Review of Linear Algebra

10-725 - Optimization
1/16/08 Recitation

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In this review

- Recall concepts we'll need in this class
- Geometric intuition for linear algebra
- Outline:
  - Matrices as linear transformations or as sets of constraints
  - Linear systems & vector spaces
  - Solving linear systems
  - Eigenvalues & eigenvectors

Basic concepts

- *Vector* in R^n is an ordered set of n real numbers.
  - e.g. v = (1,6,3,4) is in R^4
  - "(1,6,3,4)" is a column vector: $\begin{pmatrix} 1 \\ 6 \\ 3 \\ 4 \end{pmatrix}$
  - as opposed to a row vector: $\begin{pmatrix} 1 & 6 & 3 & 4 \end{pmatrix}$
- m-by-n *matrix* is an object with m rows and n columns, each entry fill with a real number: $\begin{pmatrix} 1 & 2 & 8 \\ 4 & 78 & 6 \\ 9 & 3 & 2 \end{pmatrix}$

Basic concepts

- Transpose: reflect vector/matrix on line: \

$$\begin{pmatrix} a \\ b \end{pmatrix}^T = \begin{pmatrix} a & b \end{pmatrix} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

  - Note: $(Ax)^T = x^T A^T$ (We'll define multiplication soon…)
- Vector norms:
  - L_p norm of $v = (v_1,\ldots,v_k)$ is $(\Sigma_i |v_i|^p)^{1/p}$
  - Common norms: L_1, L_2
  - $L_{infinity} = \max_i |v_i|$
- Length of a vector v is $L_2(v)$

Basic concepts

- Vector dot product: $u \bullet v = (u_1 \quad u_2) \bullet (v_1 \quad v_2) = u_1 v_1 + u_2 v_2$
  - Note dot product of u with itself is the square of the length of u.
- Matrix product:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$AB = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

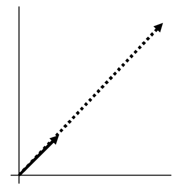
Basic concepts

- Vector products:
  - Dot product: $u \bullet v = u^T v = (u_1 \quad u_2) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = u_1 v_1 + u_2 v_2$
  - Outer product:

$$uv^T = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} (v_1 \quad v_2) = \begin{pmatrix} u_1 v_1 & u_1 v_2 \\ u_2 v_1 & u_2 v_2 \end{pmatrix}$$

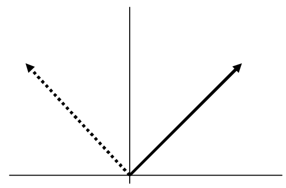
Matrices as linear transformations

$$\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$$



(stretching)

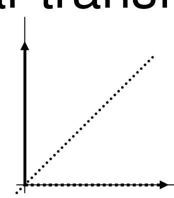
$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$



(rotation)

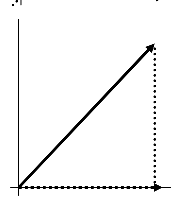
Matrices as linear transformations

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



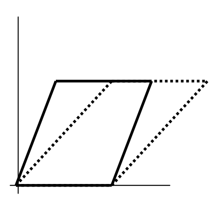
(reflection)

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



(projection)

$$\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + cy \\ y \end{pmatrix}$$



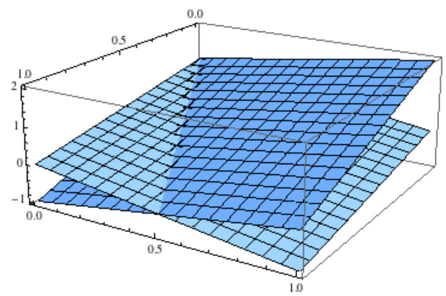
(shearing)

