10-725 Recitation 5-6-08

Submodularity

Recall definitions:

$\forall A, B$, with $A \subseteq B$, $F(A \cup \{x\}) - F(A) \geq F(B \cup \{x\}) - F(B)$

supermodularity $\{ \quad \leq$

modularity $\{ \quad =$

$F(A) = \sum_{A_i \in A} F(\{A_i\})$

"diminishing returns"

Equivalent definition: $\forall A, B$, $F(A) + F(B) \geq F(A \cup B) + F(A \cap B)$

In class, we looked at:

(1) $\begin{cases} \max_A F(A) \\ \text{s.t. } |A| \leq k \end{cases}$ where F is submodular

↳ water quality sensing problem

↳ greedy algorithm optimal unless $P = NP$

(2) $\begin{cases} \min_{A \subseteq V} F(A) \end{cases}$ F submodular

↳ solvable in polytime (but with ellipsoid algorithm)

↳ If F symmetric, Queryanne's Algorithm is $O(n^3)$

In this recitation,

- examples of submodular functions
- closer look at minimizing submodular functions

Examples of submodular functions

ex → Given graph $G=(V,E)$,
let $S=(V,F)$ be a subgraph : $F \subseteq E$.
S is a forest iff it contains no cycles.
e.g. spanning tree is forest with $|F|=|V|-1$

Define graphic rank function of $A \subseteq E$:
$r(A) = \max \{ |F| : F \subseteq A,\ (V,F) \text{ is forest} \}$
Then $r(A)$ is submodular.
Note: $nc(A) = |V| - r(A) = \#$ connected components in subgraph (V,A)
is supermodular.

ex → Let V be collection of random variables.
Entropy $H(S)$ where $S \subseteq V$ is submodular.
Proof:
$$0 \le I(A;B) = H(A) + H(B) - H(A \cup B) - H(A \cap B)$$

ex → Let $f(A)$ where $A \subseteq V$ be mutual information:
$f(A) = I(A, V \setminus A)$. Then $f(A)$ is submodular.
Proof:
$$I(A, V\setminus A) = H(A) + H(V \setminus A) - H(V) - \cancel{H(\emptyset)}^{\,0}$$
So, $f(A) + f(B) = [H(A) + H(B)] + [H(V\setminus A) + H(V \setminus B)] - 2H(V)$
Since $H(\cdot)$ is submodular,
$$H(A) + H(B) \ge H(A \cap B) + H(A \cup B)$$
$$H(V\setminus A) + H(V \setminus B) \ge H(V \setminus (A \cup B)) + H(V \setminus (A \cap B))$$
So
$$f(A) + f(B) \ge [H(A \cap B) + H(V \setminus (A \cap B)) - H(V)] + [H(A \cup B) + H(V \setminus (A \cup B)) - H(V)]$$
$$= f(A \cap B) + f(A \cup B)$$

Minimizing submodular functions

Submodular polyhedron P_F

Let: $|V| = n$ $\quad X \in \mathbb{R}^n$ where $X = \begin{pmatrix} x(v_1) \\ \vdots \\ x(v_n) \end{pmatrix}$

$$X(A) = \sum_{i \in A} X(V_i)$$

(e.g. $V = \{V_1, V_2, V_3\}$ $\quad n = 3 \quad X = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}$

$A = \{1, 3\}$

$X(A) = 7$

If F is a set function,

$$P_F = \{x \in \mathbb{R}^n \mid x(A) \le F(A), \ \forall A \subseteq V\}$$

(e.g. define F as:

V_1	V_2	V_3	$F(A)$
0	0	0	0
0	0	1	3
0	1	0	2
0	1	1	4
1	0	0	1
1	0	1	4
1	1	0	3
1	1	1	5

$\rightarrow$ e.g. $F(\{V_2\}) = 2$

Then P_F is

$\{x \in \mathbb{R}^n \mid X(V_1) \le 1,$
$X(V_2) \le 2,$
$X(V_3) \le 3,$
$X(V_2) + X(V_3) \le 4,$
$\ldots\}$

Let $c \in \mathbb{R}^n_+$ be positive cost vector. Suppose we want

$$\left\{ \begin{array}{l} \max\limits_x \ c^T x \\ \text{s.t. } x \in P_F \end{array} \right\} \rightarrow \text{LP with } 2^n \text{ constraints (in general)}$$

(e.g. $\max\limits_x \ (2, 9, 8) \cdot x$

s.t. $x \in P_F$

Solve: Order V_2, V_3, V_1 since $c(V_2) = 9 \ge c(V_3) \ge c(V_1)$

Optimal X^* is: $X^*(V_2) = F(\{V_2\}) = 2$

$X^*(V_3) = F(\{V_2, V_3\}) - F(\{V_2\}) = 4 - 2 = 2$

$X^*(V_1) = F(\{V_1, V_2, V_3\}) - F(\{V_2, V_3\}) = 5 - 4 = 1$

$$X^* = \underset{V_1 \ V_2 \ V_3}{(1, 2, 2)}$$

Suppose c_A : $c_A(i) = \begin{cases} 1 & \text{if } i \in A \\ 0 & \text{otherwise} \end{cases}$

Consider $\begin{cases} \max\limits_x & c_A^T x \\ \text{s.t.} & x \in P_F \end{cases}$

Then $c_A^T x^* = F(A)$.

e.g. Let $c_A = (1, 0, 1)$ instead of previous value. i.e. $A = \{V_1, V_3\}$

Solve for x^*:

Order V_1, V_3, V_2 since $c(V_1) \geq c(V_3) \geq c(V_2)$.

$x^*(V_1) = F(\{V_1\}) = 1$

$x^*(V_3) = F(\{V_1, V_3\}) - F(\{V_1\}) = 4 - 1 = 3$

$x^*(V_2) = F(\{V_1, V_2, V_3\}) - F(\{V_1, V_3\}) = 5 - 4 = 1$

So $c_A^T x^* = 4 = F(A)$

Note sum in $c_A^T x^*$ telescopes.

Define $\hat{f}(c_A) = \begin{cases} \max\limits_x & c_A^T x \\ \text{s.t.} & x \in P_F \end{cases}$

Now extend to general cost vectors c:

$c = \sum_i \lambda_i c_{A_i}$ where c_{A_i} is 01-cost, $A_1 \supset A_2 \supset \ldots$

e.g. $c = \begin{pmatrix} 7 \\ 1.5 \\ 2 \end{pmatrix}$ ← pick smallest $c(i)$

$c_{A_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\lambda_1 = 1.5$

Now recurse on $c - \lambda_1 c_{A_1} = \begin{pmatrix} 5.5 \\ 0 \\ .5 \end{pmatrix}$

to get:

$c_{A_2} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\lambda_2 = .5$

$c_{A_3} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\lambda_3 = 5$

Then $\hat{f}(c) = \sum_i \lambda_i \hat{f}(c_{A_i}) = \sum_i \lambda_i F(A_i)$

Thm: $\hat{f}(c)$ is convex in $c \in \mathbb{R}_+^n$ iff F is submodular.

(I'm not posting notes on Queyranne's Algorithm, so if you're interested in learning more about it, look up: Maurice Queyranne. "Minimizing symmetric submodular functions." Math. Prog. 82 (1998).