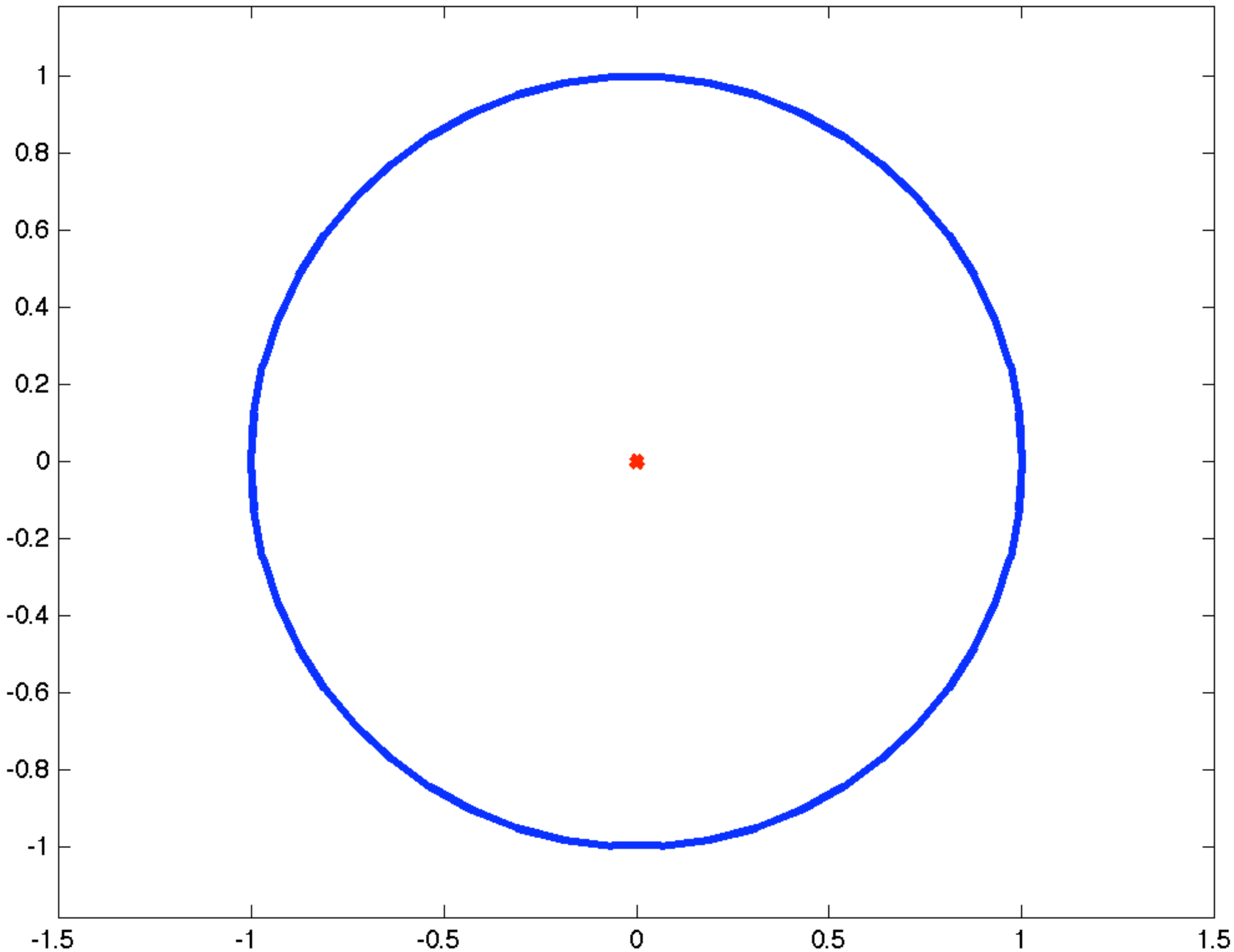
