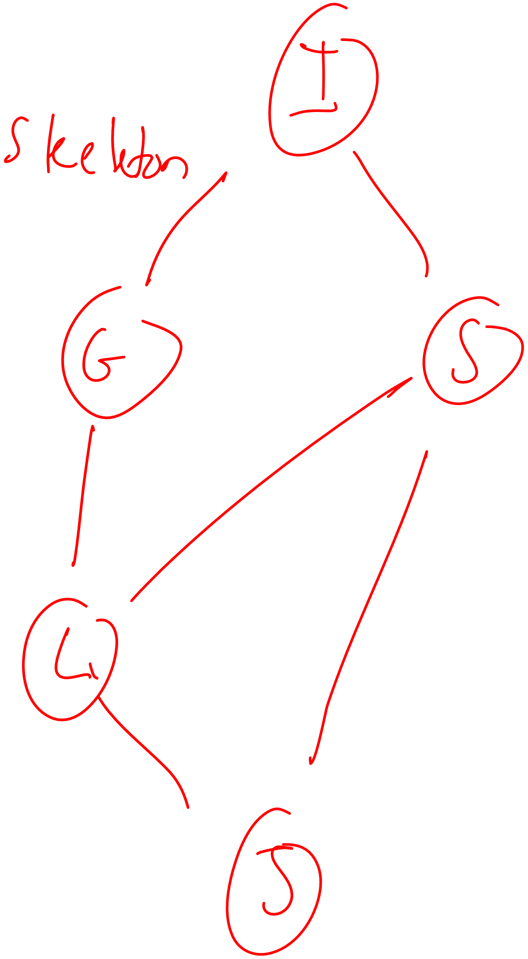
Set of indep


Intelligence
Grade
SAT
Letter
Job

$\neg L \perp S \mid J$

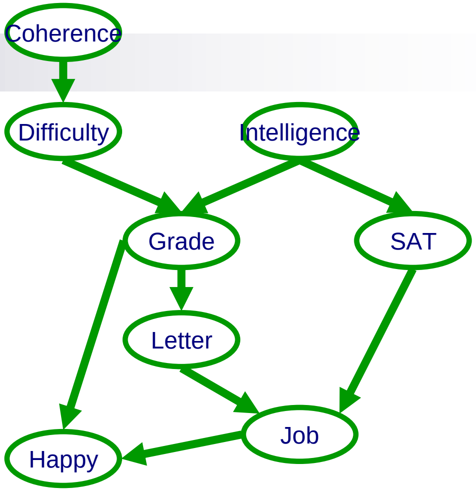
same skeleton →

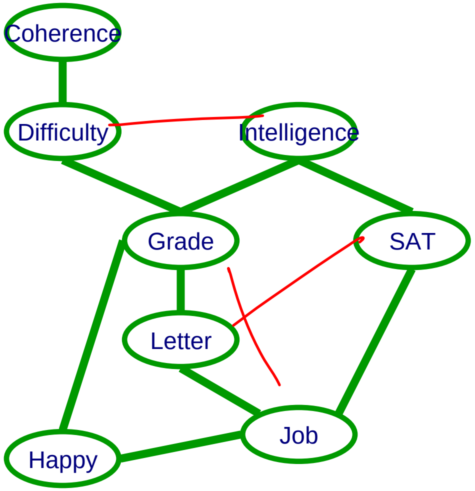
moralize



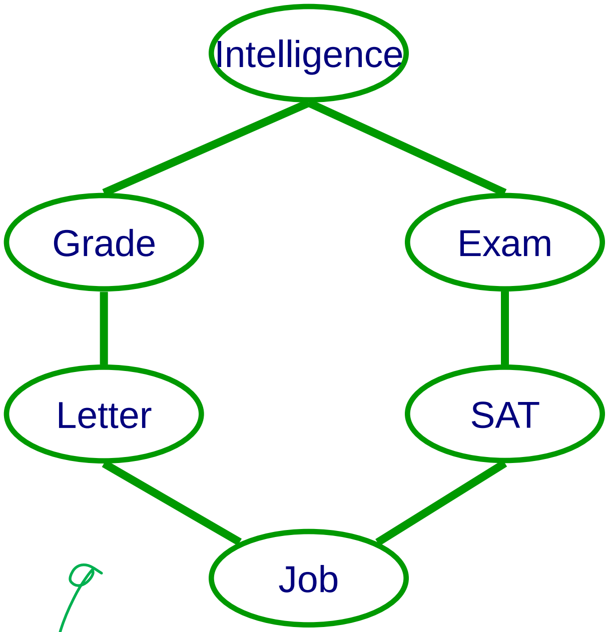
BNs ~~!~~ → MNs: Moralization

- **Theorem**: Given a BN *G* the Markov net *H* formed by moralizing *G* is the *minimal I-map* for I(*G*)
- **Intuition**:
  - □ in a Markov net, each factor must correspond to a subset of a clique
  - □ the factors in BNs are the CPTs
  - □ CPTs are factors over a node and its parents
  - □ thus node and its parents must form a clique
- **Effect**:
  - □ **some** independencies that could be read from the BN graph become hidden





From Markov nets to Bayes nets



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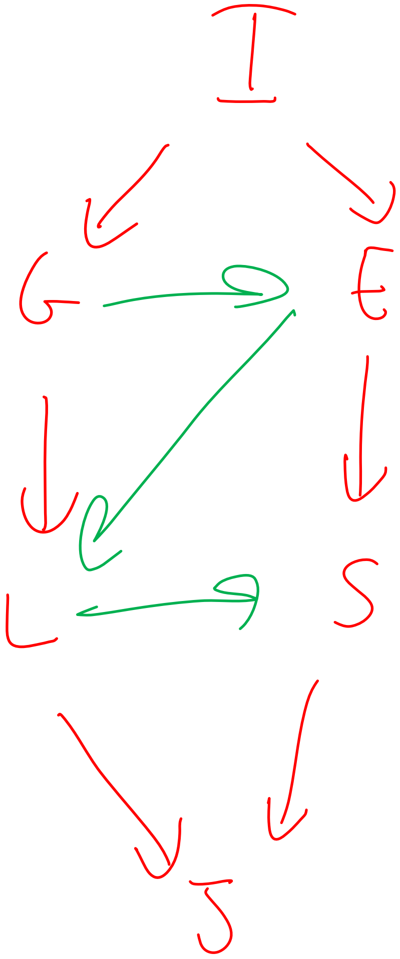
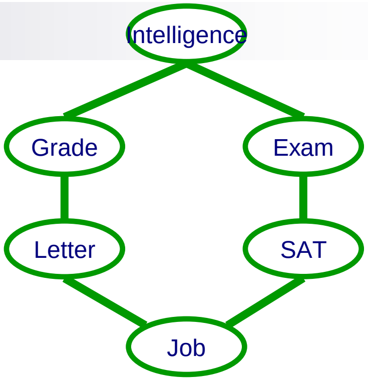