Matrices as sets of constraints

$$x + y + z = 1$$

$$2x - y + z = 2$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$



Special matrices

$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$ diagonal

$\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$ upper-triangular

$\begin{pmatrix} a & b & 0 & 0 \\ c & d & e & 0 \\ 0 & f & g & h \\ 0 & 0 & i & j \end{pmatrix}$ tri-diagonal

$\begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix}$ lower-triangular

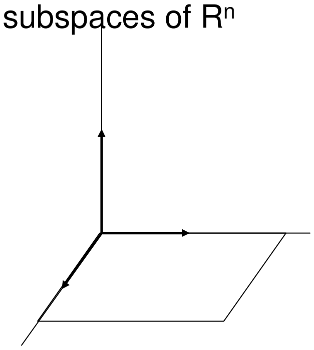
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ I (identity matrix)

Vector spaces

- Formally, a *vector space* is a set of vectors which is closed under addition and multiplication by real numbers.
- A *subspace* is a subset of a vector space which is a vector space itself, e.g. the plane z=0 is a subspace of R^3 (It is essentially R^2.).
- We'll be looking at R^n and subspaces of R^n

Our notion of planes in R^3 may be extended to *hyperplanes* in R^n (of dimension n-1)

Note: subspaces must include the origin (zero vector).

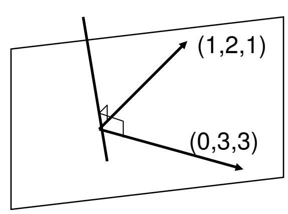


Linear system & subspaces

$$\begin{pmatrix} 1 & 0 \\ 2 & 3 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$u \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + v \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

- Linear systems define certain subspaces
- Ax = b is solvable iff b may be written as a linear combination of the columns of A
- The set of possible vectors b forms a subspace called the *column space* of A



Linear system & subspaces

The set of solutions to Ax = 0 forms a subspace called the *null space* of A.

$$\begin{pmatrix} 1 & 0 \\ 2 & 3 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

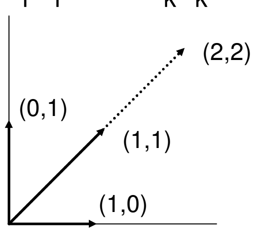
→ Null space: {(0,0)}

$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & 5 \\ 1 & 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

→ Null space: {(c,c,-c)}

Linear independence and basis

- Vectors $v_1,\ldots,v_k$ are linearly independent if $c_1v_1+\ldots+c_kv_k = 0$ implies $c_1=\ldots=c_k=0$



i.e. the nullspace is the origin

$$\begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

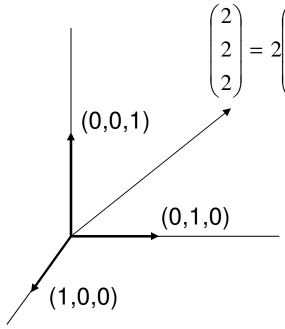
$$\begin{pmatrix} 1 & 0 \\ 2 & 3 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Recall nullspace contained only (u,v)=(0,0).
i.e. the columns are linearly independent.

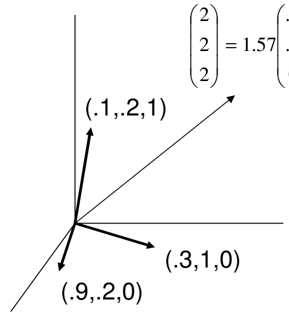
Linear independence and basis

- If all vectors in a vector space may be expressed as linear combinations of $v_1, \ldots, v_k$, then $v_1, \ldots, v_k$ *span* the space.

$$\begin{pmatrix}2\\2\\2\end{pmatrix} = 2\begin{pmatrix}1\\0\\0\end{pmatrix} + 2\begin{pmatrix}0\\1\\0\end{pmatrix} + 2\begin{pmatrix}0\\0\\1\end{pmatrix}$$



