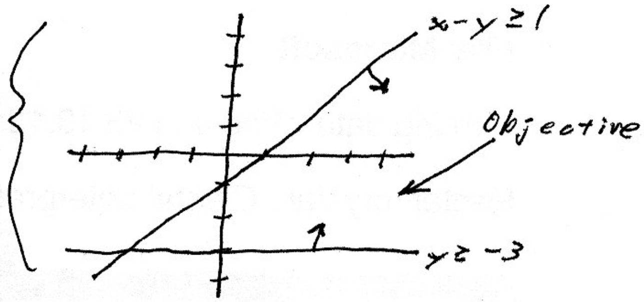
Recitation 1/31/08: LP Duality & QPs ①

Recall definition of an LP:

linear objective with linear inequality (+ equality) constraints

e.g. $\left\{\begin{array}{l} \min_{x,y} \; 3x+y \\ \text{s.t. } x-y\geq 1 \\ \qquad\; y\geq -3 \end{array}\right\} \rightarrow \left\{\begin{array}{l} \min_{x,y} \; 3x+y \\ \text{s.t. } x-y-1\geq 0 \\ \qquad\; y+3 \geq 0 \end{array}\right.$

feasible (clear from picture)



Now, change to dual:

Take general linear combination of constraints:

$$\begin{array}{l} a(x-y-1)\geq 0 \\ b(\;y+3)\geq 0 \\ \hline a(x-y-1)+b(y+3)\geq 0 \end{array} \qquad (\text{where } a,b\geq 0)$$

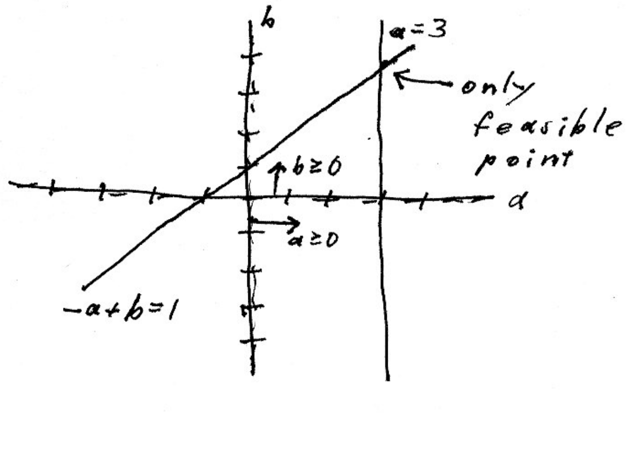
$$ax+(-a+b)y\geq a-3b$$

If $a=3$ and $-a+b=1$, then

$$ax+(-a+b)y = \underbrace{3x+y}_{\text{primal objective}} \geq \underbrace{a-3b}_{\text{lower bound}}$$

So dual problem is to maximize this lower bound.

$$\left\{\begin{array}{l} \max_{a,b} \; a-3b \\ \text{s.t. } a=3 \\ \qquad -a+b=1 \\ \qquad a,b\geq 0 \end{array}\right.$$



Now, let's go back to primal
by taking the dual of the dual:

Take general linear combination of dual constraints

[Note: Change dual objective to $\min -\alpha + 3b$ instead of $\max \alpha - 3b$ b/c the method we're using to get dual relies on minimization problem.

$$x(\alpha-3) + y(-\alpha+b-1) + z\alpha + wb \geq 0 \quad , \quad x, y \in \mathbb{R}, \; z, w \geq 0$$

[Note: (Related to above note, we have $x, y \in \mathbb{R}$ since equality constraints and $z, w \geq 0$ since inequality constraints. If we were deriving dual of a maximization problem, we would require $z, w \leq 0$.

$$(x - y + z)\alpha + (y + w)b \geq 3x + y$$

If $x - y + z = -1$ and $y + w = +3$, then

$$(x-y+z)\alpha + (y+w)b = \underbrace{-\alpha + 3b}_{\text{dual objective}} \geq \underbrace{3x + y}_{\text{lower bound}}$$

Also, we can eliminate w, z:

$$x - y + z = -1 \qquad\qquad y + w = 3$$
$$z = -1 - x + y \geq 0 \qquad\qquad w = 3 - y \geq 0$$
$$-x + y \geq 1 \qquad\qquad -y \geq -3$$

So dual of dual is:

$$\max_{x,y} 3x + y \text{ s.t. } -x + y \geq 1 \,, \; -y \geq -3$$

Set $x = -x$ and $y = -y$ to get

$$\min_{x,y} 3x + y \text{ s.t. } x - y \geq 1 \,, \; y \geq 3$$

This is our original (primal) problem!

* For LPs, the dual of the dual is the primal. *

↳ b/c strong duality holds (more later...)

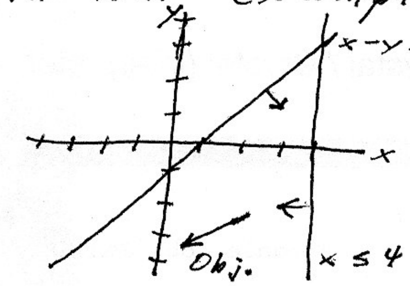
For LPs,

1. If primal is feasible + finite, dual is too.
2. If primal is unbounded, dual is infeasible.
3. If dual " " , primal " " .