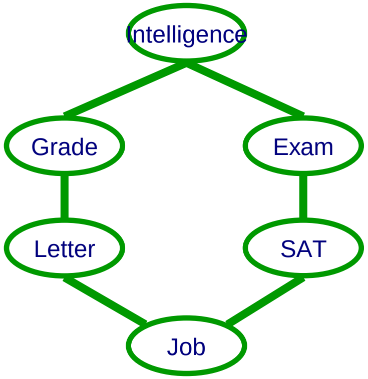


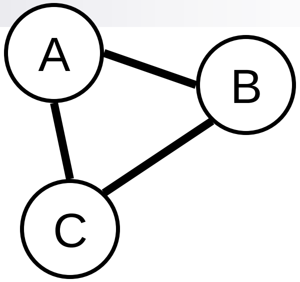
MNs ! BNs: Triangulation

- **Theorem**: Given a MN *H,* let *G* be the Bayes net that is a *minimal I-map* for I(*H)* then *G* must be **chordal**
- **Intuition**:
  - □ v-structures in BN introduce immoralities
  - □ these immoralities were not present in a Markov net
  - □ the triangulation eliminates immoralities
- **Effect**:
  - □ **many** independencies that could be read from the MN graph become hidden





Markov nets v. Pairwise MNs



- Every Markov network can be transformed into a Pairwise Markov net
  - introduce extra “variable” for each factor over three or more variables
  - domain size of extra variable is exponential in number of vars in factor
- **Effect**:
  - any local structure in factor is lost
  - a chordal MN doesn’t look chordal anymore

Overview of types of graphical models and transformations between them