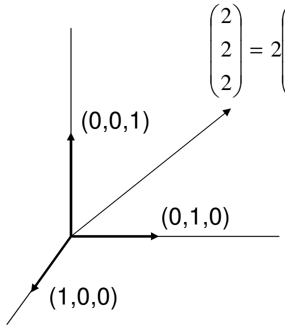
$$\begin{pmatrix}2\\2\\2\end{pmatrix} = 1.57\begin{pmatrix}.9\\.2\\0\end{pmatrix} + 1.29\begin{pmatrix}.3\\1\\0\end{pmatrix} + 2\begin{pmatrix}.1\\.2\\1\end{pmatrix}$$



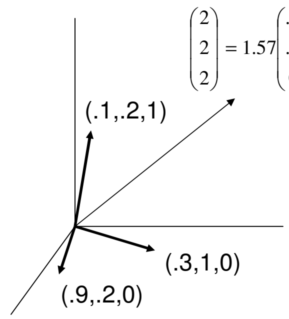
Linear independence and basis

- A *basis* is a set of linearly independent vectors which span the space.
- The *dimension* of a space is the # of "degrees of freedom" of the space; it is the number of vectors in any basis for the space.
- A basis is a maximal set of linearly independent vectors and a minimal set of spanning vectors.

$$\begin{pmatrix}2\\2\\2\end{pmatrix} = 2\begin{pmatrix}1\\0\\0\end{pmatrix} + 2\begin{pmatrix}0\\1\\0\end{pmatrix} + 2\begin{pmatrix}0\\0\\1\end{pmatrix}$$



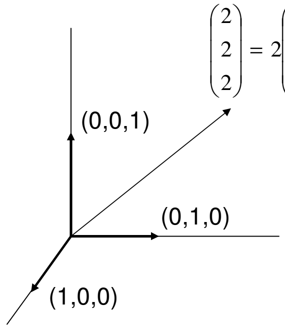
$$\begin{pmatrix}2\\2\\2\end{pmatrix} = 1.57\begin{pmatrix}.9\\.2\\0\end{pmatrix} + 1.29\begin{pmatrix}.3\\1\\0\end{pmatrix} + 2\begin{pmatrix}.1\\.2\\1\end{pmatrix}$$



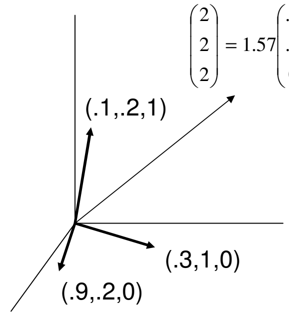
Linear independence and basis

- Two vectors are *orthogonal* if their dot product is 0.
- An *orthogonal basis* consists of orthogonal vectors.
- An *orthonormal basis* consists of orthogonal vectors of unit length.

$$\begin{pmatrix}2\\2\\2\end{pmatrix} = 2\begin{pmatrix}1\\0\\0\end{pmatrix} + 2\begin{pmatrix}0\\1\\0\end{pmatrix} + 2\begin{pmatrix}0\\0\\1\end{pmatrix}$$



$$\begin{pmatrix}2\\2\\2\end{pmatrix} = 1.57\begin{pmatrix}.9\\.2\\0\end{pmatrix} + 1.29\begin{pmatrix}.3\\1\\0\end{pmatrix} + 2\begin{pmatrix}.1\\.2\\1\end{pmatrix}$$



About subspaces

- The *rank* of A is the dimension of the column space of A.
- It also equals the dimension of the *row space* of A (the subspace of vectors which may be written as linear combinations of the rows of A).

$$\begin{pmatrix}1 & 0\\2 & 3\\1 & 3\end{pmatrix}$$

(1,3) = (2,3) – (1,0)

Only 2 linearly independent rows, so rank = 2.