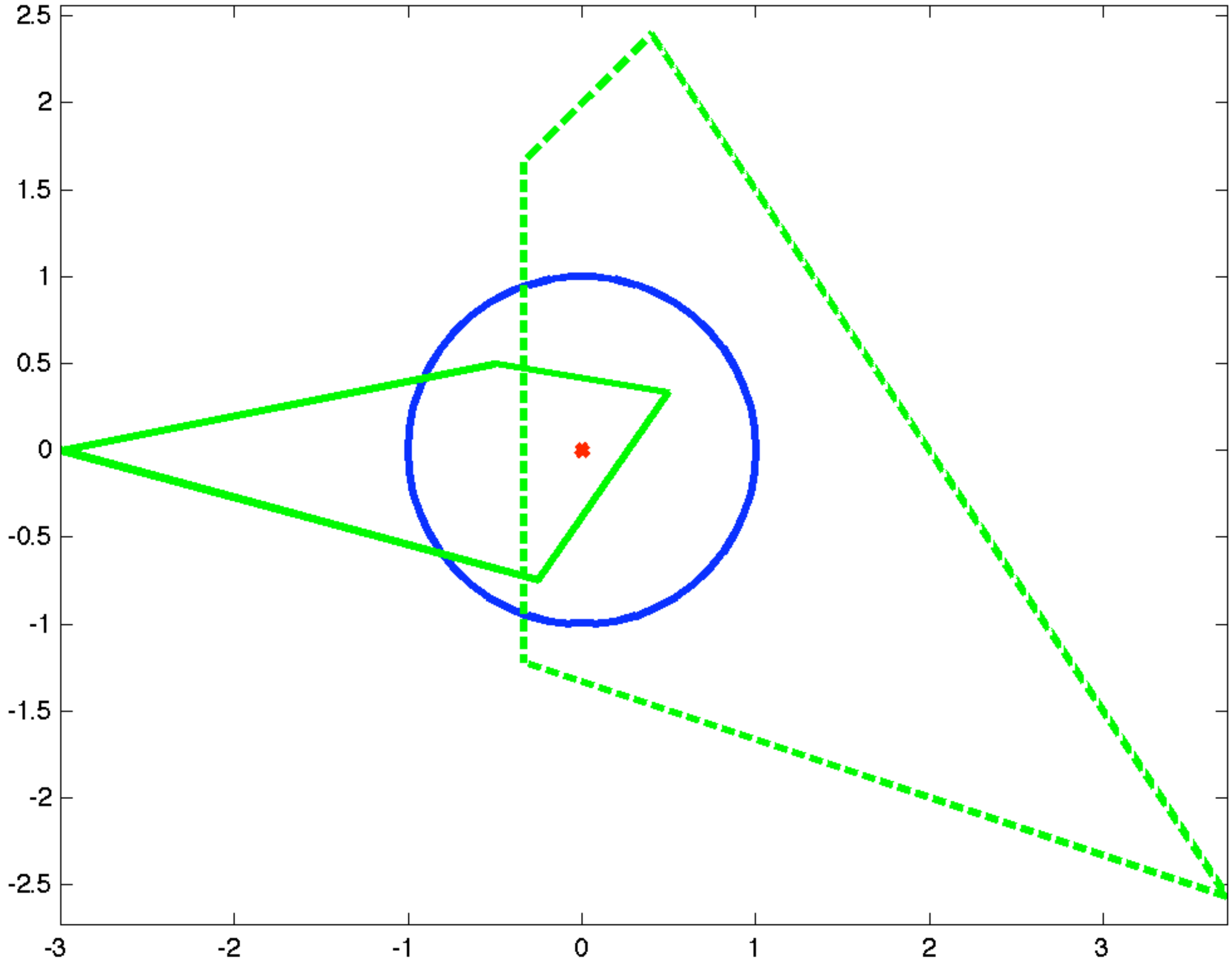
Review of duality so far