To illustrate 2:

Change 1 constraint in our example:

$$\min_{x,y} \; 3x + y \quad \text{s.t.} \quad x - y \geq 1, \quad x \leq 4$$



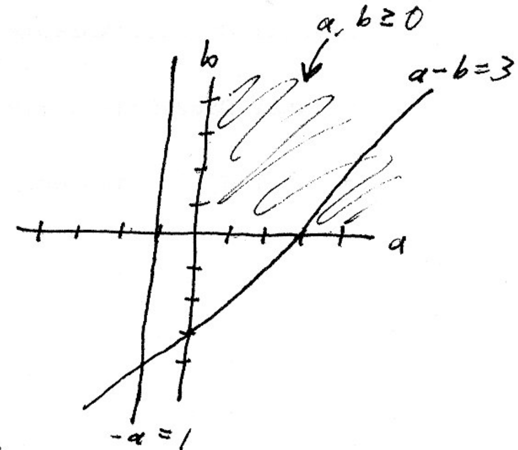
~~unbod~~ unbounded

Look at dual:

$$a(x-y-1) + b(-x+4) \geq 0$$

$$(a-b)x - ay \geq a - 4b$$

$$\text{dual:} \quad \max_{a,b} \; a - 4b \quad \text{s.t.} \quad a - b = 3, \quad -a = 1, \quad a, b \geq 0$$



infeasible

LP Duality: Lagrangians

For LPs, this is an equivalent way of looking at duality.

Say problem is: $\begin{cases} \min_{x,y} & 3x+y \\ \text{s.t.} & x-y \geq 1 \\ & y \geq -3 \end{cases}$

Then write out Lagrangian as:

$L(a,b,x,y) =$

$$\underbrace{\left[3x+y\right]}_{\substack{\text{objective}\\ \text{of primal}\\ \text{problem}}} - \underbrace{\left[a(x-y-1) + b(y+3)\right]}_{\text{constraints}}$$

constraints (same linear combination of "$\ldots \geq 0$" as in previous explanation of duality)

[Note: a, b are called Lagrange multipliers.

We now want to solve:

$$\underset{x,y}{\min} \; \underset{a,b \geq 0}{\max} \; L(a,b,x,y)$$

minimize objective (min over x,y); enforce constraints (max over $a,b \geq 0$)

To understand this, think of this as a game:

MIN player wants to choose x,y to minimize $L()$, and

MAX player wants to choose $a,b \geq 0$ to maximize $L()$.

If multiplier $= 0$, then constraint is not binding.

Suppose MIN chooses $y < -3$ (so constraint is violated). Then MAX can choose b to make $-b(y+3)$ arbitrarily large; so, MIN can't violate constraint b/c multiplier b is enforcing it.

This is complementary slackness: if Lagrange multiplier is > 0, then the corresponding constraint is binding.

To derive dual from Lagrangian, write

$$L(a,b,x,y) = \underbrace{(3-a)x + (1+a-b)y}_{\text{dual constraints}} + \underbrace{a - 3b}_{\text{dual objective}}$$

$$\left.\begin{aligned} \frac{\partial}{\partial x} L(a,b,x,y) &= 3 - a = 0 \\ \frac{\partial}{\partial y} L(a,b,x,y) &= \underbrace{1 + a - b = 0}_{\text{same dual constraints as before}} \end{aligned}\right\}$$

Set $=0$ since there must be no incentive for MIN player to deviate.

QPs

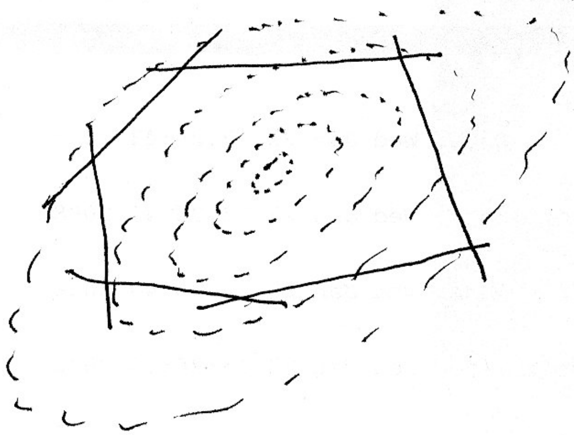
QP definition: quadratic objective,
linear inequality + equality constraints

$$\min_x \ \tfrac{1}{2} x^T H x + c^T x$$
$$\text{s.t. } Ax + b \geq 0$$
$$Ex + d = 0$$

Convex QP definition: same as QP except positive semidefinite objective
(H is +ve semidef.)

QPs are hard (NP-complete), but convex QPs are not (about same as LPs).

↳ Geometric intuition:



If ~~convex~~ general QP, then optimal solution might lie at any corner of polytope, e.g. say objective is largest at center of ellipses. There could be a lot of corners to check....

But if convex QP, solution will be in interior, or if solution is at boundary of feasible region, then duality will make problem easy to solve.