About subspaces

Fundamental Theorem of Linear Algebra:

If A is m x n with rank r,

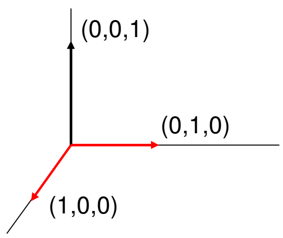
Column space(A) has dimension r

Nullspace(A) has dimension n-r (= *nullity* of A)

Row space(A) = Column space(A^T) has dimension r

Left nullspace(A) = Nullspace(A^T) has dimension m - r

Rank-Nullity Theorem: rank + nullity = n



$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

m = 3
n = 2
r = 2

Non-square matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{pmatrix}$$

m = 3
n = 2
r = 2
System Ax=b may not have a solution (x has 2 variables but 3 constraints).

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}$$

m = 2
n = 3
r = 2
System Ax=b is underdetermined (x has 3 variables and 2 constraints).

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

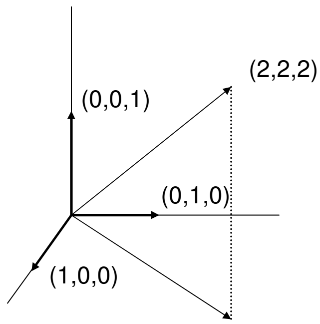
Basis transformations

- Before talking about basis transformations, we need to recall matrix inversion and projections.

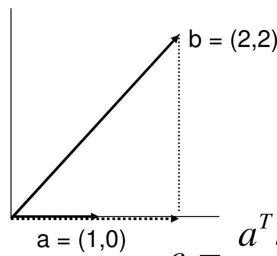
Matrix inversion

- To solve Ax=b, we can write a closed-form solution if we can find a matrix A^{-1}
  s.t. $AA^{-1} = A^{-1}A = I$ (identity matrix)
- Then Ax=b iff $x = A^{-1}b$:
  $x = Ix = A^{-1}Ax = A^{-1}b$
- A is *non-singular* iff A^{-1} exists iff Ax=b has a unique solution.
- Note: If A^{-1}, B^{-1} exist, then $(AB)^{-1} = B^{-1}A^{-1}$,
  and $(A^T)^{-1} = (A^{-1})^T$

Projections



$$\begin{pmatrix}2\\2\\0\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}\begin{pmatrix}2\\2\\2\end{pmatrix}$$

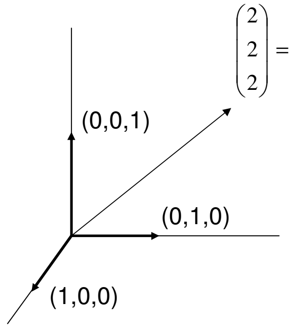


$$c=\frac{a^T b}{a^T a}a=\begin{pmatrix}2\\0\end{pmatrix}$$

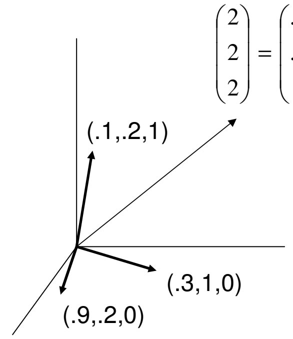
Basis transformations

We may write v=(2,2,2) in terms of an alternate basis:

$$\begin{pmatrix}2\\2\\2\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}2\\2\\2\end{pmatrix}$$



$$\begin{pmatrix}2\\2\\2\end{pmatrix}=\begin{pmatrix}.9&.3&.1\\.2&1&.2\\0&0&1\end{pmatrix}\begin{pmatrix}1.57\\1.29\\2\end{pmatrix}$$



Components of (1.57,1.29,2) are projections of v onto new basis vectors, normalized so new v still has same length.

Basis transformations

Given vector v written in standard basis, rewrite as v_Q in terms of basis Q.