- LP/QP duality, cone duality, set duality
- All are **halfspace bounds**
  - on a cone
  - on a set
  - on objective of LP/QP

Set duality



Set duality

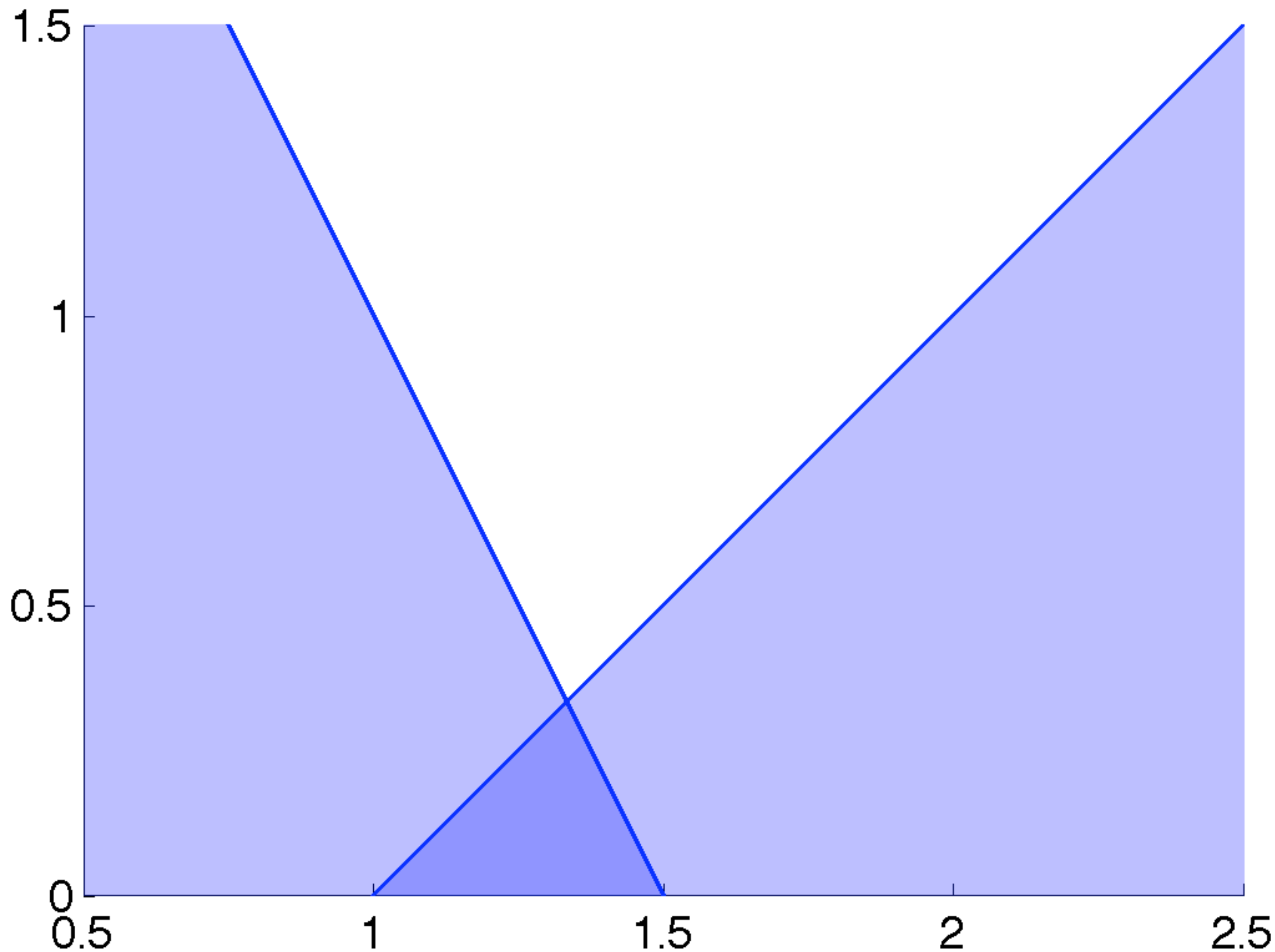


LP/QP objective

min z s.t.

z ≥ x - 1

z ≥ 3 - 2x



Dual functions

- Arbitrary function $F(x)$
- Dual is $F^*(y) =$

- For example: $F(x) = x^T x/2$
- $F^*(y) =$

Fenchel's inequality

- $F^*(y) = \sup_x [x^T y - F(x)]$

Duality and subgradients

- Suppose $F(x) + F^*(y) - x^T y = 0$

Duality examples

- $1/2 - \ln(-x)$
- e^x
- $x \ln(x) - x$

More examples

- $F(x) = x^T Q x/2 + c^T x$, Q psd:

- $F(X) = -\ln |X|$, X psd:

Indicator functions

- Recall: for a set S,

  $I_S(x) =$

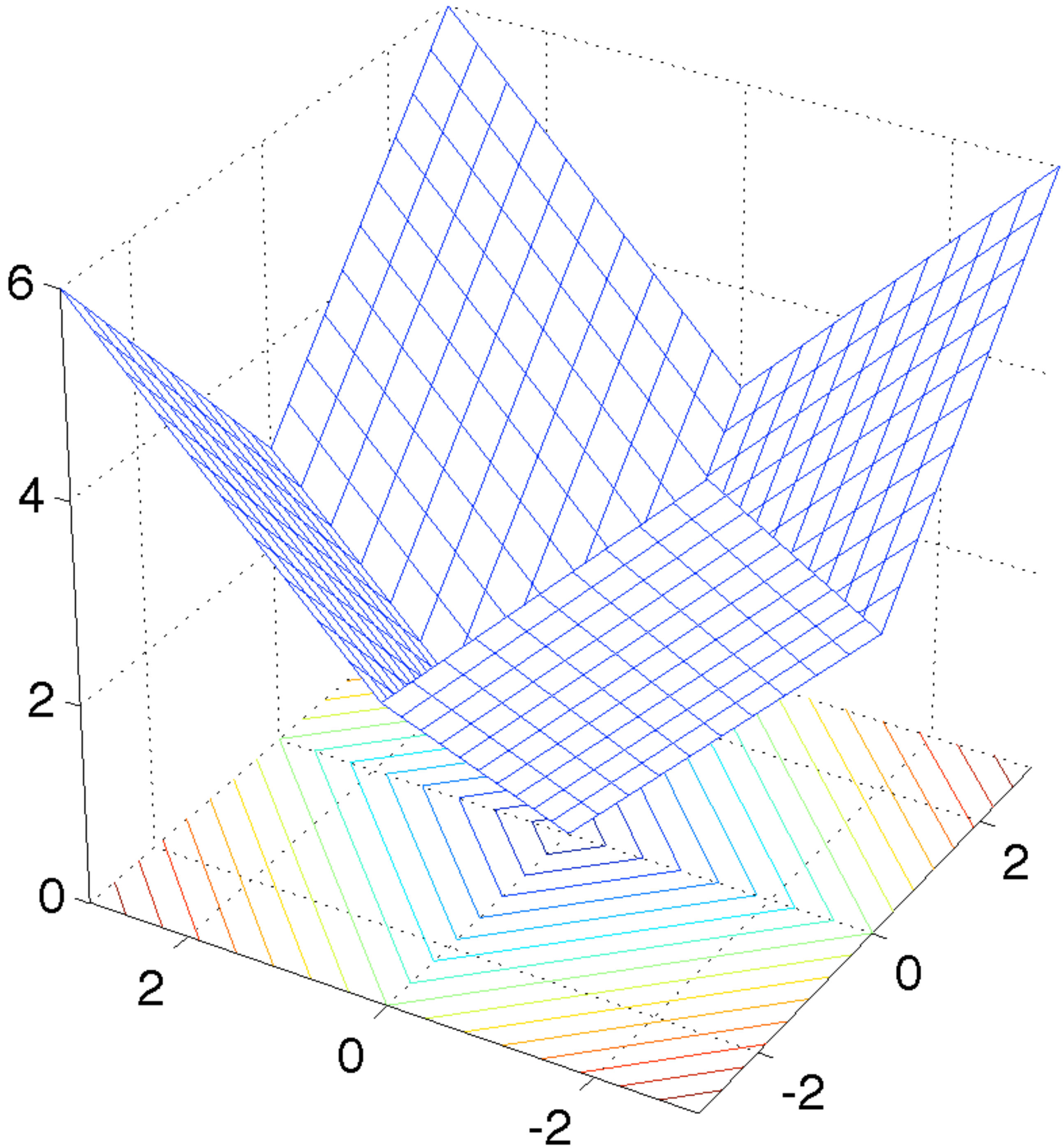
- E.g., $I_{[-1,1]}(x)$:

Duals of indicators

- $I_a(x)$, point a:

- $I_K(x)$, cone K:

- $I_C(x)$, set C:



6
4
2
0
2
0
-2
2
0
-2

Properties

- $F(x) \geq G(x)$ $F^*(y)$ $G^*(y)$

- F^* is closed, convex
- F^{**} = cl conv F (= F if F closed, convex)

- If F is differentiable:

Working with dual functions

- $G(x) = F(x) + k$
- $G(x) = k\,F(x) \qquad k > 0$
- $G(x) = F(x) + a^T x$

Working with dual functions

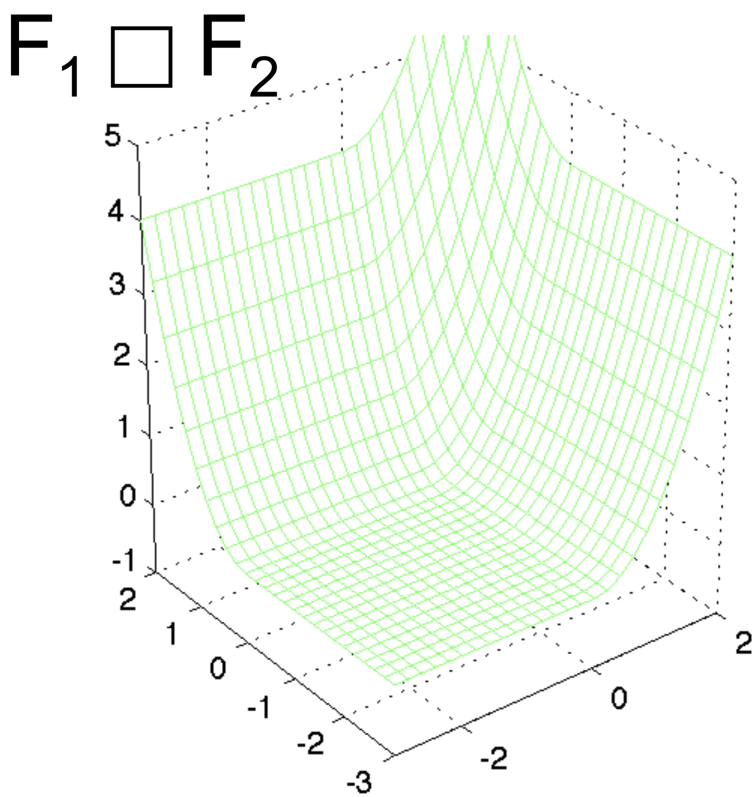
- $G(x_1, x_2) = F_1(x_1) + F_2(x_2)$

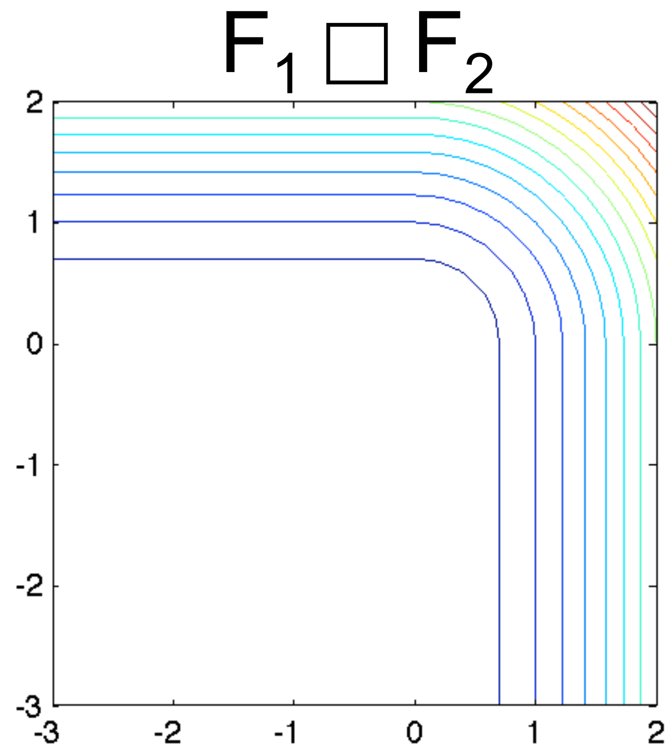
An odd-looking operation

- Definition: **infimal convolution**

- E.g., $F_1(x) = I_{[-1,1]}(x)$, $F_2(x) = |x|$

Infimal convolution example





- $F_1(x) = I_{\leq 0}(x)$, $F2(x) = x^2$

Dual of infimal convolution

- $G(x) = F_1(x) \square F_2(x)$
- $G^*(y) =$

- $G(x) = F_1(x) + F_2(x)$ $\qquad$ $G^*(y) =$

Convex program duality

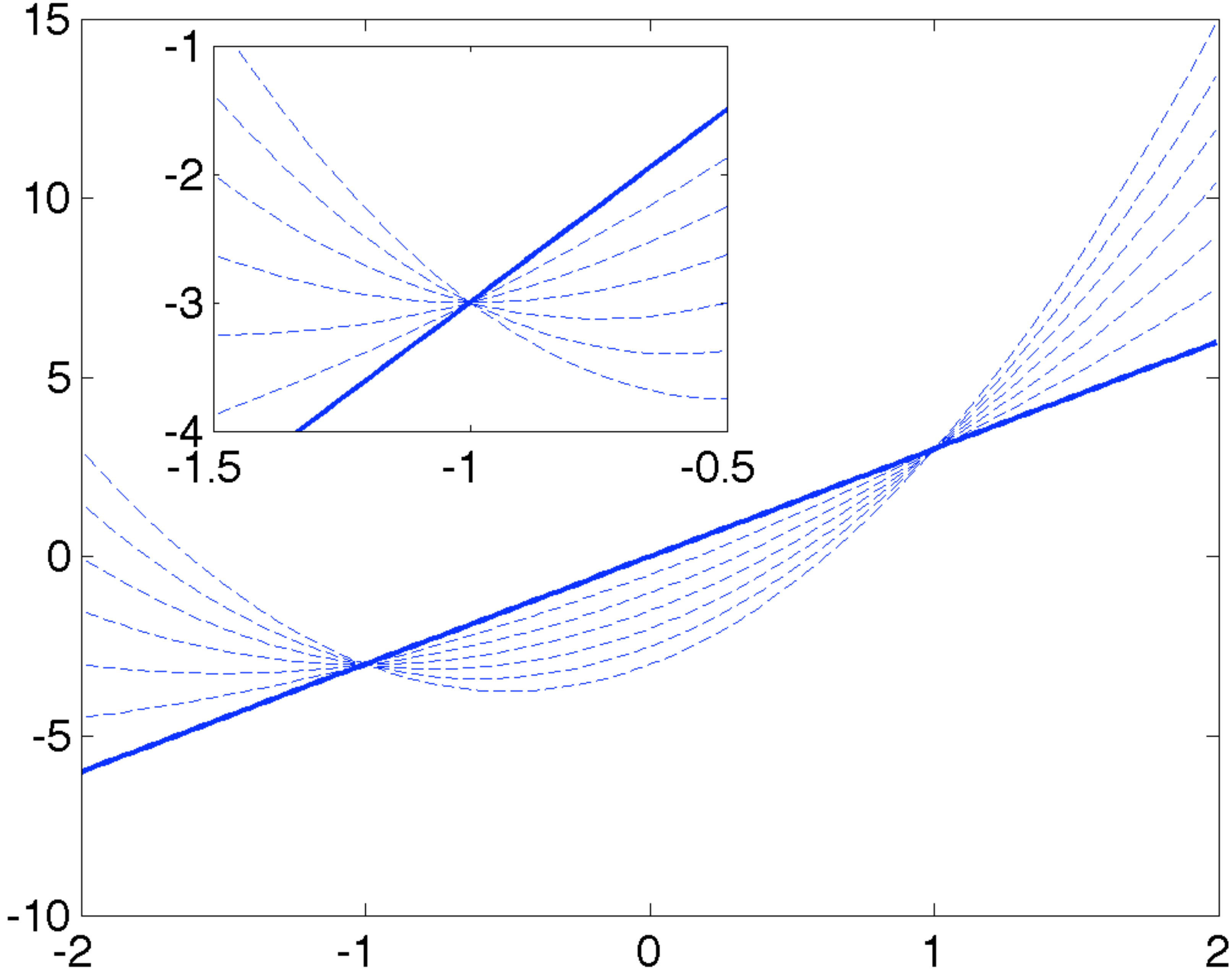
- min $f(x)$ s.t.

  $Ax = b$

  $g_i(x) \leq 0 \qquad i \in I$

Duality example

- min $3x$ s.t. $x^2 \le 1$
- $L(x, y) = 3x + y(x^2 - 1)$



15
10
5
0
-5
-10
-2
-1
0
1
2
-1
-2
-3
-4
-1.5
-1
-0.5

Dual function

- $L(y) = \inf_x L(x,y) = \inf_x 3x + y(x^2 - 1)$

Dual function