If columns of Q are orthonormal, $v_Q = Q^T v$

Otherwise, $v_Q = (Q^T Q) Q^T v$

Special matrices

- Matrix A is *symmetric* if $A = A^T$
- A is *positive definite* if $x^T A x > 0$ for all non-zero x (*positive semi-definite* if inequality is not strict)

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a^2 + b^2 + c^2$$

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a^2 - b^2 + c^2$$

Special matrices

- Matrix A is *symmetric* if $A = A^T$
- A is *positive definite* if $x^TAx>0$ for all non-zero x (*positive semi-definite* if inequality is not strict)

- Useful fact: Any matrix of form A^TA is positive semi-definite.

To see this, $x^T(A^TA)x = (x^TA^T)(Ax) = (Ax)^T(Ax) \geq 0$

Determinants

- If det(A) = 0, then A is singular.
- If det(A) ≠ 0, then A is invertible.
- To compute:
  - Simple example: $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$
  - Matlab: det(A)

Determinants

- m-by-n matrix A is *rank-deficient* if it has rank r < m (≤ n)
- Thm: rank(A) < r iff

  det(A) = 0 for all t-by-t submatrices,

  r ≤ t ≤ m

Eigenvalues & eigenvectors

- How can we characterize matrices?
- The solutions to Ax = λx in the form of eigenpairs (λ,x) = (eigenvalue,eigenvector) where x is non-zero
- To solve this, (A – λI)x = 0
- λ is an eigenvalue iff det(A – λI) = 0

Eigenvalues & eigenvectors

(A – λI)x = 0

λ is an eigenvalue iff det(A – λI) = 0

Example:

$$A = \begin{pmatrix} 1 & 4 & 5 \\ 0 & 3/4 & 6 \\ 0 & 0 & 1/2 \end{pmatrix}$$

$$\det(A - \lambda I) = \begin{pmatrix} 1-\lambda & 4 & 5 \\ 0 & 3/4-\lambda & 6 \\ 0 & 0 & 1/2-\lambda \end{pmatrix} = (1-\lambda)(3/4-\lambda)(1/2-\lambda)$$

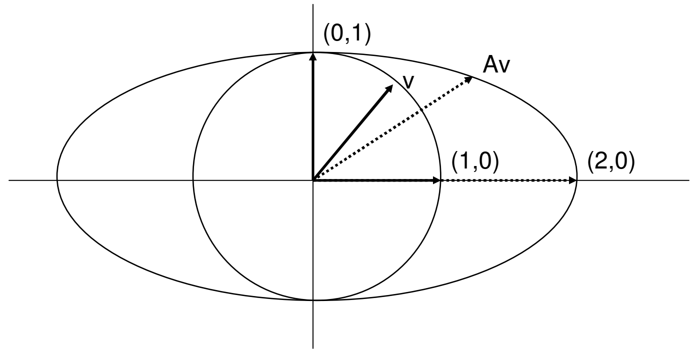
$$\lambda = 1, \lambda = 3/4, \lambda = 1/2$$

Eigenvalues & eigenvectors

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

Eigenvalues λ = 2, 1 with eigenvectors (1,0), (0,1)

Eigenvectors of a linear transformation A are not rotated (but will be scaled by the corresponding eigenvalue) when A is applied.



Solving Ax=b

x + 2y + z = 0
y - z = 2
x +2z= 1

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 2 \\ 1 & 0 & 2 & 1 \end{pmatrix}$$

Write system of equations in matrix form.

----

x + 2y + z = 0
y - z = 2
-2y + z= 1

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & -2 & 1 & 1 \end{pmatrix}$$

Subtract first row from last row.

----

x + 2y + z = 0
y - z = 2
- z = 5

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & -1 & 5 \end{pmatrix}$$

Add 2 copies of second row to last row.

Now solve by back-substitution: z = -5, y = 2-z = 7, x = -2y-z = -9

Solving Ax=b & condition numbers

- Matlab: linsolve(A,b)
- How stable is the solution?
- If A or b are changed slightly, how much does it effect x?
- The *condition number* c of A measures this:

  $c = \lambda_{max} / \lambda_{min}$
- Values of c near 1 are good.